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THE BAIRE CATEGORY OF THE HYPERSPACE OF NONTRIVIAL CONVERGENT SEQUENCES

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ABSTRACT. Assume that X is a regular space. We study topological properties of the family $S_c(X)$ of nontrivial convergent sequences in X equipped with the Vietoris topology. In particular, we show that if X has no isolated points, then $S_c(X)$ is a space of the first category which answers the question posed by S. García-Ferreira and Y.F. Ortiz-Castillo in [1].

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1. INTRODUCTION

Let X be a topological Hausdorff space. The Vietoris topology on the family $K(X)$ of all compact subsets of X is generated by a base consisting of sets

$$(1.1)\quad \langle V_1, \ldots, V_n \rangle := \left\{ K \in K(X)\colon K \subset \bigcup_{i=1}^{n} V_i \text{ and } K \cap V_i \neq \emptyset \text{ for } 1 \leq i \leq n \right\},$$

where n runs over $\mathbb{N}$ and $V_1, \ldots, V_n$ are nonempty open sets in X.

Our notation is consistent with that used in [1]. We say that $S \subset X$ is a nontrivial convergent sequence in X if $S = \{x_n\}_{n\in\mathbb{N}} \cup \{\lim S\}$ for some injective sequence $(x_n)_{n\in\mathbb{N}}$ in X which is convergent to some $\lim S \in X \setminus \{x_n\}_{n\in\mathbb{N}}$. In other words, S is a set of terms of an injective convergent sequence with its limit. Obviously, S is compact for any space X, and $S_c(X)$ is empty for a discrete space X. Hence each closed subset F of S is compact. Consequently, the spaces $K(X)$ and the family $CL(X)$ of all closed subsets of X with the topology generated by an analogous base, given by (1.1), introduce the same topology in their subspace $S_c(X)$.

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In general, separation axioms of the spaces $CL(X)$ and $K(X)$ depend on X. More precisely, $CL(X)$ is normal if and only if X is compact (see [5]), and $K(X)$ is metrizable if and only if X is metrizable ([3]).

The main aim of this paper is to show that $S_c(X)$ is of first category in itself under the assumption that X is regular and crowded (i.e. has no isolated points). This result gives a positive answer to Question 3.4 in [1]. The authors of [1] asked whether $S_c(X)$ is of the first category in itself if X is a metric crowded space. Note that in [2] authors used other methods to prove this result. From now on, sets of the first category will be called also meager. In our considerations we will use the Banach Category Theorem (see [4]) which states that in any topological space the union of a family of open meager sets is meager, too. Thanks to this fact, it suffices to construct a meager open neighbourhood of any $S \in S_c(X)$. Such a construction is possible in the case of regular crowded space X. For reader's convenience we repeat or modify some arguments from [1].

2. Main result

We will follow some ideas taken from the paper [1] while considering some specific subsets of $S_c(X)$. For given $k, m \in \mathbb{N}, 1 \leq i \leq k$ and pairwise disjoint nonempty closed sets $C_1, \ldots, C_k$, we denote

$$\left\{ S \in S_c(X) \colon S \subset \bigcup_{j=1}^{k} C_j \text{ and } 1 \leq \operatorname{card}(S \cap C_j) \leq m \text{ for all } j \leq k,\ j \neq i \right\}$$

by $N_k^i(m, \{C_j \colon j \leq k\})$ and $\bigcup_{i=1}^{k} N_k^i(m, \{C_j \colon j \leq k\})$ by $N_k(m, \{C_j \colon j \leq k\})$.

It can be immediately seen that, if $S \in N_k^i(m, \{C_j \colon j \leq k\})$, then $\operatorname{card}(S \cap C_i) = \omega$.

The following lemma is from [1, Lemma 3.1].

Lemma 2.1. *Let X be a crowded topological space. Then the set*

$$N_k^i(m, \{C_j \colon j \leq k\})$$

is nowhere dense, closed in $S_c(X)$ whenever $k, m \in \mathbb{N}, 1 \leq i \leq k$ and $C_1, \ldots, C_k$ are pairwise disjoint, nonempty and closed sets.

Note that the assertion of the above fact can be lost if X is not crowded.

Example 2.2. Consider sets $X := [0,1],\ Y := [0,1] \cup \{2\},\ Z := [0,1] \cup [2,3]$ with the Euclidean topology. Note that X is a closed subspace of Y, Y is a closed subspace of Z and both spaces X and Z are crowded but Y has one isolated point. Take $k := 2,\ m := 1,\ C_1 := [0,1],\ C_2 := \{2\},\ i := 2$. By Lemma 2.1, the set $N_2^2(1, \{C_1, C_2\})$ is nowhere dense in the space $S_c(Z)$. Nevertheless, this set is not nowhere dense in $S_c(Y)$. To check it, set $V_1 := [0,1],\ V_2 := \{2\}$, and consider the open set $\langle V_1, V_2 \rangle$. Obviously, each $S \in \langle V_1, V_2 \rangle$ has exactly one point in $S \cap V_2$. Thus $S \in N_2^2(1, \{C_1, C_2\})$ and consequently, $\langle V_1, V_2 \rangle \subset N_2^2(1, \{C_1, C_2\})$.

Moreover, all sets $N_k^i(m, \{C_j : j \le k\})$ as in Lemma 2.1 are nowhere dense in $S_c(X)$.

The next result (see [1, Theorem 3.2]) is an application of the previous lemma.

Lemma 2.3. *Suppose that U_1, U_2 are nonempty, closed and disjoint subsets of a crowded space X. Then $\langle \mathrm{Int}(U_1), \mathrm{Int}(U_2)\rangle$ is meager in $S_c(X)$.*

Proof. Thanks to Lemma 2.1 it suffices to observe that

$$\langle \mathrm{Int}(U_1), \mathrm{Int}(U_2)\rangle \cap S_c(X) \subset \bigcup_{m\in\mathbb{N}} N_2(m, \{U_1, U_2\}).$$

But this follows from the disjointness of closed sets U_1, U_2. □

Now, we will construct the respective neighbourhoods of nontrivial convergent sequences.

Lemma 2.4. *Suppose X is a Hausdorff space and $S = \{x_n\}_{n\in\mathbb{N}} \cup \{\lim S\} \in S_c(X)$. There are neighbourhoods $V_n, n \in \mathbb{N}$ of x_n's and a neighbourhood V_S of $\lim S$ such that*

$$V_1 \cap V_n = \emptyset \textit{ for all } n \ge 2 \quad \textit{ and } V_1 \cap V_S = \emptyset.$$

Proof. Use the Hausdorff axiom to find disjoint neighbourhoods V_1' of x_1, and V_S of $\lim S$. Since $(x_n)_{n\in\mathbb{N}}$ is convergent to $\lim S$, the set

$$M := \{m \ge 2 \colon x_m \notin V_S\}$$

is finite. Again, for each $m \in M$ use the Hausdorff axiom to find disjoint neighbourhoods $V_{1,m}'$ of x_1, and V_m of x_m. The intersection $V_1 = V_1' \cap \bigcap_{m\in M} V_{1,m}$ satisfies

$$V_1 \cap V_m = \emptyset \text{ for each } m \in M.$$

Then it suffices to define $V_k := V_S$ for all $k \notin M \cup \{1\}$. □

Proposition 2.5. Suppose X is a regular space and

$$S = \{x_n\}_{n\in\mathbb{N}} \cup \{\lim S\} \in S_c(X).$$

Then there are nonempty, closed and disjoint sets U_1, U_2 with $S \in \langle \mathrm{Int}(U_1), \mathrm{Int}(U_2)\rangle$.

Proof. Let V_S, V_n, $n \in \mathbb{N}$, be the respective neighbourhoods of $\lim S$, x_n, $n \in \mathbb{N}$ considered in Lemma 2.4. Since $V_1 \cap (V_S \cup \bigcup_{n\ge 2} V_n) = \emptyset$, we have

$$V_1 \cap \mathrm{cl}(V_S \cup \bigcup_{n\ge 2} V_n) = \emptyset.$$

Then we use the regularity of X to find a neighbourhood W_1 of x_1 such that $\mathrm{cl}(W_1) \subset V_1$. Put $U_1 := \mathrm{cl}(W_1)$, $U_2 := \mathrm{cl}(V_S \cup \bigcup_{n\ge 2} V_n)$. Obviously, these sets are nonempty, closed and disjoint. We need to show that $S \in \langle \mathrm{Int}(U_1), \mathrm{Int}(U_2)\rangle$. Indeed, $x_1 \in W_1 \subset \mathrm{Int}(U_1)$ and $\{\lim S\} \cup \bigcup_{n\ge 2}\{x_n\} \subset V_S \cup \bigcup_{n\ge 2} V_n \subset \mathrm{Int}(U_2)$. □

Theorem 2.6. *Suppose that X is a regular crowded space. Then the space $S_c(X)$ is of the first category in itself.*

Proof. Take $S \in S_c(X)$ and a neighbourhood of S of the form $\langle \mathrm{Int}(U_1), \mathrm{Int}(U_2)\rangle$ considered in Proposition 2.5. Then by Lemma 2.3, this neighbourhood is meager. Therefore, by the Banach Category Theorem, $S_c(X)$ is of the first category as a union of meager neighbourhoods of its points. □

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