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ON THE MONOID OF COFINITE PARTIAL ISOMETRIES OF $\mathbb{N}$ WITH A BOUNDED FINITE NOISE

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ABSTRACT. In the paper we study algebraic properties of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ of cofinite partial isometries of the set of positive integers $\mathbb{N}$ with the bounded finite noise j. For the monoids $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ we prove counterparts of some classical results of Eberhart and Selden describing the closure of the bicyclic semigroup in a locally compact topological inverse semigroup. In particular we show that for any positive integer j every Hausdorff shift-continuous topology τ on $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is discrete and if $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a proper dense subsemigroup of a Hausdorff semitopological semigroup S, then $S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a closed ideal of S, and moreover if S is a topological inverse semigroup then $S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a topological group. Also we describe the algebraic and topological structure of the closure of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in a locally compact topological inverse semigroup.

KEYWORDS: partial isometry, inverse semigroup, partial bijection, bicyclic monoid, closure, locally compact, topological inverse semigroup

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1. INTRODUCTION AND PRELIMINARIES

In this paper we shall follow the terminology of [9, 12, 27, 29]. We shall denote the first infinite cardinal by ω and the cardinality of a set A by $|A|$. By $\mathrm{cl}_X(A)$ we denote the closure of subset A in a topological space X.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\mathrm{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

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If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S).

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S and it induces the *natural partial order* on the semilattice $E(S)$ [30].

An inverse subsemigroup T of an inverse semigroup S is called *full* if $E(T) = E(S)$.

A congruence $\mathfrak{C}$ on a semigroup S is called a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group. Any inverse semigroup S admits the *minimum group congruence* $\mathfrak{C}_{\mathbf{mg}}$:

$$a\mathfrak{C}_{\mathbf{mg}}b \quad \text{if and only if} \quad \text{there exists} \quad e \in E(S) \quad \text{such that} \quad ea = eb.$$

Also, we say that a semigroup homomorphism $\mathfrak{h}\colon S \to T$ is a *group homomorphism* if the image $(S)\mathfrak{h}$ is a group, and $\mathfrak{h}\colon S \to T$ is *trivial* if it is either an isomorphism or annihilating.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l \cdot q^mp^n = q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [12].

If $\alpha\colon X \rightharpoonup Y$ is a partial map, then we shall denote the domain and the range of α by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. A partial map $\alpha\colon X \rightharpoonup Y$ is called *cofinite* if both sets $X \setminus \operatorname{dom}\alpha$ and $Y \setminus \operatorname{ran}\alpha$ are finite.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of a non-zero cardinal λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha\colon y\alpha \in \operatorname{dom}\beta\}, \quad \text{for } \alpha,\beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse* (*monoid*) *semigroup* over the cardinal λ (see [12]). The symmetric inverse semigroup was introduced by Wagner [30] and it plays a major role in the theory of semigroups. By $\mathscr{I}^{\mathrm{cf}}_\lambda$ is denoted a subsemigroup of injective partial selfmaps of λ with cofinite domains and ranges in $\mathscr{I}_\lambda$. Obviously, $\mathscr{I}^{\mathrm{cf}}_\lambda$ is an inverse submonoid of the semigroup $\mathscr{I}_\lambda$. The semigroup $\mathscr{I}^{\mathrm{cf}}_\lambda$ is called the *monoid of injective partial cofinite selfmaps* of λ [20].

A partial transformation $\alpha\colon (X,d) \rightharpoonup (X,d)$ of a metric space (X,d) is called *isometric* or a *partial isometry*, if $d(x\alpha, y\alpha) = d(x,y)$ for all $x,y \in \operatorname{dom}\alpha$. It is obvious that the composition of two partial isometries of a metric space (X,d) is a partial isometry, and the converse partial map to a partial isometry is a partial isometry, too. Hence the set of partial isometries of a metric space (X,d) with the operation of composition of partial isometries is an inverse submonoid of the symmetric inverse monoid over the cardinal $|X|$. Also, it is obvious that the set of partial cofinite isometries of a metric space (X,d) with the operation the composition of partial isometries is an inverse submonoid of the monoid of injective partial cofinite selfmaps of the cardinal $|X|$.

We endow the sets $\mathbb{N}$ and $\mathbb{Z}$ with the standard linear order.

The semigroup $\mathbf{ID}_\infty$ of all partial cofinite isometries of the set of integers $\mathbb{Z}$ with the usual metric $d(n,m)=|n-m|$, $n,m\in\mathbb{Z}$, was studied in the papers [7, 8, 21].

Let $\mathbf{I}\mathbb{N}_\infty$ be the set of all partial cofinite isometries of the set of positive integers $\mathbb{N}$ with the usual metric $d(n,m)=|n-m|$, $n,m\in\mathbb{N}$. Then $\mathbf{I}\mathbb{N}_\infty$ with the operation of composition of partial isometries is an inverse submonoid of $\mathscr{I}_\omega$. The semigroup $\mathbf{I}\mathbb{N}_\infty$ of all partial cofinite isometries of positive integers is studied in [22]. There we described the Green relations on the semigroup $\mathbf{I}\mathbb{N}_\infty$, its band and proved that $\mathbf{I}\mathbb{N}_\infty$ is a simple E-unitary F-inverse semigroup. Also in [22], the least group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathbf{I}\mathbb{N}_\infty$ is described and there it is proved that the quotient-semigroup $\mathbf{I}\mathbb{N}_\infty/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$. An example of a non-group congruence on the semigroup $\mathbf{I}\mathbb{N}_\infty$ is presented. Also it is proved that a congruence on the semigroup $\mathbf{I}\mathbb{N}_\infty$ is a group congruence if and only if its restriction onto an isomorphic copy of the bicyclic semigroup in $\mathbf{I}\mathbb{N}_\infty$ is a group congruence. In [24] it was shown that the monoid $\mathbf{I}\mathbb{N}_\infty$ does not embed isomorphically into the semigroup $\mathbf{ID}_\infty$. Moreover every non-annihilating homomorphism $\mathfrak{h}\colon \mathbf{I}\mathbb{N}_\infty\to\mathbf{ID}_\infty$ has the following property: the image $(\mathbf{I}\mathbb{N}_\infty)\mathfrak{h}$ is isomorphic either to $\mathbb{Z}_2$ or to $\mathbb{Z}(+)$. Also it is proved that $\mathbf{I}\mathbb{N}_\infty$ does not have a finite set of generators, and moreover it does not contain a minimal generating set.

Later by $\mathbb{I}$ we denote the unit elements of $\mathbf{I}\mathbb{N}_\infty$.

Remark 1.1. We observe that the bicyclic semigroup is isomorphic to the semigroup $\mathscr{C}_\mathbb{N}$ which is generated by partial transformations α and β of the set of positive integers $\mathbb{N}$, defined as follows:

$$\operatorname{dom}\alpha=\mathbb{N},\qquad \operatorname{ran}\alpha=\mathbb{N}\setminus\{1\},\qquad (n)\alpha=n+1$$

and

$$\operatorname{dom}\beta=\mathbb{N}\setminus\{1\},\qquad \operatorname{ran}\beta=\mathbb{N},\qquad (n)\beta=n-1$$

(see Exercise IV.1.11(ii) in [28]). It is obvious that $\mathbb{I}=\alpha\beta$ and $\mathscr{C}_\mathbb{N}$ is a submonoid of $\mathbf{I}\mathbb{N}_\infty$.

The semigroup of monotone (order preserving) injective partial transformations φ of $\mathbb{N}$ such that the sets $\mathbb{N}\setminus\operatorname{dom}\varphi$ and $\mathbb{N}\setminus\operatorname{ran}\varphi$ are finite was introduced in [18] and there it was denoted by $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$. The semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is called *the semigroup of cofinite monotone partial bijections* of $\mathbb{N}$. In [18] Gutik and Repovš studied properties of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. In particular, they showed that $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an inverse bisimple semigroup and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. It is obvious that $\mathbf{I}\mathbb{N}_{\infty}$ is an inverse submonoid of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$.

A partial map $\alpha\colon \mathbb{N} \rightharpoonup \mathbb{N}$ is called *almost monotone* if there exists a finite subset A of $\mathbb{N}$ such that the restriction $\alpha|_{\mathbb{N}\setminus A}\colon \mathbb{N}\setminus A \rightharpoonup \mathbb{N}$ is a monotone partial map. By $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ we shall denote the semigroup of almost monotone injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N}\setminus \operatorname{dom}\varphi$ and $\mathbb{N}\setminus \operatorname{ran}\varphi$ are finite for all $\varphi \in \mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_{\omega}$ and the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ too. The semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is called *the semigroup of cofinite almost monotone injective partial transformations* of $\mathbb{N}$. In the paper [11] the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is studied. In particular, it was shown that the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is inverse, bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. In the paper [23] we showed that every automorphism of a full inverse subsemigroup of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ which contains the semigroup $\mathscr{C}_{\mathbb{N}}$ is the identity map. Also there we constructed a submonoid $\mathbf{I}\mathbb{N}_{\infty}^{[\underline{1}]}$ of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ with the following property: if S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_{\infty}^{[\underline{1}]}$ as a submonoid, then every non-identity congruence $\mathfrak{C}$ on S is a group congruence. We show that if S is an inverse submonoid of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid then S is simple and the quotient semigroup $S/\mathfrak{C}_{\mathbf{mg}}$, where $\mathfrak{C}_{\mathbf{mg}}$ is minimum group congruence on S, is isomorphic to the additive group of integers. Also, topologizations of inverse submonoids of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ and embeddings of such semigroups into compact-like topological semigroups established in [11, 23]. Similar results for semigroups of cofinite almost monotone partial bijections and cofinite almost monotone partial bijections of $\mathbb{Z}$ were obtained in [19].

Next we need some notions defined in [22] and [23]. For an arbitrary positive integer n_0 we denote $[n_0) = \{n \in \mathbb{N}\colon n \geqslant n_0\}$. Since the set of all positive integers is well ordered, the definition of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ implies that for every $\gamma \in \mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ there exists the smallest positive integer $n_{\gamma}^{\mathbf{d}} \in \operatorname{dom}\gamma$ such that the restriction $\gamma|_{[n_{\gamma}^{\mathbf{d}})}$ of the partial map $\gamma\colon \mathbb{N} \rightharpoonup \mathbb{N}$ onto the set $[n_{\gamma}^{\mathbf{d}})$ is an element of the semigroup $\mathscr{C}_{\mathbb{N}}$, i.e., $\gamma|_{[n_{\gamma}^{\mathbf{d}})}$ is a some shift of $[n_{\gamma}^{\mathbf{d}})$. For every $\gamma \in \mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ we put $\overrightarrow{\gamma} = \gamma|_{[n_{\gamma}^{\mathbf{d}})}$, i.e.

$$\operatorname{dom}\overrightarrow{\gamma} = [n_{\gamma}^{\mathbf{d}}), \quad (x)\overrightarrow{\gamma} = (x)\gamma \quad \text{for all } x \in \operatorname{dom}\overrightarrow{\gamma} \quad \text{and} \quad \operatorname{ran}\overrightarrow{\gamma} = (\operatorname{dom}\overrightarrow{\gamma})\,\gamma.$$

Also, we put

$$\underline{n}_{\gamma}^{\mathbf{d}} = \min\operatorname{dom}\gamma \qquad \text{for} \quad \gamma \in \mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N}).$$

It is obvious that $\underline{n}_{\gamma}^{\mathbf{d}} = n_{\gamma}^{\mathbf{d}}$ when $\gamma \in \mathscr{C}_{\mathbb{N}}$, and $\underline{n}_{\gamma}^{\mathbf{d}} < n_{\gamma}^{\mathbf{d}}$ when $\gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \setminus \mathscr{C}_{\mathbb{N}}$. Also for any $\gamma \in \mathbf{I}\mathbb{N}_{\infty}$ we denote

$$\underline{n}_{\gamma}^{\mathbf{r}} = (\underline{n}_{\gamma}^{\mathbf{d}})\gamma \qquad \text{and} \qquad n_{\gamma}^{\mathbf{r}} = (n_{\gamma}^{\mathbf{d}})\gamma.$$

The results of Section 3 of [24] imply that $n_{\gamma}^{\mathbf{r}} - \underline{n}_{\gamma}^{\mathbf{r}} = n_{\gamma}^{\mathbf{d}} - \underline{n}_{\gamma}^{\mathbf{d}}$ for any $\gamma \in \mathbf{I}\mathbb{N}_{\infty}$, and moreover for any non-negative integer j

$$\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} = \left\{\gamma \in \mathbf{I}\mathbb{N}_{\infty} \colon n_{\gamma}^{\mathbf{d}} - \underline{n}_{\gamma}^{\mathbf{d}} \leqslant j\right\}$$

is a simple inverse subsemigroup of $\mathbf{I}\mathbb{N}_{\infty}$ such that $\mathbf{I}\mathbb{N}_{\infty}$ admits the following infinite semigroup series

$$\mathscr{C}_{\mathbb{N}} = \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[0]} = \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[1]} \subsetneqq \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[2]} \subsetneqq \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[3]} \subsetneqq \cdots \subsetneqq \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[k]} \subsetneqq \cdots \subset \mathbf{I}\mathbb{N}_{\infty}.$$

For any positive integer k the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[k]}$ is called the *monoid of cofinite isometries of positive integers with the noise k*.

A *(semi)topological semigroup* is a topological space with a (separately) continuous semigroup operation. An inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S, τ) is a topological semigroup;
- an *inverse semigroup* topology if (S, τ) is a topological inverse semigroup;
- a *shift-continuous* topology if (S, τ) is a semitopological semigroup.

The bicyclic monoid admits only the discrete semigroup Hausdorff topology [13]. Bertman and West in [6] extended this result for the case of Hausdorff semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic monoid [1, 25, 26]. The problem of embedding the bicyclic monoid into compact-like topological semigroups was studied in [3, 4, 5, 17].

In this paper we study algebraic properties of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and extend results of the papers [13] and [6] to the semigroups $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, $j \geqslant 0$. In particular we show that for any positive integer j every Hausdorff shift-continuous topology τ on $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is discrete and and if $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a proper dense subsemigroup of a Hausdorff semitopological semigroup S, then $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a closed ideal of S, and moreover if S is a topological inverse semigroup then $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a topological group. Also we describe the algebraic and topological structure of the closure of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in a locally compact topological inverse semigroup.

Latter in this paper without loss of generality we may assume that j is an arbitrary positive integer $\geqslant 2$.

2. Algebraic properties of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$

The following simple proposition describes Green's relations on the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.

Proposition 2.1. *For elements γ and δ of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ the following statements hold:*

(*i*) *$\gamma\mathscr{L}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $\operatorname{dom}\gamma = \operatorname{dom}\delta$;*

(*ii*) *$\gamma\mathscr{R}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $\operatorname{ran}\gamma = \operatorname{ran}\delta$;*

(*iii*) *$\gamma\mathscr{H}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $\gamma = \delta$;*

(*iv*) *$\gamma\mathscr{D}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $\operatorname{dom}\gamma$ $(\operatorname{ran}\gamma)$ and $\operatorname{dom}\delta$ $(\operatorname{ran}\delta)$ are isometric subsets of $\mathbb{N}$, i.e., there exists an isometry from $\operatorname{dom}\gamma$ $(\operatorname{ran}\gamma)$ onto $\operatorname{dom}\delta$ $(\operatorname{ran}\delta)$;*

(*v*) *$\gamma\mathscr{J}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, i.e., $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a simple semigroup.*

Proof. Statements (*i*), (*ii*) and (*iii*) immediately follow from Proposition 3.2.11 of [27] and corresponding statements of Proposition 1 of [22].

Statement (*iv*) follows from the definition of the monoid and Proposition 3.2.5 of [27].

Statement (*v*) follows from Theorem 5 of [23]. □

Proposition 2.2 follows from the definition of the natural partial order $\preccurlyeq$ on an inverse semigroup and the statement that every element of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a partial shift of the integers (see [22, Lemma 1]).

Proposition 2.2. *Let γ and δ be elements of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Then the following conditions are equivalent:*

(*i*) *$\gamma \preccurlyeq \delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$*

(*ii*) *$n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}$ and $\operatorname{dom}\gamma \subseteq \operatorname{dom}\delta$;*

(*iii*) *$n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}$ and $\operatorname{ran}\gamma \subseteq \operatorname{ran}\delta$.*

It is obvious that in statements (*ii*) and (*iii*) of Proposition 2.2 we may replace the symbols $n_{\gamma}^{\mathbf{r}}$ and $n_{\gamma}^{\mathbf{d}}$ by $\underline{n}_{\gamma}^{\mathbf{r}}$ and $\underline{n}_{\gamma}^{\mathbf{d}}$, respectively.

The definition of the minimum group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and Proposition 2.2 imply the following proposition.

Proposition 2.3. *Let γ and δ be elements of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Then $\gamma\mathfrak{C}_{\mathbf{mg}}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}$. Moreover, the quotient semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$ by the map*

$$\pi_{\mathfrak{C}_{\mathbf{mg}}}\colon \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \to \mathbb{Z}(+), \quad \gamma \mapsto n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}.$$

Example 2.4. We put $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} = \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \sqcup \mathbb{Z}(+)$ and extend the multiplications from $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and $\mathbb{Z}(+)$ onto $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in the following way:

$$k\cdot\gamma = \gamma\cdot k = k + (\gamma)\pi_{\mathfrak{C}_{\mathbf{mg}}} \in \mathbb{Z}(+), \qquad \text{for all} \quad k \in \mathbb{Z}(+) \quad \text{and} \quad \gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}.$$

By Theorem 2.17 from [9, Vol. 1, p. 77] so defined binary operation is a semigroup operation on $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ such that $\mathbb{Z}(+)$ is an ideal in $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Also, this semigroup operation extends the natural partial order $\preccurlyeq$ from $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ onto $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in the following way:

(i) all distinct elements of $\mathbb{Z}(+)$ are pair-wise incomparable;

(ii) $k \preccurlyeq \gamma$ if and only if $n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = k$ for $k \in \mathbb{Z}(+)$ and $\gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.

For any $x \in \mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ we denote ${\uparrow}_{\preccurlyeq} x = \left\{y \in \mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \colon x \preccurlyeq y\right\}$.

By Proposition 7 of [22] the map $\mathfrak{h} \colon \mathbf{I}\mathbb{N}_{\infty} \to \mathscr{C}_{\mathbb{N}}$, $\gamma \mapsto \overrightarrow{\gamma}$ is a homomorphism. Then its restriction $\mathfrak{h}|_{\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}} \colon \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \to \mathscr{C}_{\mathbb{N}}$ is homomorphism, too.

A *homomorphic retraction* of a semigroup S is a map from S into S which is both a retraction and a homomorphism. The image of the homomorphic retraction is called a *homomorphic retract*. These terms seem to have first appeared in [10].

Since $(\gamma)\mathfrak{h} = \overrightarrow{\gamma} = \gamma$ for any $\gamma \in \mathscr{C}_{\mathbb{N}}$ we get the following proposition.

Proposition 2.5. *The map $\mathfrak{h} \colon \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \to \mathscr{C}_{\mathbb{N}}$, $\gamma \mapsto \overrightarrow{\gamma}$ is a homomorphic retraction, and hence the monoid $\mathscr{C}_{\mathbb{N}}$ is a homomorphic retract of $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.*

For any subset $M \subseteq \{2, \ldots, j\}$ we denote

$$\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M] = \left\{\gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \colon n_{\gamma}^{\mathbf{d}} - x \in M \cup \{0\} \text{ for all } x \in \operatorname{dom}\gamma \text{ such that } x \leqslant n_{\gamma}^{\mathbf{d}}\right\}.$$

For arbitrary $M_1, M_2 \subseteq \{2, \ldots, j\}$ it is obvious that $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M_1] \subseteq \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M_2]$ if and only if $M_1 \subseteq M_2$, and moreover we have that $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M] = \mathscr{C}_{\mathbb{N}}$ when $M = \varnothing$ and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M] = \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ when $M = \{2, \ldots, j\}$.

Remark 2.6. By Lemma 1 of [22] we get that

$$\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M] = \left\{\gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \colon n_{\gamma}^{\mathbf{r}} - x \in M \cup \{0\} \text{ for all } x \in \operatorname{ran}\gamma \text{ such that } x \leqslant n_{\gamma}^{\mathbf{r}}\right\}.$$

Proposition 2.7. *$\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M]$ is an inverse semigroup of $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ for any $M \subseteq \{2, \ldots, j\}$.*

Proof. Fix any $\gamma, \delta \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M]$. We consider the following cases.

(1) If $n_{\gamma}^{\mathbf{r}} \leqslant n_{\delta}^{\mathbf{d}}$ then $n_{\gamma\delta}^{\mathbf{r}} = n_{\delta}^{\mathbf{r}}$ and $\operatorname{ran}(\gamma\delta) \subseteq \operatorname{ran}\delta$, because by Lemma 1 from [22] all elements of $\mathbf{I}\mathbb{N}_{\infty}$ are partial shifts of the set $\mathbb{N}$. This and Remark 2.6 imply that $\gamma\delta \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M]$.

(2) If $n_{\gamma}^{\mathbf{r}} > n_{\delta}^{\mathbf{d}}$ then by similar arguments as in the previous case we get that $n_{\gamma\delta}^{\mathbf{d}} = n_{\gamma}^{\mathbf{d}}$ and $\operatorname{dom}(\gamma\delta) \subseteq \operatorname{dom}\delta$. This implies that $\gamma\delta \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M]$.

Remark 2.6 implies that if $\gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M]$ then so is γ^{-1}. □

3. On a topologization and a closure of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$

Later in the paper by $\mathbb{I}$ we denote the identity map of $\mathbb{N}$, and assume that α and β are the elements of the submonoid $\mathscr{C}_{\mathbb{N}}$ in $\mathbf{I}\mathbb{N}_{\infty}$ which are defined in Remark 1.1.

It is obvious that $\alpha\beta = \mathbb{I}$ and $\beta\alpha$ is the identity map of $\mathbb{N} \setminus \{1\}$. This implies the following lemma.

Lemma 3.1. *If* $\gamma \in \mathbf{I}\mathbb{N}_\infty$, *then*

(*i*) $\beta\alpha \cdot \gamma = \gamma$ *if and only if* $\operatorname{dom}\gamma \subseteq \mathbb{N} \setminus \{1\}$;
(*ii*) $\gamma \cdot \beta\alpha = \gamma$ *if and only if* $\operatorname{ran}\gamma \subseteq \mathbb{N} \setminus \{1\}$.

For any positive integer i let $\varepsilon^{[i]}$ be the identity map of the set $\mathbb{N} \setminus \{i\}$.

The following theorem generalized the results on the topologizabily of the bicyclic monoid obtained in [13] and [6].

Theorem 3.2. *For any positive integer j every Hausdorff shift-continuous topology τ on $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is discrete.*

Proof. Since τ is Hausdorff, every retract of $\left(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}, \tau\right)$ is its closed subset. It is obvious that $\beta\alpha \cdot \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ and $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cdot \beta\alpha$ are retracts of the topological space $\left(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}, \tau\right)$, because $\beta\alpha$ is an idempotent of $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$. Later we shall show that the set $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \setminus \left(\beta\alpha \cdot \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cup \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cdot \beta\alpha\right)$ is finite.

By Lemma 3.1, $\gamma \in \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \setminus \left(\beta\alpha \cdot \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cup \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cdot \beta\alpha\right)$ if and only if $1 \in \operatorname{dom}\gamma$, $1 \in \operatorname{ran}\gamma$, and $n_\gamma^{\mathbf{d}} - \underline{n}_\gamma^{\mathbf{d}} \leqslant j$. Then by Lemma 1 of [22], γ is a partial shift of the set of integers, and hence γ is an idempotent of $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ such that $1 \in \operatorname{dom}\gamma$ and $\varepsilon^{[2]} \cdot \ldots \cdot \varepsilon^{[j-1]} \preccurlyeq \gamma$. It is obvious that such idempotents γ are finitely many in $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$, and hence the set $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \setminus \left(\beta\alpha \cdot \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cup \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]} \cdot \beta\alpha\right)$ is finite. This implies that the point $\mathbb{I}$ has a finite open neighbourhood and hence $\mathbb{I}$ is an isolated point of the topological space $\left(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}, \tau\right)$.

We observe that $\mathbf{I}\mathbb{N}_\infty$, and hence $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$, is a submonoid of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ of cofinite monotone partial bijections of $\mathbb{N}$ [22]. By Proposition 2.2 of [18] every right translation and every left translation by an element of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is a finite-to-one map, and hence such conditions hold for the semigroup $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$. Also by Theorem 5 of [23], $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is a simple semigroup. This implies that for any $\chi \in \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ there exist $\alpha, \beta \in \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ such that $\alpha\chi\beta = \mathbb{I}$, and moreover the equality $\alpha\chi\beta = \mathbb{I}$ has finitely many solutions. Since $\mathbb{I}$ is an isolated point of $\left(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}, \tau\right)$, the separate continuity of the semigroup operation in $\left(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}, \tau\right)$ and the above arguments imply that $\left(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}, \tau\right)$ is the discrete space. $\square$

The following proposition generalized results obtained for the bicyclic monoid in [13] and [16].

Proposition 3.3. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff semitopological semigroup S. Then $I = S \setminus \mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is a closed ideal of S.*

Proof. By Theorem 3.2, $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is a discrete subspace of S, and hence by Lemma 3 of [21], $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is an open subspace of S.

Fix an arbitrary element $y \in I$. If $xy = z \notin I$ for some $x \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ then there exists an open neighbourhood $U(y)$ of the point y in the space S such that $\{x\} \cdot U(y) = \{z\} \subset \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. The neighbourhood $U(y)$ contains infinitely many elements of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. This contradicts Proposition 2.2 of [18], which states that for each $v, w \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ both sets $\left\{u \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \colon vu = w\right\}$ and $\left\{u \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \colon uv = w\right\}$ are finite. The obtained contradiction implies that $xy \in I$ for all $x \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and $y \in I$. The proof of the statement that $yx \in I$ for all $x \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and $y \in I$ is similar.

Suppose to the contrary that $xy = w \notin I$ for some $x, y \in I$. Then $w \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and the separate continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S, respectively, such that $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, both equalities $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ contradict mentioned above Proposition 2.2 from [18]. The obtained contradiction implies that $xy \in I$. $\square$

Lemma 3.4. *Let j be any positive integer $\geqslant 2$. Then the element $\varepsilon \cdot (\beta\varepsilon)^j \cdot \alpha^j$ is an idempotent of the submonoid $\mathscr{C}_{\mathbb{N}}$ for any idempotent ε of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.*

Proof. Since $\mathbb{I} = \alpha\beta$, we have that

$$\varepsilon \cdot \beta\varepsilon\alpha \cdot \beta^2\varepsilon\alpha^2 \cdot \ldots \cdot \beta^j\varepsilon\alpha^j = \varepsilon \cdot (\mathbb{I}\beta\varepsilon)^j \cdot \alpha^j = \varepsilon \cdot (\beta\varepsilon)^j \cdot \alpha^j$$

and

$$\beta^k\varepsilon\alpha^k \cdot \beta^k\varepsilon\alpha^k = \beta^k\varepsilon\mathbb{I}\varepsilon\alpha^k = \beta^k\varepsilon\varepsilon\alpha^k = \beta^k\varepsilon\alpha^k,$$

for any positive integer k. Also, $\varepsilon(\beta\varepsilon)^j\alpha^j$ is an idempotent of $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, because $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is an inverse semigroup and the product of idempotents in an inverse semigroup is an idempotent as well.

By definitions of the partial transformations α and β and the above part of the proof we get that

$$n^{\mathbf{d}}_{\beta^k\varepsilon\alpha^k} = n^{\mathbf{d}}_{\varepsilon} + k \qquad \text{and} \qquad \underline{n}^{\mathbf{d}}_{\beta^k\varepsilon\alpha^k} = \underline{n}^{\mathbf{d}}_{\varepsilon} + k, \tag{3.1}$$

and hence

$$n^{\mathbf{d}}_{\beta^k\varepsilon\alpha^k} - \underline{n}^{\mathbf{d}}_{\beta^k\varepsilon\alpha^k} = n^{\mathbf{d}}_{\varepsilon} - \underline{n}^{\mathbf{d}}_{\varepsilon}, \tag{3.2}$$

for any positive integer k. Then equalities (3.1) and (3.2) imply that for any $k = 1, \ldots, j$ the idempotent

$$\varepsilon_k = \varepsilon(\beta\varepsilon)^k\alpha^k$$

has the following properties:

$$n^{\mathbf{d}}_{\varepsilon_k} = n^{\mathbf{d}}_{\beta^k\varepsilon\alpha^k}, \quad \underline{n}^{\mathbf{d}}_{\varepsilon_k} = \underline{n}^{\mathbf{d}}_{\beta^k\varepsilon\alpha^k},$$

and

$$1, \ldots, \underline{n}^{\mathbf{d}}_{\varepsilon}, \ldots, \underline{n}^{\mathbf{d}}_{\varepsilon} + k - 1, n^{\mathbf{d}}_{\varepsilon} - 1, n^{\mathbf{d}}_{\varepsilon}, \ldots, n^{\mathbf{d}}_{\varepsilon} + k - 1 \notin \operatorname{dom}\varepsilon_k.$$

Hence we get that ε_j is the identity map of $\left[n_\varepsilon^{\mathbf{d}}+j\right)$, which implies the statement of the lemma. $\square$

Lemma 3.5. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then there exists an idempotent $e\in S\setminus\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ such that $V(e)\cap E(\mathscr{C}_\mathbb{N})$ is an infinite subset for any open neighbourhood $V(e)$ of e in S.*

Proof. By Proposition 3.3, $S\setminus\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is an ideal of S. Since S is an inverse semigroup, $S\setminus\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ contains an idempotent.

Put f be an arbitrary idempotent of $S\setminus\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$. Since the unit element of a Hausdorff topological monoid is again the unit element of its closure in a topological semigroup, for an arbitrary positive integer k by Proposition 3.3 we have that

$$\beta^k f\alpha^k\cdot\beta^k f\alpha^k=\beta^k f\mathbb{I}f\alpha^k=\beta^k ff\alpha^k=\beta^k f\alpha^k,$$

and hence $\beta^k f\alpha^k\in E(S)\setminus E\big(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}\big)$. This implies that $e=f\cdot\beta f\alpha\cdot\ldots\cdot\beta^j f\alpha^j$ is an idempotent in S because S is an inverse semigroup. The continuity of the semigroup operation in S implies that for every open neighbourhood $V(e)$ of the point e in S there exists an open neighbourhood $W(f)$ of the point f in S such that

$$W(f)\cdot\beta\cdot W(f)\alpha\cdot\ldots\cdot\beta^j\cdot W(f)\cdot\alpha^j\subseteq V(e).$$

By Proposition II.3 of [13] the set $W(f)\cap E\big(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}\big)$ is infinite. Since for any positive integer n_0 there exist finitely many idempotents $\varepsilon\in\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ such that $n_\varepsilon^{\mathbf{d}}=n_0$, we conclude that the set $\big\{n_\varepsilon^{\mathbf{d}}\colon\varepsilon\in W(f)\cap E\big(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}\big)\big\}$ is infinite, too. Then there exists an infinite sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ of idempotents of $W(f)\cap E\big(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}\big)$ such that $n_{\varphi_{i_1}}^{\mathbf{d}}\neq n_{\varphi_{i_2}}^{\mathbf{d}}$ for any distinct positive integers i_1 and i_2. Lemma 3.5 implies that $\varphi_i\cdot(\beta\varphi_i)^j\cdot\alpha^j$ is an idempotent of the submonoid $\mathscr{C}_\mathbb{N}$ which belongs to $V(e)$ for any positive integer i. Since the set $\big\{n_\varepsilon^{\mathbf{d}}\colon\varepsilon\in W(f)\cap E\big(\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}\big)\big\}$ is infinite, the set $V(e)\cap E(\mathscr{C}_\mathbb{N})$ is infinite, too. $\square$

Theorem 3.6. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then $I=S\setminus\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ is a topological group.*

Proof. We claim that the ideal I contains a unique idempotent.

Suppose to the contrary that I has at least two distinct idempotent e and f. By Lemma 3.5 without loss of generality we may assume that the set $V(e)\cap E(\mathscr{C}_\mathbb{N})$ is infinite for any open neighbourhood $V(e)$ of e in S. Since S is an inverse semigroup $ef=fe=h$ for some $h\in I\cap E(S)$. Fix an arbitrary open neighbourhood $U(h)$ in S. Then there exist disjoint open neighbourhoods $W(e)$ and $W(f)$ of the points e and f in S, respectively, such that $W(e)\cdot W(f)\subseteq U(h)$. Since S is Hausdorff, we can additionally assume that $W(e)\cap U(h)=\varnothing$ if $e\neq h$ and $W(f)\cap U(h)=\varnothing$

if $f \neq h$. Since $e \neq f$ we conclude that $W(e) \cap U(h) = \varnothing$ or $W(f) \cap U(h) = \varnothing$. Since the set $W(f) \cap E\big(\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}\big)$ is infinite and for any positive integer n_0 there exist finitely many idempotents $\iota \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ such that $n_{\iota}^{\mathbf{d}} = n_0$, we conclude that the set $\big\{\underline{n}_{\iota}^{\mathbf{d}} \colon \iota \in W(f) \cap E\big(\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}\big)\big\}$ is infinite as well. Also, the choice of the neighbourhood $W(e)$ implies that the set $\big\{\underline{n}_{\iota}^{\mathbf{d}} = n_{\iota}^{\mathbf{d}} \colon \iota \in W(e) \cap E\big(\mathscr{C}_{\mathbb{N}}\big)\big\}$ is infinite, too. Then the semigroup operation in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ implies that there exist idempotents $\iota_e \in W(e)$ and $\iota_f \in W(f)$ such that $\iota_e \in \iota_e \cdot W(f)$ and $\iota_f \in \iota_f \cdot W(e)$, which implies $W(e) \cap U(h) \neq \varnothing \neq W(f) \cap U(h)$. But this contradicts the choice of the neighbourhoods $W(e)$, $W(f)$, $U(h)$.

Since S is an inverse semigroup, we have that $xx^{-1} = x^{-1}x = e$ for any $x \in I$. This implies that I is a subgroup of S with the unit element e. Also, the continuity of semigroup operation and the inversion in S implies that I is a topological group with the induced topology from S. $\square$

Lemma 3.7 follows from the definition of an element $\overrightarrow{\gamma}$ for an arbitrary $\gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$.

Lemma 3.7. *For any $\gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ the following statements hold:*

(i) $\overrightarrow{\gamma} \in \mathscr{C}_{\mathbb{N}}$;
(ii) $\overrightarrow{\gamma}^{-1} = \overrightarrow{\gamma^{-1}}$;
(iii) $\gamma\overrightarrow{\gamma}^{-1} = \overrightarrow{\gamma}\,\overrightarrow{\gamma}^{-1}$;
(iv) $\overrightarrow{\gamma}^{-1}\gamma = \overrightarrow{\gamma}^{-1}\overrightarrow{\gamma}$.

Proposition 3.8. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then the unique idempotent of $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ commutes with all elements of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.*

Proof. By Theorem 3.6, $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a group. Put e_0 be the unique idempotent of $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Also, by Lemma 3.5 the set $U(e_0) \cap E(\mathscr{C}_{\mathbb{N}})$ is infinite for any open neighbourhood $U(e_0)$ of the point e_0 in S. This implies that $e_0 \in \operatorname{cl}_S(\mathscr{C}_{\mathbb{N}})$. Then by Proposition III.2 of [13], $e_0 \cdot \gamma = \gamma \cdot e_0$ for any $\gamma \in \mathscr{C}_{\mathbb{N}}$.

Fix an arbitrary $\gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. By Lemma 3.7 we have that

$$\overrightarrow{\gamma} \cdot \overrightarrow{\gamma}^{-1} \cdot \gamma = \gamma \cdot \overrightarrow{\gamma}^{-1} \cdot \overrightarrow{\gamma} = \overrightarrow{\gamma} \in \mathscr{C}_{\mathbb{N}}.$$

Since S is an inverse semigroup and $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is an ideal of S, Lemma 3.7 implies that

$$\begin{aligned} e_0 \cdot \gamma &= \big(e_0 \cdot \overrightarrow{\gamma} \cdot \overrightarrow{\gamma}^{-1}\big) \cdot \gamma = e_0 \cdot \big(\overrightarrow{\gamma} \cdot \overrightarrow{\gamma}^{-1} \cdot \gamma\big) = \\ &= e_0 \cdot \overrightarrow{\gamma} = \overrightarrow{\gamma} \cdot e_0 = \big(\gamma \cdot \overrightarrow{\gamma}^{-1} \cdot \overrightarrow{\gamma}\big) \cdot e_0 = \\ &= \gamma \cdot \big(\overrightarrow{\gamma}^{-1} \cdot \overrightarrow{\gamma} \cdot e_0\big) = \gamma \cdot e_0. \end{aligned}$$

This completes the proof of the proposition. $\square$

Corollary 3.9. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then the group $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ contains a dense cyclic subgroup.*

Proof. By Proposition 3.8, the unique idempotent e_0 of $S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ commutes with all elements of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and hence the map $\mathfrak{h}\colon S \to S\setminus\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, $(\gamma)\mathfrak{h} = e_0\cdot\gamma$ is a homomorphisms. Since $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a subgroup of S, by Corollary 1.32 of [12] the image $(\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]})\mathfrak{h}$ is a cyclic group. Also, since $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a dense subset of a topological semigroup S, Proposition 1.4.1 of [14] implies that the image $(\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]})\mathfrak{h}$ is a dense subset of $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. □

4. On a closure of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in a locally compact topological inverse semigroup

In [13] Eberhart and Selden described the closure of the bicyclic monoid in a locally compact topological inverse semigroup. We give this description in the terms of the monoid $\mathscr{C}_{\mathbb{N}}$.

Example 4.1. The definition of the bicyclic monoid, its algebraic properties (see [12, Section 1.12]) and Remark 1.1 imply that the following relation

$$\gamma \sim \delta \quad \text{if and only if} \quad n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}, \qquad \gamma, \delta \in \mathscr{C}_{\mathbb{N}},$$

coincides with the minimum group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathscr{C}_{\mathbb{N}}$. Moreover, the quotient semigroup $\mathscr{C}_{\mathbb{N}}/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$ by the map

$$\pi_{\mathfrak{C}_{\mathbf{mg}}}\colon \mathscr{C}_{\mathbb{N}} \to \mathbb{Z}(+), \quad \gamma \mapsto n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}.$$

The minimum group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathscr{C}_{\mathbb{N}}$ defines the natural partial order $\preccurlyeq$ on the monoid $\mathscr{C}_{\mathbb{N}}$ in the following way:

$$\gamma \preccurlyeq \delta \quad \text{if and only if} \quad n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}} \quad \text{and} \quad n_{\gamma}^{\mathbf{d}} \geqslant n_{\delta}^{\mathbf{d}}, \qquad \gamma, \delta \in \mathscr{C}_{\mathbb{N}}.$$

We put $\mathcal{C}\mathscr{C}_{\mathbb{N}} = \mathscr{C}_{\mathbb{N}} \sqcup \mathbb{Z}(+)$ and extend the multiplications from the semigroup $\mathscr{C}_{\mathbb{N}}$ and the group $\mathbb{Z}(+)$ onto $\mathcal{C}\mathscr{C}_{\mathbb{N}}$ in the following way:

$$k \cdot \gamma = \gamma \cdot k = k + (\gamma)\pi_{\mathfrak{C}_{\mathbf{mg}}} \in \mathbb{Z}(+), \qquad \text{for all} \quad k \in \mathbb{Z}(+) \quad \text{and} \quad \gamma \in \mathscr{C}_{\mathbb{N}}.$$

Then so defined binary operation is a semigroup operation on $\mathcal{C}\mathscr{C}_{\mathbb{N}}$ such that $\mathbb{Z}(+)$ is an ideal in $\mathcal{C}\mathscr{C}_{\mathbb{N}}$. Also, this semigroup operation extends the natural partial order $\preccurlyeq$ from $\mathscr{C}_{\mathbb{N}}$ onto $\mathcal{C}\mathscr{C}_{\mathbb{N}}$ in the following way:

(*i*) all distinct elements of $\mathbb{Z}(+)$ are pair-wise incomparable;
(*ii*) $k \preccurlyeq \gamma$ if and only if $n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = k$ for $k \in \mathbb{Z}(+)$ and $\gamma \in \mathscr{C}_{\mathbb{N}}$.

For any $x \in \mathcal{C}\mathscr{C}_{\mathbb{N}}$ we denote ${\uparrow_{\preccurlyeq}}x = \{y \in \mathcal{C}\mathscr{C}_{\mathbb{N}} \colon x \preccurlyeq y\}$.

We define the topology τ_{lc} on $\mathcal{C}\mathscr{C}_{\mathbb{N}}$ in the following way:

(*i*) all elements of the monoid $\mathscr{C}_{\mathbb{N}}$ are isolated points in $(\mathcal{C}\mathscr{C}_{\mathbb{N}}, \tau_{\mathsf{lc}})$;

(ii) for any $k\in\mathbb{Z}(+)$ the family $\mathscr{B}_{\mathsf{lc}}(k)=\{U_i(k)\colon i\in\mathbb{N}\}$, where
$$U_i(k)=\{k\}\cup\left\{\gamma\in\mathscr{C}_{\mathbb{N}}\colon k\preccurlyeq\gamma \text{ and } n_\gamma^{\mathbf{d}}\geqslant i\right\},$$
is the base of the topology τ_{lc} at the point $k\in\mathbb{Z}(+)$.

In [13] Eberhart and Selden proved that τ_{lc} is the unique Hausdorff locally compact semigroup inverse topology on $\mathcal{C}\mathscr{C}_{\mathbb{N}}$. Moreover, they shown that if $\mathscr{C}_{\mathbb{N}}$ is a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup S, then S is topologically isomorphic to $(\mathcal{C}\mathscr{C}_{\mathbb{N}},\tau_{\mathsf{lc}})$.

Example 4.2. Let $\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ be a semigroup defined in Example 2.4. Put M be an arbitrary subset of $\{2,\ldots,j\}$.

We define the topology τ_{lc}^M on $\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ in the following way:

(i) all elements of the monoid $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ are isolated points in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]},\tau_{\mathsf{lc}}^M\right)$;

(ii) for any $k\in\mathbb{Z}(+)$ the family $\mathscr{B}_{\mathsf{lc}}^M(k)=\left\{U_i^M(k)\colon i\in\mathbb{N}\right\}$, where
$$U_i^M(k)=\{k\}\cup\left\{\gamma\in\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}[M]\colon k\preccurlyeq\gamma \text{ and } n_\gamma^{\mathbf{d}}\geqslant i\right\},$$
is the base of the topology τ_{lc}^M at the point $k\in\mathbb{Z}(+)$.

Remark 4.3.

1. We observe that a simple verifications show that the following conditions hold:
   (i) if $k=0$ then $U_i^M(k)=U_i^M(0)=\{0\}\cup\left\{\gamma\in\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}[M]\colon k\preccurlyeq\gamma \text{ and } \gamma\notin{\uparrow_{\preccurlyeq}}\beta^{i-2}\alpha^{i-2}\right\}$;
   (ii) if $k>0$ then $U_i^M(k)=\{0\}\cup\left\{\gamma\in\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}[M]\colon k\preccurlyeq\gamma \text{ and } \gamma\notin{\uparrow_{\preccurlyeq}}\beta^{i-2}\alpha^{i-2+k}\right\}$;
   (iii) if $k<0$ then $U_i^M(k)=\{0\}\cup\left\{\gamma\in\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}[M]\colon k\preccurlyeq\gamma \text{ and } \gamma\notin{\uparrow_{\preccurlyeq}}\beta^{i-2-k}\alpha^{i-2}\right\}$.
2. Since all elements of the monoid $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ are isolated points in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]},\tau_{\mathsf{lc}}^M\right)$ and all distinct elements of the subgroup $\mathbb{Z}(+)$ are incomporable with the respect to the natural partial order on $\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$, Proposition 2.2 implies that τ_{lc}^M is a Hausdorff topology on $\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$. Also, since for any $\gamma\in\mathscr{C}_{\mathbb{N}}$ the set ${\uparrow_{\preccurlyeq}}\gamma$ is finite we get that $U_i^M(k)$ is compact for any $k\in Z(+)$ and any positive integer i. This implies that the space $\left(\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]},\tau_{\mathsf{lc}}^M\right)$ is locally comapct, and hence by Theorems 3.3.1, 4.2.9 and Corollary 3.3.6 from [14] it is metrizable.

Proposition 4.4. $\left(\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]},\tau_{lc}^M\right)$ *is a topological inverse semigroup.*

Proof. Since all elements of the monoid $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ are isolated points in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]},\tau_{\mathsf{lc}}^M\right)$ and all distinct elements of the subgroup $\mathbb{Z}(+)$ commute with elements of $\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$, it is suffices to check the continuity of the semigroup operation at the pairs (γ,k_1) and (k_1,k_2) where $\gamma\in\mathbf{I}\mathbb{N}_\infty^{\boldsymbol{g}[j]}$ and $k_1,k_2\in\mathbb{Z}(+)$.

Fix any $\gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and $k \in \mathbb{Z}(+)$. Then $\overrightarrow{\gamma} = \beta^p\alpha^r$ for some fixed non-negative integers p and r. Hence

$$\gamma \cdot k = (\gamma)\pi_{\mathfrak{C}_{\mathbf{mg}}} + k = \big(\overrightarrow{\gamma}\big)\pi_{\mathfrak{C}_{\mathbf{mg}}} + k = r - p + k,$$

and for any positive integer $i > \max\{p, r\} + j$ we have that

$$\gamma \cdot U_i^M(k) \subseteq U_i^M(r - p + k).$$

Fix any $k_1, k_2 \in \mathbb{Z}(+)$. Then for any positive integer $i > j$ by Proposition 1.4.7 of [27] and Proposition 2.7 we have that $U_i^M(k_1) \cdot U_i^M(k_2) \subseteq U_i^M(k_1 + k_2)$.

The above arguments and the equality $\big(U_i^M(k)\big)^{-1} = U_i^M(-k)$ complete the proof of the proposition. □

Lemma 4.5. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup S. Then $G = S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is topologically isomorphic to the discrete additive group of integers $\mathbb{Z}(+)$.*

Proof. By Corollary 3.9, G is a subgroup of $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ which contains a dense cyclic subgroup. By Theorem 3.2, $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a discrete subspace of S, and hence by Theorem 3.3.9 of [14], G is a closed subspace of S. Then Theorem 3.3.8 of [14] and Theorem 3.6 imply that G with the induced topology from S is a locally compact topological group. By the Weil Theorem (see [31]) the topological group G is either compact or discrete. By Lemma 3.5 the remainder $\operatorname{cl}_S(\mathscr{C}_{\mathbb{N}}) \setminus \mathscr{C}_{\mathbb{N}}$ of the subsemigroup $\mathscr{C}_{\mathbb{N}}$ in S is non-empty. Then by Theorem 3.3.8 of [14], $\operatorname{cl}_S(\mathscr{C}_{\mathbb{N}})$ is a locally compact space. Theorem V.7 of [13] implies that $H = \operatorname{cl}_S(\mathscr{C}_{\mathbb{N}}) \setminus \mathscr{C}_{\mathbb{N}}$ is a group, which is topologically isomorphic to the discrete additive group of integers $\mathbb{Z}(+)$. By Proposition 1.4.19 of [2], H is a closed discrete subgroup of G, and hence by Theorem 1.4.23 of [2] the topological group G is topologically isomorphic to the discrete additive group of integers $\mathbb{Z}(+)$. □

A partial order $\leq$ on a topological space X is called *closed* (or *continuous*) if the relation $\leq$ is a closed subset of $X \times X$ in the product topology [15]. A topological space with a closed partial order is called a *pospace*.

Later we assume that $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup S and we identify the topological group G with the discrete additive group of integers $\mathbb{Z}(+)$.

We observe that equality ${\uparrow_{\preccurlyeq}}k = \{\gamma \in S \colon \gamma \cdot 0 = k\}$ implies that ${\uparrow_{\preccurlyeq}}k$ is an open-and-closed subset of S for any $k \in \mathbb{Z}(+)$. Since $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a discrete subspace of S the above arguments and Lemma 4.5 imply the following lemma:

Lemma 4.6. *The natural partial order $\preccurlyeq$ on S is closed, and moreover ${\uparrow_{\preccurlyeq}}x$ is open-and-closed subset of S for any $x \in S$.*

Lemma 4.7. *For any $k,l \in \mathbb{Z}(+)$ the subspace $\uparrow_{\preccurlyeq}k$ and $\uparrow_{\preccurlyeq}l$ of S are homeomorphic. Moreover, the map $P_{\alpha^k}\colon \uparrow_{\preccurlyeq}0 \to \uparrow_{\preccurlyeq}k$, $x \mapsto x\cdot\alpha^k$ is a homeomorphism for $k>0$, and the map $\Lambda_{\beta^k}\colon \uparrow_{\preccurlyeq}0 \to \uparrow_{\preccurlyeq}k$, $x \mapsto \beta^k\cdot x$ is a homeomorphism for $k<0$.*

Proof. Proposition 1.4.7 from [27] implies that the maps P_{α^k} and Λ_{β^k} are well defined. It is obvious that complete to prove that the second part of the lemma holds. We shall show that the map P_{α^k} determines a homeomorphism from $\uparrow_{\preccurlyeq}0$ onto $\uparrow_{\preccurlyeq}k$. In the case of the map Λ_{β^k} the proof is similar.

We define a map $P_{\beta^k}\colon \uparrow_{\preccurlyeq}k \to \uparrow_{\preccurlyeq}0$ by the formula $(x)P_{\beta^k} = x\cdot\beta^k$. Then we have that $(0)P_{\alpha^k} = k$ and $(k)P_{\beta^k} = 0$. Moreover, we have that $(x)P_{\alpha^k}P_{\beta^k} = x$ for any $x \in \uparrow_{\preccurlyeq}0$ and $(y)P_{\beta^k}P_{\alpha^k} = y$ for any $y \in \uparrow_{\preccurlyeq}k$. Therefore the compositions of maps $P_{\alpha^k}P_{\beta^k}\colon \uparrow_{\preccurlyeq}0 \to \uparrow_{\preccurlyeq}0$ and $P_{\beta^k}P_{\alpha^k}\colon \uparrow_{\preccurlyeq}k \to \uparrow_{\preccurlyeq}k$ are identity maps of the sets $\uparrow_{\preccurlyeq}0$ and $\uparrow_{\preccurlyeq}k$, respectively. Hence the maps P_{α^k} and P_{β^k} are bijections, and hence P_{β^k} is inverse of P_{α^k}. Since right translations in the topological semigroup S are continuous, the maps $P_{\alpha^k}\colon \uparrow_{\preccurlyeq}0 \to \uparrow_{\preccurlyeq}k$ and $P_{\beta^k}\colon \uparrow_{\preccurlyeq}k \to \uparrow_{\preccurlyeq}0$ are homeomorphisms. □

By Lemma 3.5 the remainder $\operatorname{cl}_S(\mathscr{C}_{\mathbb{N}}) \setminus \mathscr{C}_{\mathbb{N}}$ of the subsemigroup $\mathscr{C}_{\mathbb{N}}$ in S is non-empty. Also, Theorem V.7 of [13] implies that the remainder $\operatorname{cl}_S(\mathscr{C}_{\mathbb{N}}) \setminus \mathscr{C}_{\mathbb{N}}$ is a group, which is topologically isomorphic to the discrete additive group of integers $\mathbb{Z}(+)$. This and results of [13, Section V] (see Example 4.1) imply the following proposition:

Proposition 4.8. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$. Then τ induces the topology τ_{lc} on the semigroup $\mathcal{C}\mathscr{C}_{\mathbb{N}}$.*

If $M=\varnothing$ then we denote the locally compact semigroup inverse topology τ_{lc}^{M} on the monoid $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ by $\tau_{\mathsf{lc}}^{\varnothing}$. Also in the case when $M=\{2,\ldots,j\}$ we denote the topology τ_{lc}^{M} on $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ by $\tau_{\mathsf{lc}}^{[2:j]}$.

Proposition 4.8 implies the following:

Proposition 4.9. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$. Then $\tau_{\mathsf{lc}}^{\varnothing} \subseteq \tau \subseteq \tau_{\mathsf{lc}}^{[2:j]}$.*

Theorem 4.10. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup (S,τ). Then (S,τ) topologically isomorphic to the topological inverse semigroup $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau_{\mathsf{lc}}^{M}\right)$ for some subset M of $\{2,\ldots,j\}$.*

Proof. Lemma 4.5 implies that the inverse semigroup S is isomorphic to the monoid $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Also, by the definition of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, Lemma 4.7 and Proposition 4.9 we get that there exists a maximal subset M_1 of $\{2,\ldots,j\}$ such that the following condition holds:

($*$) for every open neighbourhood V_0 of the point $0 \in \mathbb{Z}(+)$ in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$ there exists an open neighbourhood $U_i^{M_1}(0)$ of 0 in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau_{\mathsf{lc}}^{M_1}\right)$ such that $U_i^{M_1}(0) \subseteq V_0$ and $V_0 \setminus U_i^{M_1}(0)$ is infinite.

Since the topology τ is locally compact and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a discrete subsemigroup of $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$, without loss of generality we may assume that the open neighbourhood V_0 is compact.

The maximality of M_1 and condition ($*$) imply that there exists a subset $M_1^1 \subseteq \{2, \ldots, j\}$ such that $M_1 \subset M_1^1$, $\left|M_1^1 \setminus M_1\right| = 1$ and for every open neighbourhood V_0 of the point $0 \in \mathbb{Z}(+)$ in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$ the following conditions hold:

$$\left|\left(V_0 \cap U_i^{M_1^1}(0)\right) \setminus U_i^{M_1}(0)\right| = \infty \quad \text{and} \quad \left|U_i^{M_1^1}(0) \setminus \left(V_0 \cap U_i^{M_1^1}(0)\right)\right| = \infty. \tag{4.1}$$

By continuity of the semigroup operation in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$ there exists a compact-and-open neighbourhood $U_0 \subseteq V_0$ of the point $0 \in \mathbb{Z}(+)$ in the space $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$ such that $\beta \cdot U_0 \cdot \alpha \subseteq V_0$. Then the semigroup operation of $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$, the above inclusion and conditions (4.1) imply that the set $V_0 \setminus U_0$ is infinite, which contradicts the compactness of V_0. This and maximality of M_1 imply that the set $V_0 \setminus U_i^{M_1}(0)$ is finite for every open neighbourhood V_0 of the point $0 \in \mathbb{Z}(+)$ in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau\right)$ and any open neighbourhood $U_i^{M_1}(0)$ of 0 in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau_{\mathsf{lc}}^{M_1}\right)$. Then the bases of τ and $\tau_{\mathsf{lc}}^{M_1}$ at the point $0 \in \mathbb{Z}(+)$ coincide, and hence by Lemma 4.7 we get that $\tau = \tau_{\mathsf{lc}}^{M_1}$. $\square$

Corollary 4.11. *For any positive integer j there exists exactly 2^{j-1} pairwise topologically non-isomorphic Hausdorff locally compact semigroup inverse topologies on the monoid $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.*

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