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CANAL SURFACES AND FOLIATIONS – A SURVEY

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1. Introduction

Here, *conformal geometry* is seen as the theory of invariants of Möbius transformations of space forms $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$. Since the Euclidean space $\mathbb{R}^3$ and the hyperbolic space $\mathbb{H}^3$ are conformally equivalent to open subspaces of the sphere $\mathbb{S}^3$, the case of $\mathbb{S}^3$ is of greatest interest.

All the Möbius transformations of one of the spaces under consideration can be presented as a composition of spherical inversions and spherical inversions map spheres (or planes) onto spheres (or planes), the notion of sphere is conformally invariant. Since all these transformations are smooth, they map spheres osculating a surface onto spheres osculating its images. Also, they transform envelopes of families of spheres onto envelopes of corresponding families of their images.

Canal surfaces considered in this article are defined as envelopes of one parameter families of spheres. Surfaces of this sort can be found easily in nature: water pipes, hoses for vacuum cleaners and blood vessels being some of them. Their analytic models play an important role in 3D computer graphics. The simplest examples of canal surfaces are provided by surfaces of revolution and their images by Möbius transformations, spherical inversions in particular.

Foliations (here, of 3-dimensional manifolds) are particular families of connected and pairwise disjoint surfaces (called *leaves*) filling the manifolds. In differential geometry, several authors discuss existence, properties and classification of foliations by leaves satisfying particular geometric conditions, for example being

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totally geodesic, minimal, umbilical, of either constant, or positive, or negative curvatures (sectional, Ricci, scalar, mean etc.) and so on (see [30, 31] and the bibliographies therein).

In this article, we provide a survey of results concerning canal surfaces and *canal foliations*, that is foliations of 3-dimensional manifolds of constant curvature by leaves being canal surfaces, obtained recently (2000 – 2020) by the author and his collaborators. In Section 2, we describe (after [10] and [19]) a useful and well known interpretation of 2-dimensional oriented spheres $S \subset \mathbb{S}^3$ as points of the 4-dimensional de Sitter space. Section 3 is devoted to a description of local conformal invariants of surfaces used in the description of canal surfaces. Section 4 contains a survey of results on canal surfaces while Section 5 those on canal foliations.

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2. Space of spheres

Consider the 5-dimensional *Lorenz space* $\mathbb{L}^5$, that is the vector space $\mathbb{R}^5$ equipped with the *Lorentz quadratic form* $\mathcal{L}$ and the associated *Lorentz bilinear form* $\mathcal{L}(\cdot,\cdot)$ given by

$$\mathcal{L}(x_0, \cdots, x_4) = -x_0^2 + x_1^2 + \cdots + x_4^2$$

and

$$\mathcal{L}(u, v) = -u_0 v_0 + u_1 v_1 + \cdots + u_4 v_4$$

when $u = (u_0, y_1, \ldots, u_4)$ and $v = (v_0, v_1, \ldots, v_4)$.

The isotropy cone $\mathcal{L}_{iso} = \{v \in \mathbb{R}^5\,;\, \mathcal{L}(v) = 0\}$ of $\mathcal{L}$ is called the *light cone*. Its non-zero vectors are also called *light-like*. The light cone splits the set of nonzero vectors $v \in \mathbb{L}^5$ into two classes: A vector v in $\mathbb{R}^5$ is called *space-like* if $\mathcal{L}(v) > 0$ and *time-like* if $\mathcal{L}(v) < 0$. A straight line is called space-like (respectively, time-like) if it contains a space-like (respectively, time-like) vector.

The *de Sitter space* Λ^4 is defined as the set of all the points $x = (x_0, x_1, \ldots, x_5)$ of $\mathbb{R}^5$ for which $\mathbb{L}(x) = 1$.

The points at infinity of the light cone in the upper half space $\{x_0 > 0\}$ form a 3-dimensional sphere. Let us denote it by S^3_∞. Since it can be considered as the set of lines through the origin in the light cone, it is identified with the intersection S^3_1 of the upper half light cone and the hyperplane $\{x_0 = 1\}$, which is given by $S^3_1 = \{(x_1, \cdots, x_4) \,|\, x_1^2 + \cdots + x_4^2 = 1\}$.

To each point $\sigma \in \Lambda^4$ there corresponds a sphere $\Sigma = \sigma^\perp \cap S^3_\infty$ (or, $\Sigma = \sigma^\perp \cap S^3_1$), where $\sigma^\perp$ is the 4-dimensional vector subspace of $\mathbb{R}^5$ orthogonal (with respect to the form $\mathbb{L}$) to the straight line passing through σ and the origin (see Figure 1,

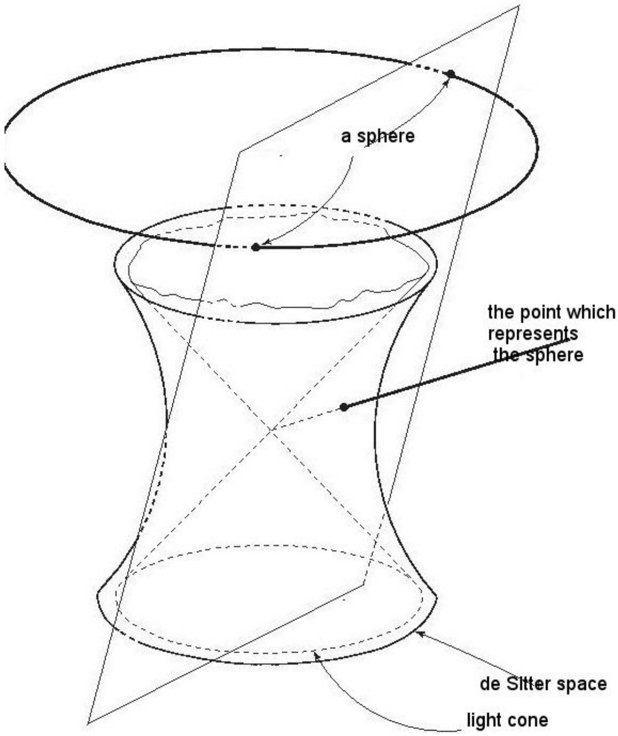

FIGURE 1. Spheres as points of de Sitter space.

where – by obvious technical reasons – the dimensions of all the geometric objects are reduced by 2: $5 \mapsto 3$, $4 \mapsto 2$, $3 \mapsto 1$ and $2 \mapsto 0$).

Since $(-\sigma)^{\perp} = \sigma^{\perp}$, we get two copies of the same sphere $\Sigma \subset \mathbb{S}^3$ equipped with two opposite orientations. This is why de Sitter space Λ^4 can be identified with the space of oriented 2-spheres contained in $\mathbb{S}^3$.

A regular curve $\gamma : I \to \mathbb{R}^5$ is called space-like if, at each point t ot the inerval I, its tangent vector $\dot{\gamma}(t)$ is space-like, that is $\mathbb{L}(\dot{\gamma}(t)) > 0$. If $\gamma(I) \subset \Lambda^4$, γ can be considered as the one-parameter family of the corresponding spheres $\Sigma(t)$. If both these conditions are satisfied, the family of spheres $\Sigma(t)$ associated to the points $\gamma(t)$, $t \in I$, defines an envelope. An extra condition is necessary to guarantee that the envelope is immersed: all the geodesic acceleration vectors $\vec{k_g}(t) = \ddot{\gamma}(t) + \gamma(t)$, $t \in I$, should be time-like.

Certain motivation for considering the correspondence $\sigma \longleftrightarrow \Sigma$ between points σ of de Sitter space Λ^4 and spheres $\Sigma \subset \mathbb{S}^3$ arises from the following.

Given two two-dimensional spheres S and Σ in $\mathbb{R}^3$ (or, $\mathbb{S}^3$), the spherical inversion of Σ with respect to S is another sphere (or, a plane considered as a sphere of infinite radius) $\tilde{\Sigma} = \iota_S(\Sigma)$. The corresponding points s, σ and $\tilde{\sigma}$ of Λ^4 are related by

$$\tilde{\sigma} = 2\mathbb{L}(s,\sigma)s - \sigma. \tag{2.1}$$

From (2.1), several properties of spherical inversion follow easily. For example,

$$\iota_S(S) = S,\ \iota_S^2(\Sigma) = \Sigma,\ \mathbb{L}(\iota_s(\sigma_1), \iota_s(\sigma_2)) = \mathbb{L}(\sigma_1, \sigma_2)$$

for all the spheres $S, \Sigma, \Sigma_1, \Sigma_2 \subset \mathbb{S}^3$ (and the corresponding points σ_1, σ_2 of Λ^4). Therefore, spherical inversions considered in Λ^4 appear to be isometries with respect to the Lorenz scalar product. Since, all the conformal transformations of $\mathbb{S}^3$ are compositions of such inversions, we arrive at

Proposition 2.1. *All the conformal transformations of the sphere $\mathbb{S}^3$ become Lorenzian isometries of Λ^4 under the correspondence between 2-spheres $\Sigma \subset \mathbb{S}^3$ and points of de Sitter space Λ^4 we discussed here.*

Remark 2.2. Note that the analytic description of the inversive image $\tilde{\Sigma}$ in Euclidean coordinates is significantly more complicated than that of $\tilde{\sigma}$ in (2.1):

$$\tilde{\Sigma} = S\Big(o + \frac{\rho^2(c-o)}{\|c-o\}^2 - r}, \frac{\rho^2 r}{(\|c-o\|^2 - r)^2)^2}\Big)$$

when $\Sigma = S(c,r)$ and $S = S(o,\rho)$ is the sphere of inversion.

3. Conformal invariants

Assume now that S is a surface which is *umbilic free*, that is, that the principal curvatures $k_1(x)$ and $k_2(x)$ of S are different at any point x of S. Let X_1 and X_2 be unit vector fields tangent to the curvature lines corresponding to, respectively, k_1 and k_2. Throughout the paper, we assume that $k_1 > k_2$. Put $\mu = (k_1 - k_2)/2$. For a long time, it is known ([34], see also [7]) that the vector fields $\xi_i = X_i/\mu$ and the coefficients θ_i $(i = 1, 2)$ in

$$[\xi_1, \xi_2] = -\frac{1}{2}\left(\theta_2\xi_1 + \theta_1\xi_2\right)$$

are invariant under arbitrary (orientation preserving) conformal transformation of the Euclidean space $\mathbb{R}^3$. (In fact, they are invariant under arbitrary conformal change of the Riemannian metric on $\mathbb{R}^3$.) Elementary calculations involving Codazzi equations show that

$$\theta_1 = \frac{1}{\mu^2} \cdot X_1(k_1) \quad \text{and} \quad \theta_2 = \frac{1}{\mu^2} \cdot X_2(k_2).$$

The quantities θ_i $(i = 1, 2)$ are called *conformal principal curvatures* of S.

Another conformally invariant scalar quantity Ψ can be derived from the derivation of the Bryant's *conformal Gauss map* β which maps a point x of S to the sphere tangent at x to S that has the same mean curvature as S at x (see [4]).

The sphere $\beta(x)$ can be seen as a point of $\Lambda^4 \subset \mathbb{L}^5$. We get (with all the scalar products $\langle\cdot,\cdot\rangle$ below denoting the Lorentz bilinear form $\mathcal{L}$ in $\mathbb{L}^5$)

$$\begin{aligned}&\frac{1}{2}\big(\langle\xi_1(\xi_1(\beta)),\xi_1(\xi_1(\beta))\rangle - \langle\xi_2(\xi_2(\beta)),\xi_2(\xi_2(\beta))\rangle \\ &\qquad -\langle\xi_1(\xi_1(\beta)),\xi_2(\beta)\rangle^2 + \langle\xi_2(\xi_2(\beta)),\xi_1(\beta)\rangle\big) \\ &\qquad = \Psi + \frac{1}{2}\left(\theta_1^2 - \theta_2^2 + \xi_1(\theta_1) + \xi_2(\theta_2)\right). \end{aligned} \tag{3.1}$$

The quantities on both sides of the above equality are equal to

$$\frac{1}{\mu^3}\left(\triangle H + 2\mu^2 H\right), \tag{3.2}$$

where H is the mean curvature of S and $\triangle$ is the Laplace operator on S equipped with the Riemannian metric induced from the ambient space. Moreover, this quantity appears in the Euler-Lagrange equation for *Willmore functional*

$$S \mapsto \int_S \mu^2 d\,area.$$

The vector fields ξ_1, ξ_2 (or, the dual 1-forms ω_1, ω_2) together with quantities θ_1, θ_2 and Ψ generate all the local conformal invariants for surfaces and determine a surface locally up to conformal transformations of $\mathbb{R}^3$ ([16], see also [7] again).

Define (5×5) matrices A_1 and A_2 by

$$A_1 = \begin{pmatrix} \theta_1/2 & -(1+\Psi)/2 & b/2 & \theta_1/2 & 0 \\ 1 & 0 & 0 & -1 & (1+\Psi)/2 \\ 0 & 0 & 0 & 0 & -b/2 \\ 0 & 1 & 0 & 0 & -\theta_1/2 \\ 0 & -1 & 0 & 0 & -\theta_1/2 \end{pmatrix} \tag{3.3}$$

and

$$A_2 = \begin{pmatrix} -\theta_2/2 & -c/2 & -(1-\Psi)/2 & \theta_2/2 & 0 \\ 0 & 0 & 0 & 0 & c/2 \\ 1 & 0 & 0 & 1 & (1-\Psi)/2 \\ 0 & 0 & -1 & 0 & -\theta_2/2 \\ 0 & 0 & -1 & 0 & \theta_2/2 \end{pmatrix}, \tag{3.4}$$

where $b = -\theta_1\theta_2 + \xi_2(\theta_1)$ and $c = \theta_1\theta_2 + \xi_1(\theta_2)$.

Proposition 3.1 (Fialkov, [16]). *Given a simply connected domain $U \subset \mathbb{R}^2$, linearly independent 1-forms ω_1 and ω_2 and smooth functions θ_1, θ_2 and Ψ defined on U for which the matrix valued 1-form ω,*

$$\omega = A_1\omega_1 + A_2\omega_2, \tag{3.5}$$

satisfies the structural equation

$$d\omega + (1/2)[\omega,\omega] = 0, \tag{3.6}$$

there exists an immersion $\iota : U \to \mathbb{R}^3$ for which $S = \iota(U)$ realizes these forms and functions as local conformal invariants.

The integrability condition (3.6) implies the following.

Proposition 3.2 ([2])**.** *Any surface $S \subset \mathbb{R}^3$ with constant conformal principal curvatures has at least one of these curvatures equal to zero.*

Moreover, for arbitrary constant c, the family of all the immersed in $\mathbb{R}^3$ surfaces $S = \iota(\mathbb{R}^2)$ with constant conformal principal curvatures 0 and c is nonempty and parametrized by triples (f_2, C_1, C_2), where f_2 and C_1 are smooth real functions of one variable while C_2 is a real constant. The corresponding surface S admits the conformally invariant 1-forms $\omega_i = f_i dx_i$ with f_1 given by

$$f_1(x_1, x_2) = C_1(x_1) \cdot e^{-\frac{1}{2}c\int_0^{x_2} f_2(t)dt}, \tag{3.7}$$

where $C_1 : \mathbb{R} \to \mathbb{R}$ is arbitrary smooth function while the Bryant conformal invariant Ψ of S is given by

$$\Psi(x_1, x_2) = C_2 \cdot \exp\Big(- c \cdot \int_0^{x_2} f_2(t)dt\Big) - 2. \tag{3.8}$$

Example 3.3. Torus, cylinder and cone of revolution, and their images under arbitrary Möbius transformations are the only surfaces with both conformal principal curvatures equal to zero. All these surfaces are called *Dupin cyclides.*

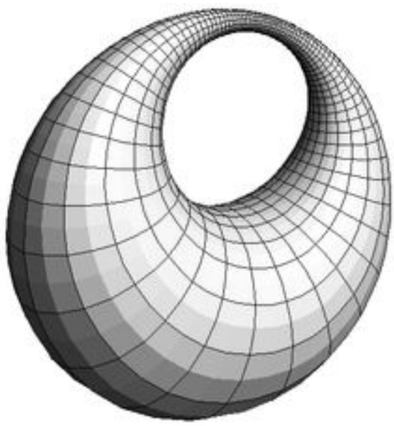

FIGURE 2. A Dupin cyclide.

In [11, 12, 13], Darboux mentioned several results concerning Dupin cyclides. Among them, one can find the following.

Proposition 3.4 (Darboux)**.** *Dupin cyclides are the only surfaces that are in two different ways envelopes of one-parameter families of spheres as well as the only surfaces that have circles as all the lines of curvature.*

4. Canal surfaces

As mentioned in Introduction, *canal surfaces* in space forms are defined as the envelopes of one-parameter families of spheres. Therefore, they can be seen as space-like curves in de Sitter space Λ^4 (of time-like geodesic curvature vectors when regularly immersed).

Since all the Möbius transformations map spheres to spheres, the notion of canal belongs to conformal geometry: conformal image of a canal surface is a canal surface. The simplest examples of canal surfaces are provided by Dupin cyclides (see Section 3), surfaces of revolution and their images under Möbius transformations.

By definition, any canal surface S in $\mathbb{R}^3$ is the solution of a system of equations

$$\begin{cases} (x-x(t))^2+(y-y(t))^2+(z-z(t))^2=r(t)^2, \\ x'(t)(x-x(t))+y'(t)(y-y(t))+z'(t)(z-z(t))=r'(t)r(t), \end{cases} \tag{4.1}$$

where – obviously – the first of these equations defines a sphere enveloped by S while the second one – a plane. The intersection of the osculating sphere with the corresponding plane is – in general – a circle contained in S and is called a *characteristic circle* of S.

Proposition 4.1. *Characteristic circles of canal surfaces appear to be their lines of curvature corresponding to the principal conformal curvature, say θ_1, vanishing identically along the surface: $\theta_1(p)=0$ for any point p of the canal surface under consideration.*

In some sense, canal surfaces admit osculating Dupin cyclides called in [1] *necklaces.* (For a discussion of osculation of Dupin cyclides and arbitary surfaces see [3].)

Theorem 4.2 ([1])**.** *The osculating spheres $\sigma_2(\phi)$ for the principal curvature k_2 along a characteristic circle Γ (being a line of principal curvature for k_1 parametrized by ϕ) have an envelope which is a Dupin cyclide $\mathcal{D}$; in other terms the corresponding points $\sigma_2(\phi)\in\Lambda^4$ form a circle.*

Since osculating spheres have order of tangency 2 with the corresponding canal, the Bryant invariants Ψ_S and Ψ_N of a canal surface S and its necklace N are equal along the characteristic circle of their tangency. Since Ψ_N is constant, we get the following.

Corollary 4.3 ([1])**.** *The Bryant invariant of a canal surface is constant along its characteristic circles.*

The above Corollary and the condition $\theta_1\equiv 0$ satisfied for all the canals motivates the interest in canal surfaces for which θ_2 is constant along the characteristic circles. Such canals are called in [1] *special.*

Special canals. Let us classify special canals, that is canal surfaces satisfying the conditions

$$\theta_1 = 0, \quad \theta_2 \text{ is constant along characteristic circles.} \tag{4.2}$$

Theorem 4.4 ([1])**.** *A surface S in $\mathbb{R}^3$ is a special canal if and only if it is a conformal image of either a surface of revolution or a cylinder over a planar curve, or a cone.*

Note that the three classes of surfaces listed in the above theorem can be characetrized in terms of de Sitter space Λ^4:

Proposition 4.5 ([1])**.**

(i) *Special canal surfaces are envelopes of spheres which form a curve γ contained in $\Lambda^4 \cap V$, V being a 3-dimensional vector subspace of $\mathbb{R}^5$.*
(ii) *V is of mixed type (resp., degenerate, resp. space-like) whenever S is a conformal image of a surface of revolution (resp., a cylinder, resp., a cone).*
(iii) *in this case, the intersection $\Lambda^4 \cap V$ looks like a 2-dimensional de Sitter subspace Λ^2 of Λ^4, resp., a cylinder generated by parallel light-rays, resp., a sphere.*

Now, let us recall that a surface S in $\mathbb{R}^3$ is said to be *isothermic* whenever it is obtained locally by charts (x_1, x_2) such that the curves $x_i = \text{const.}, i = 1, 2$, coincide with the curvature lines on S. In other words, S is isothermic whenever there exists a positive function f such that the Lie bracket $[f\xi_1, f\xi_2]$ vanishes identically. From the definition of conformal principal curvatures in Section 3 it follows that this condition is equivalent to the following one:

$$\xi_1(f) = (1/2)f\xi_2, \quad \xi_2(f) = (-1/2)f\xi_1.$$

Note that this is a conformally invariant notion: the image of an isothermic surface by a Möbius transformation in again isothermic. For more information about such surfaces (and, more general, submanifolds of higher dimension) see, for example, [5], [6] and the bibliographies therein.

Finally, if S is an isothermic canal, then, say, $\theta_1 = 0$, $\xi_2(f) = 0$, f is constant along characteristic circles, the same holds for $\xi_2(f)$ and for $\theta_2 = -2\xi_2(f)/f$. This yields the following.

Theorem 4.6 ([1])**.** *Any isothermic canal surface is special.*

In [28], it has been shown that any *Willmore canal surface*, that is a canal surface which is a critical point of the *Willmore functional*

$$S \mapsto \int_S \mu_2 dA,$$

is isothermic. Therefore,

Corollary 4.7. *Any Willmore canal is special.*

5. Canal foliations

Recall (see, for example, [9] and[18]) that a p-dimensional C^r-*foliation* $\mathcal{F}$ ($r = 0, 1, \ldots, \infty$) on an n-dimensional manifold M is a decomposition of M into connected submanifolds – called *leaves* – such that for any $x \in M$ there exists a C^r-differentiable chart $\phi = (\phi', \phi'') : U \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ defined on a neighbourhood U of x and satisfying the condition

($*$) for any L of $\mathcal{F}$, the connected components – called *plaques* – of $L \cap U$ are given by the equation $\phi'' =$ const.

The simplest examples of foliations are provided by submersions $F : M \to N$ with leaves being the connected components of the fibers $F^{-1}(y)$, $y \in N$, in particular by products $M = M_1 \times M_2$ of connected manifolds with leaves $M_1 \times \{y\}$, $y \in M_2$. The first non-trivial example of a foliation of the sphere $\mathbb{S}^3 = \{(w, z) \in \mathbb{C}^2; |w|^2 + |z|2 = 1\}$ has been provided by Reeb [29]: the *Reeb foliation* is obtained by gluing along the common toral boundary $T^2 = \{(w, z); |w|^2 = |z|^2 = 1/2\}$ the two *Reeb components* (see Figure 3) which can be produced from the strip $[-1, 1] \times \mathbb{R}$ foliated by the boundary lines and the graphs of smooth functions $f + c$, $c \in \mathbb{R}$, where $f : (-1, 1) \to \mathbb{R}$ satisfies $f(-t) = f(t)$, $f''(t) > 0$ and $\lim_{t\to\pm 1} f(t) + \infty$, by: first, the rotation around the axis $t = 0$ of symmetry of the strip, then passing to the quotient $D^2 \times \mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$ being the group of translations generated by $(w, s) \mapsto (w, s + 1)$, $|w| \le 1$, $s \in \mathbb{R}$.

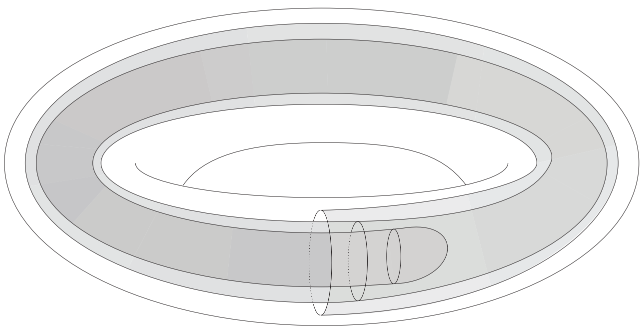

Figure 3. A Reeb component.

Observe that Reeb components can be foliated by canal surfaces, therefore the same happens to Reeb foliations of $\mathbb{S}^3$. Foliations of 3-manifolds of constant curvature by canal surfaces are called here *canal foliations*.

One of the procedures which allows to produce more complicated foliations from given ones is called *turbulization*.

Let us begin with a 2-dimensional foliation $\mathcal{F}$ of a manifold M, $\dim M = 3$. Find a loop Γ transverse to $\mathcal{F}$ and its tubular neighbourhood $N(\Gamma) \approx D^2 \times S^1$. Replace $\mathcal{F}$ outside $N(\Gamma)$ by the foliation shown in Figure 4 and fill the interior of $N(\Gamma)$ with a Reeb component. The resulting foliation $\mathcal{F}'$ is the turbulized $\mathcal{F}$.

In Figure 4, L is a piece of a leaf of $\mathcal{F}$ and γ a curve on L while L' is a piece of a leaf of $\mathcal{F}'$ and γ' a curve on L'.

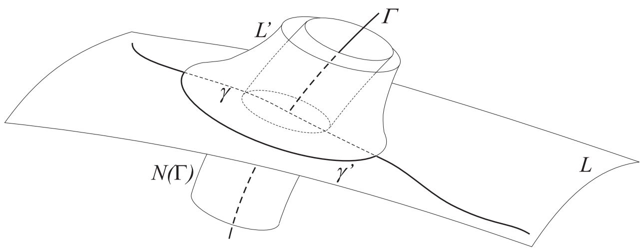

FIGURE 4. Turbulization.

Certainly, this procedure can be repeated as long as we can find new loops transverse to the foliation under consideration and turbulizations of canal foliations can sometimes provide new canal foliations.

Similarly to the construction of Reeb components, one can foliate the zone $Z = T^2 \times [0,1]$ by a family of tori $T^2 \times \{t\}$, $t \in A$, $A \subset [0,1]$ being closed, and filling the zones between two consecutive tori with cylinders spiralling from one boundary component of such zone towards the other one, Figure 5.

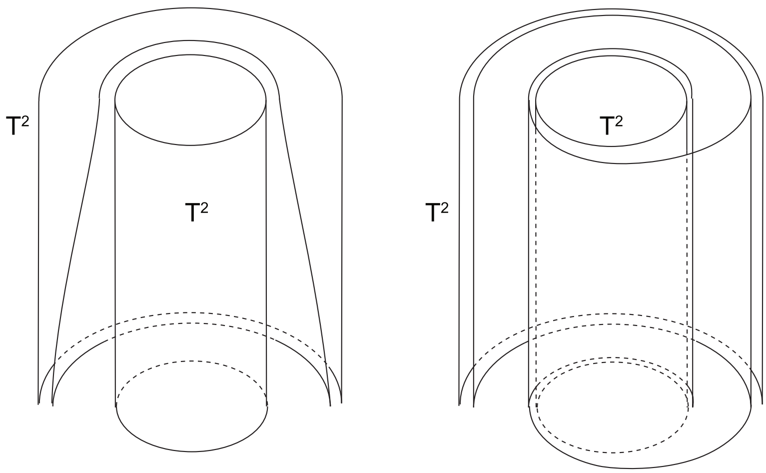

FIGURE 5. Two ways of spiraling.

Canal foliations of the sphere $\mathbb{S}^3$ have been classified:

Theorem 5.1 ([26]). *Any foliation $\mathcal{F}$ of $\mathbb{S}^3$ by canal surfaces is either*

- *a Reeb foliation with the toral leaf being a Dupin cyclide or*
- *is obtained from such a Reeb foliation inserting a zone $Z \simeq \mathbf{T}^2 \times [0,1]$ foliated by toral and cylindrical leaves (as described above).*

A question about the existence of canal foliations on other 3-manifolds of constant curvature arises in a natural way.

The first step is to ask about existence of foliations by the simplest canals: Dupin cyclides. Such foliations are called *Dupin foliations* here.

Theorem 5.2 ([25]). (i) *Dupin foliations of $\mathbb{S}^3$ do not exist.*
(ii) *The only Dupin foliations of $\mathbb{R}^3$ are these by parallel planes.*
(iii) *Closed hyperbolic manifolds admit no Dupin foliations.*

Certainly, one can imagine several different examples of canal foliations of the Euclidean 3-space $\mathbb{R}^3$ and of the hyperbolic 3-space $\mathbb{H}^3$. The most interesting case is that of closed hyperbolic 3-manifolds. It occurs, that the best way towards this goal is to find a topological version of the "canalicity".

Following [17], we shall say that a *griddled structure* on a surface L is a 1-dimensional foliation $\mathcal{C}$ (with singularities) such that any singularity of $\mathcal{C}$ is isolated and any regular leaf of $\mathcal{C}$ is homeomorphic to $\mathbb{S}^1$. Similarly, a *griddled structure* on a foliated 3-manifold $(M, \mathcal{F})$ will be an orientable subfoliation $\mathcal{C}$ of the codimension 1 foliation $\mathcal{F}$ which induces by restriction a griddled structure on each leaf L of $\mathcal{F}$. In both cases, we will say that L and $(M, \mathcal{F})$ are *griddled*.

First, we are going to classify griddled surfaces. To this end, recall that an action of $\mathbb{S}^1$ is called *semi-free* if any non trivial isotropy subgroup coincides with $\mathbb{S}^1$. Now a semi-free action of $\mathbb{S}^1$ on a surface defines a griddled structure provided that its singularities are isolated; we call such a structure *canonical*.

Example 5.3. Natural examples of canonical griddled structures on surfaces are provided by the following semi-free actions:

(1) the action of the group of rotations around the origin of the plane $\mathbb{R}^2$ or the unit disk $\mathbb{D}^2$; the action of the group of rotations of the unit sphere S^2 around the vertical axis $\overrightarrow{Oz}$ of $\mathbb{R}^3$; these structures have either one or two singularities,

(2) the natural free action of the first factor on the annulus $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$, the cylinders $\mathbb{S}^1 \times \mathbb{R}$ or $\mathbb{S}^1 \times [0, 1]$, the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, all the four structures being *regular*, that is without singularities.

Lemma 5.4. *Any singularity of a connected griddled surface $(L, \mathcal{C})$ is a center and $(L, \mathcal{C})$ is topologically conjugate to one of the canonical griddled surfaces of Example 5.3.*

The above lemma together with several technical steps allows to classify griddled foliated 3-manifolds (either with boundary or without):

Theorem 5.5 ([17]). *A compact 3-manifold M supports a griddled foliation if and only if M is one of the following manifolds:*
i) $D^2 \times [0, 1]$, $D^2 \times S^1$, $S^2 \times [0, 1]$ or $T^2 \times [0, 1]$ if $\partial M \neq \emptyset$,
ii) $S^2 \times S^1$, T^3 or an S^1-bundle over T^2 if $\partial M = \emptyset$.

Certainly, canal surfaces and many canal foliations are griddled with the griddled structure consisting of all the characteristic circles of the surface/leaves.

However, the standard Reeb foliation on $\mathbb{S}^3$ is not griddled: the griddled structures of the two Reeb components induce two transverse griddlings of the toral leaf. This is why we need to accept the following definitions:

Consider a finite family $\mathcal{D} = \{L_1, L_2, \ldots, L_k\}$ of compact leaves of a codimension 1 compact foliated manifold $(M, \mathcal{F})$. Cutting M along these leaves, we produce a finite family $\{N_1, N_2, \ldots, N_l\}$ of compact foliated manifolds and a foliation preserving submersive map

$$\psi : \coprod_j N_j \to M.$$

The restriction of ψ either to the interior or to any boundary component of any N_j is injective. We call ψ a *foliated decomposition* of $(M, \mathcal{F})$ defined by $\mathcal{D}$. Now, we introduce our last concept:

A codimension 1 foliation $\mathcal{F}$ on a compact connected 3-manifold M (possibly with boundary) is a *topological canal foliation* if there exists a foliated decomposition $\psi : \coprod_j N_j \to M$ of $(M, \mathcal{F})$ defined by a finite family $\mathcal{D}$ of compact leaves verifying the following two conditions:

(i) for each j, the foliation $\mathcal{F}_j$ induced by $\mathcal{F}$ on N_j is tangent to the boundary ∂N_j, admits a griddled structure $\mathcal{C}_j$ and any component of ∂N_j is a regularly griddled torus,

(ii) any torus $L_j \in \mathcal{D}$ being the image by ψ of two boundary components of $\coprod_j N_j$ is endowed with two griddled structures which are mutually transverse.

The elements of $\mathcal{D}$ are called the *turning leaves* of $\mathcal{F}$ and the manifolds N_j are its *griddled components*.

Theorem 5.6 ([17])**.** *A compact* 3*-manifold M supports a topological canal foliation if and only if M is one of the following:*

(i) $D^2 \times [0,1]$, $D^2 \times S^1$, $S^2 \times [0,1]$ *or* $T^2 \times [0,1]$ *if* $\partial M \neq \emptyset$,

(ii) $S^2 \times S^1$, S^3 *or any Lens space,* T^3 *or any* S^1*-bundle over* T^2 *or any* T^2*-bundle over* S^1 *if* $\partial M = \emptyset$.

None of the manifolds M listed in the theorem above admits a hyperbolic structure. Indeed, the fundamental group of each of them (and its doubling $M\#M$ in the case $\partial M \neq \emptyset$) has the growth of polynomial type while – as shown in [27] and [33] – the fundamental groups of all closed hyperbolic manifolds have exponential type of growth. (For more about types of growth, see – for instance – [35], Chapter 2.) As mentioned before, many geometric canal foliations admit griddled structures and, in fact, all of them admit structures of topological canal foliations. Therefore, we can conclude with the following.

Corollary 5.7. *Closed hyperbolic* 3*-manifolds do not admit (neither geometric nor topological) canal foliations.*

6. Epilogue

Certainly, this article does not exhaust the list of recent result on canals. Here, we rewiev briefly some of those which were not mentioned in previous sections.

1. In [24], the authors find the minimal value of the length (in de Sitter space) of closed space-like curves with non-vanishing non-space-like geodesic curvature vector. These curves are in correspondence with closed almost regular canal surfaces, and their length is a natural quantity in conformal geometry.

2. As shown in Section 2, the space of 2-dimensional spheres in the 3-dimensional space form has dimension 4. Similarly, the space of Dupin cyclides can be shown to be of dimension 9. Given two *contact conditions*, that is two planes equipped with two points (supposed to be points of tangency), in general there is no sphere satisfying them (that is, tangent to the planes at given points). Similarly (see [20, 23]), given three such tangency conditions, in general there is no Dupin cyclide satisfying them. But, one can find a codimension-one subspace of triples of contact conditions such that for any its element there exists a one-parameter family of Dupin cyclides satisfying the three contacts.

3. In [15], the authors provide an algorithm to compute in de Sitter space a characteristic circle of a Dupin cyclide given a point and the tangent line. They provide also iterative algorithms (in the space of spheres) to compute (in 3D space) some characteristic circles of a Dupin cyclide which blends two particular canal surfaces. In [14], tools from geometric algebra are used to study Dupin cyclides.

4. As mentioned in Theorem 5.2, regular foliations by Dupin cyclides do not exist neither on $\mathbb{S}^3$ nor on closed hyperbolic manifolds. However, singular Dupin foliations, like that on $\mathbb{S}^3 \subset \mathbb{C}^2$ by tori $|w| = a,\ |z| = b$ with $a, b > 0$ satisfying $a^2 + b^2 = 1$, may be considered. In [21], the authors constructed several examples of such foliations on the unite sphere and illustrated them with very interesting pictures produced by computer-drawing.

Also, cyclides different than these called Dupin can be considered: A general *cyclide* is a closed surface in $\mathbb{S}^3$ which is spanned by two transverse families of circles such that exactly one circle of each family passes through each point of the surface. Singular foliations by general cyclides on $\mathbb{S}^3$ are studied in [22].

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