Counting Stirling permutations by number of pushes

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Abstract. Let $\mathcal{T}_n^{(k)}$ denote the set of k -Stirling permutations having n distinct letters. Here, we consider the number of steps required (i.e., *pushes*) to rearrange the letters of a member of $\mathcal{T}_n^{(k)}$ so that they occur in non-decreasing order. We find recurrences for the joint distribution on $\mathcal{T}_n^{(k)}$ for the statistics recording the number of levels (i.e., occurrences of equal adjacent letters) and pushes. When $k = 2$, an explicit formula for the ordinary generating function of this distribution is also found. In order to do so, we determine the LU -decomposition of a certain infinite matrix having polynomial entries which enables one to compute explicitly the inverse matrix.

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1 Introduction

A *Stirling* sequence $\pi = \pi_1\pi_2\cdots$ is one that avoids the pattern 212 (i.e., there exist no indices $i < j < k$ such that $\pi_i = \pi_k > \pi_j$). Orderings of the multiset $1^k 2^k \cdots n^k$ where $k \geq 2$ that avoid the pattern 212 are called *k-Stirling permutations* (of order n). When $k = 2$, these are the *Stirling permutations* originally considered by Gessel and Stanley [3]. The k -Stirling permutations have been provided various refined counts in the case $k = 2$ (see, e.g., [1, 2, 5, 6]) and for general k (see [7]). It is seen that there are $\prod_{i=1}^{n-1} (ki + 1)$ k -Stirling permutations of order n . In this paper, we consider a new refinement of this number and in particular, when $k = 2$, a new q -analogue of the odd double factorial sequence (see, e.g., [10, A001147]).

We now describe the combinatorial basis of our refinement. Given a word $w = w_1w_2\cdots$ on the alphabet of positive integers, we seek the number of letters that must be moved to the left so that the resulting word is non-decreasing. Consider the smallest index $i > 1$ (if it exists), denoted by i_0 , such that there exists at least one $j \in [i - 1]$ with $w_j > w_i$. We then shift w_{i_0} within w , one position at a

time to the left, stopping at the point when no letter occurs to the left of w_{i_0} that is strictly greater than w_{i_0} within the new word w' . We will refer to this shifting of the letter w_{i_0} as a *push*. We then repeat this procedure using w' and so on until a non-decreasing word results. By the *number of pushes* of a word $w = w_1 w_2 \cdots$, we will mean the number of steps required to achieve a non-decreasing word. Let $\mathcal{T}_n^{(k)}$ denote the set of k -Stirling permutations of order n . We will consider here the number of pushes as a statistic on members of $\mathcal{T}_n^{(k)}$, represented sequentially.

For example, if $n = 5$, $k = 3$ and $\pi = 112444221355533 \in \mathcal{T}_5^{(3)}$, then π has six pushes altogether corresponding to the letters 2, 2, 1, 3 between the 4's and the 5's along with the two terminal 3's. By a *weak left-right maximum* (wlrms) within a sequence $s = s_1 s_2 \cdots s_m$, we mean an index i such that $s_i \geq s_j$ for all $1 \leq j < i$. Note that the number of pushes of s is given by $m - (\# \text{ wlrms})$ since each letter of s not corresponding to a wlrms must be shifted to the left so as to achieve sorted order. Henceforth, we will consider equivalently the distribution on $\mathcal{T}_n^{(k)}$ for the statistic recording the number of wlrms. Note that in our example above, the permutation π has nine wlrms altogether.

In the next section, we find recurrences satisfied by the joint distribution on $\mathcal{T}_n^{(k)}$ for all $k \geq 2$ for the statistics recording the number of wlrms and number of levels by appropriately decomposing the set $\mathcal{T}_n^{(k)}$ into disjoint parts for which a recursive structure can be found. Further recurrences are noted in the case $k = 2$ corresponding to Stirling permutations. In the third section, we make use of the recurrences to compute an explicit formula for the ordinary generating function of the aforementioned joint distribution on $\mathcal{T}_n^{(k)}$ in the case $k = 2$. Some special cases of our formula are observed.

We now briefly describe our strategy for computing the generating function in the third section. First, we introduce an infinite matrix M to represent a family of recurrences satisfied by some auxiliary generating functions each involving a differential equation. We seek the inverse of M . While it is difficult to determine M^{-1} directly, it turns out that when M is expressed using the LU -decomposition that the individual matrices L and U have inverses that can be computed explicitly. Note that some guesswork is required in determining the LU -decomposition of M and also in finding L^{-1} and U^{-1} . Once this is done, we then have $M^{-1} = U^{-1}L^{-1}$ and are thus able to complete our derivation of the generating function formula.

2 Recurrences for k -Stirling permutations

In this section, we consider the joint distribution of the levels and weak left-right maxima statistics on $\mathcal{T}_n^{(k)}$. Given $\pi = \pi_1 \pi_2 \cdots \in \mathcal{T}_n^{(k)}$, we define what will be referred to as a *unit* of π recursively as follows. Let i_1 be the first letter of π . Let u_1 consist of all letters of π up to and including the last occurrence of i_1 . Let i_2 (if it exists) be the letter that directly follows the last occurrence of i_1 , and define u_2 similarly using the letters of π that are not part of u_1 . Successively define the units $u_1, u_2, \dots$ until all of the letters of π have been exhausted. For example, $\pi \in \mathcal{T}_5^{(3)}$ above has two units, the second starting with the first 3. Let $\mathcal{T}_{n,\ell}^{(k)}$ denote the subset of $\mathcal{T}_n^{(k)}$ comprising those members that contain exactly ℓ units for $1 \leq \ell \leq n$. See [9] for an equivalent refinement of $\mathcal{T}_n^{(k)}$ when $k = 2$.

Let $a_k(n, \ell) = a_k(n, \ell; p, q)$ denote the joint distribution on $\mathcal{T}_{n,\ell}^{(k)}$ for the statistics recording the

number of levels and weak left-right maxima. That is,

$$a_k(n, \ell) = \sum_{\pi \in \mathcal{T}_{n, \ell}^{(k)}} p^{\text{lev}(\pi)} q^{\text{max}(\pi)},$$

where $\text{lev}(\pi)$ is the number of *levels* (i.e., the number of indices i such that $\pi_i = \pi_{i+1}$) and $\text{max}(\pi)$ is the number of *wlrm*. The levels statistic has been previously considered on $\mathcal{T}_n^{(k)}$ in the case $k = 2$ (see [6], where they are referred to as plateaus, and also the related paper [8]); here we consider a new bivariate statistic with the number of *wlrm*. We have the following recurrence formula for $a_k(n, \ell)$.

THEOREM 2.1 *If $k \geq 2$ and $n \geq \ell \geq 1$, then*

$$\begin{aligned} a_k(n, \ell) = & p^{k-1}(\ell - 1 + q^k) a_k(n - 1, \ell - 1) + (\ell - 1) \sum_{j=\ell}^{n-1} \sum_{r=0}^{k-2} p^{k-2-r} \binom{j-\ell}{r} \binom{k-1}{r+1} a_k(n-1, j) \\ & + \sum_{j=\ell}^{n-1} \sum_{i=0}^{k-2} \sum_{r=0}^{k-2-i} p^{k-2-r} q^{i+1} \binom{j-\ell}{r} \binom{k-2-i}{r} a_k(n-1, j), \end{aligned} \quad (1)$$

where $a_k(n, 0) = \delta_{n,0}$.

Proof. Suppose $\lambda \in \mathcal{T}_{n, \ell}^{(k)}$ where $1 \leq \ell \leq n$. Consider forming λ from a precursor $\lambda' \in \mathcal{T}_{n-1}^{(k)}$ (on $[2, n]$) by inserting k copies of 1 into λ' at various places. Note that the 1's must be inserted between units of λ' or at the very beginning or end of λ' , lest there be an occurrence of 212. First consider the case in which the k 1's are inserted into λ' as a single run (which is seen to increase the number of units by one). Then there are $p^{k-1}(\ell - 1) a_k(n - 1, \ell - 1)$ possibilities if the run of 1's does not occur at the beginning of λ and $p^{k-1} q^k a_k(n - 1, \ell - 1)$ possibilities if it does. Combining the two cases then gives the first term on the right side of (1) if $\ell \geq 2$. If $\ell = 1$, then one gets a contribution of $p^{n-1} q^n \delta_{n,1}$, which coincides with the first term in this case since $a_k(n, 0) = \delta_{n,0}$.

Now assume that λ contains more than one run of 1's and does not start with 1. Let λ' contain j units. Then $j \geq \ell$ since insertion of the 1's in this case either preserves or reduces the number of units. Suppose that the leftmost 1 is inserted immediately prior to the s -th unit of λ' , where $2 \leq s \leq \ell$. Then the rightmost 1 must directly follow the $(s + j - \ell)$ -th unit of λ' , which ensures that λ will contain $(s - 1) + (j - (s + j - \ell)) + 1 = \ell$ units, as required. Suppose further that 1's are to occur in exactly r of the $j - \ell$ possible positions between the i -th and the $(i + 1)$ -st units of λ' for $s \leq i < s + j - \ell$. After inserting a 1 in each of these r positions (as well as before the s -th unit and after the $(s + j - \ell)$ -th), there are $k - 2 - r$ 1's left to be distributed, each of which produces a level when inserted into its respective position, which accounts for the factor of p^{k-2-r} . Finally, there are $\binom{k-2-r+(r+1)}{r+1} = \binom{k-1}{r+1}$ ways in which to insert these $k - 2 - r$ 1's into $r + 2$ positions between units of λ' . Considering all possible $\ell \leq j \leq n - 1$ and $0 \leq r \leq k - 2$, and noting that each choice of s yields the same number of possibilities, then gives the second part of (1).

Now assume that λ contains more than one run of 1's and starts with 1. In this case, we must also keep track of the number of 1's inserted at the beginning of λ' . Let this number be denoted by $i + 1$, where $0 \leq i \leq k - 2$, which accounts for the factor of q^{i+1} . Let j and r be as before. Then there

are $\binom{k-2-i}{r}$ possible ways in which to insert the additional 1's into $r+1$ positions other than at the beginning. Once i is fixed, then reasoning as before implies that there are

$$q^{i+1} \sum_{j=\ell}^{n-1} \sum_{r=0}^{k-2-i} p^{k-2-r} \binom{j-\ell}{r} \binom{k-2-i}{r} a_k(n-1, j)$$

possible λ . Considering all i then gives the third and final part of (1), which completes the proof. $\square$

Let $a_k(n, \ell) = a_k(n, \ell; 1, q)$. Taking $p = 1$ in (1), and applying twice Vandermonde's identity (see, e.g., [4, Formula 5.23]), gives

$$\begin{aligned} a_k(n, \ell) &= (\ell - 1 + q^k) a_k(n-1, \ell-1) + (\ell - 1) \sum_{j=\ell}^{n-1} \binom{j-\ell+k-1}{k-2} a_k(n-1, j) \\ &\quad + \sum_{j=\ell}^{n-1} \sum_{i=0}^{k-2} q^{i+1} \binom{j-\ell+k-i-2}{k-i-2} a_k(n-1, j). \end{aligned} \quad (2)$$

Let $a(n, \ell) = a_2(n, \ell)$. Taking $k = 2$ in (2) yields

$$a(n, \ell) = (\ell - 1 + q^2) a(n-1, \ell-1) + (\ell - 1 + q) \sum_{j=\ell}^{n-1} a(n-1, j), \quad n \geq \ell \geq 1. \quad (3)$$

In the next section, we study further properties of $a(n, \ell)$. Note that $a(n) = \sum_{\ell=1}^n a(n, \ell)$ gives the distribution of the statistic on $\mathcal{T}_n = \mathcal{T}_n^{(2)}$ which records the number of wlrms.

In the remainder of this section, we develop an alternative recursive approach to finding $a(n)$. To do so, first let $\mathcal{D}_{n,i,m}$ denote the set of sequences $s = s_1 s_2 \cdots s_{n-i+1}$ such that $s_j \in [2n - 2j + 1]$ for $1 \leq j \leq n - i + 1$ with $m = \min(s_1 s_2 \cdots s_{n-i+1})$, where $n \geq 1$, $1 \leq i \leq n$ and $1 \leq m \leq 2i - 1$. Let $d(n, i, m) = |\mathcal{D}_{n,i,m}|$. We define a q -generalization of $d(n, i, m)$ as follows. By a *substantial* left-right minimum within a sequence $t = t_1 t_2 \cdots$, we mean an index i such that $t_i < t_j - 1$ for all $1 \leq j < i$, with this vacuously holding if $i = 1$. By a *marginal* left-right minimum, we mean an index such that $t_i < t_j$ for all $1 \leq j < i$ with $t_i = t_j - 1$ for at least one such j . We will use similar terminology when referring to right-left minima. Let $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ denote the number of substantial and marginal left-right minima, respectively, of $\lambda \in \mathcal{D}_{n,i,m}$. Define the distribution $d_q(n, i, m)$ on $\mathcal{D}_{n,i,m}$ by

$$d_q(n, i, m) = \sum_{\lambda \in \mathcal{D}_{n,i,m}} q^{2\sigma_1(\lambda) + \sigma_2(\lambda)}.$$

Then the polynomials $d_q(n, i, m)$ are determined recursively as follows.

PROPOSITION 2.2 *If $1 \leq i < n$ and $1 \leq m \leq 2i - 1$, then*

$$d_q(n, i, m) = (2i - m) d_q(n, i+1, m) + q a_q(n, i+1, m+1) + q^2 \sum_{j=m+2}^{2i+1} d_q(n, i+1, j), \quad (4)$$

with $d_q(n, n, m) = q^2$ for all $n \geq 1$ and $1 \leq m \leq 2n - 1$.

Proof. Since the first letter of any word is regarded as a substantial but not a marginal left-right minimum, the initial condition when $i = n$ follows from the definitions. Let $\delta \in \mathcal{D}_{n,i,m}$ where $1 \leq i < n$. Then δ may be obtained from a precursor $\delta' = d_1 d_2 \cdots d_{n-i} \in \mathcal{D}_{n,i+1,j}$ for some j by appending a final letter x . If $x < j - 1$, then x contributes a substantial left-right minimum and $x = m < j - 1$, i.e., $m + 2 \leq j \leq 2i + 1$. Considering all possible j yields $q^2 \sum_{j=m+2}^{2i+1} d_q(n, i+1, j)$ possibilities for δ in this case. If $x < d_r$ for all r with $x = d_r - 1$ for at least one $r \in [n-i]$, then $x = m$ is a marginal left-right minimum and $j = m + 1$, which yields $q d_q(n, i+1, m+1)$ possibilities. Finally, if $x \geq j$, then $j = m$ and there are $(2i-1) - (m-1) = 2i - m$ choices for x , which gives the remaining term on the right side of (4) and completes the proof. $\square$

The special case of $d_q(n, i, m)$ when $i = m = 1$ corresponds to the distribution of wlrms on $\mathcal{T}_n$.

PROPOSITION 2.3 *If $n \geq 1$, then $d_q(n, 1, 1) = \sum_{\ell=1}^n a(n, \ell)$.*

Proof. Let $s \in \mathcal{D}_{n,1,1}$ and $s' = s'_1 s'_2 \cdots s'_n$ denote the reversal of s ; note that $s'_i \in [2i-1]$ for $1 \leq i \leq n$. We form a member $w \in \mathcal{T}_n$ using s' by first writing two copies of 1. For each successive $i \in [2, n]$, starting with $i = 2$, insert two copies of i into the current member of $\mathcal{T}_{i-1}$ by inserting two i 's in one of $2i-1$ possible positions according to the value of s'_i . Note that these are the only possibilities regarding the placement of the i 's since they must be adjacent, for otherwise there is an occurrence of the pattern 212. Note that if the letter y in the ℓ -th position of s' corresponds to a substantial right-left minimum, which means all letters beyond the ℓ -th position are greater than $y+1$, then both copies of the letter ℓ in w are wlrms since all copies of letters in $[\ell+1, n]$ must occur to the right of the second ℓ . If the letter y in s' corresponds to a marginal right-left minimum, then no letter greater than ℓ appears to the left of ℓ in w with at least one letter in $[\ell+1, n]$ occurring between the two copies of ℓ , which implies only the first ℓ is a wlrms. Finally, if the letter y is neither a substantial nor a marginal right-left minimum, then some letter greater than ℓ appears to the left of ℓ in w , which implies neither copy of ℓ is a wlrms. Thus, since the mapping $s \mapsto w$ is seen to be a bijection, it follows that the distribution of $2\sigma_1 + \sigma_2$ on $\mathcal{D}_{n,1,1}$ is the same as the wlrms statistic on $\mathcal{T}_n$, which implies the desired result. $\square$

3 Generating function for distribution on Stirling permutations

In this section, we find an explicit formula for the generating function of $a_k(n, \ell; 1, q)$ and $a_k(n, \ell; p, q)$ in the case $k = 2$. Recall that $a(n, \ell)$ denotes the quantity $a_2(n, \ell; 1, q)$. Define $A(x; \ell) = \sum_{n \geq \ell} a(n, \ell) \frac{x^n}{n!}$. Then recurrence (3) above can be rewritten in terms of generating functions as

$$\frac{d}{dx} A(x; \ell) = (\ell - 1 + q^2) A(x; \ell - 1) + (\ell - 1 + q) \sum_{j \geq \ell} A(x; j), \quad \ell \geq 1, \quad (5)$$

with $A(x; 0) = 1$. Before considering $A(x; \ell)$ for all q , note that when $q = 1$ in (5), we have

$$\frac{d}{dx} A(x; \ell) \big|_{q=1} = \ell \sum_{j \geq \ell-1} A(x; j) \big|_{q=1}, \quad \ell \geq 1, \quad (6)$$

with $A(x; 0) \big|_{q=1} = 1$. Using (6), one may verify

$$A(x; \ell) \big|_{q=1} = (1 - \sqrt{1 - 2x})^\ell, \quad \ell \geq 0.$$

Summing this over ℓ implies $\sum_{\ell \geq 0} A(x; \ell) \mid_{q=1} = \frac{1}{\sqrt{1-2x}}$, which coincides with the well-known formula for the exponential generating function for the number of perfect matchings of $[2n]$.

Let $\mathbf{v}(x)$ be the column vector $(A(x; 0), A(x; 1), \dots)^T$, where q now is arbitrary. Define $\mathbf{A}$ to be the infinite matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ q^2 & q & q & q & \cdots \\ 0 & 1+q^2 & 1+q & 1+q & \cdots \\ 0 & 0 & 2+q^2 & 2+q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Note that $\frac{d}{dx}\mathbf{v}(x) = \mathbf{A}\mathbf{v}(x)$.

PROPOSITION 3.1 *The generating functions*

$$A(x, y) = \sum_{n \geq 0} \sum_{\ell=0}^n a(n, \ell) x^n y^\ell \text{ and } E(x, y) = \sum_{n \geq 0} \sum_{\ell=0}^n a(n, \ell) \frac{x^n y^\ell}{n!}$$

are given by

$$A(x, y) = (1, y, y^2, \dots)(1 - \mathbf{A}x)^{-1}e_1^T$$

and

$$E(x, y) = (1, y, y^2, \dots)e^{\mathbf{A}x}e_1^T,$$

where $e_1 = (1, 0, 0, \dots)$.

Proof. From the fact $\frac{d}{dx}\mathbf{v}(x) = \mathbf{A}\mathbf{v}(x)$, we have $\frac{d^m}{dx^m}\mathbf{v}(x) \mid_{x=0} = \mathbf{A}^m\mathbf{v}(0)$. Note that $\mathbf{v}(0) = e_1^T = (1, 0, 0, \dots)^T$, which implies

$$A(x, y) = (1, y, y^2, \dots) \left(\sum_{m \geq 0} \mathbf{A}^m x^m \right) e_1^T = (1, y, y^2, \dots)(1 - \mathbf{A}x)^{-1}e_1^T$$

and

$$E(x, y) = (1, y, y^2, \dots) \left(\sum_{m \geq 0} \mathbf{A}^m \frac{x^m}{m!} \right) e_1^T = (1, y, y^2, \dots)e^{\mathbf{A}x}e_1^T,$$

as required. □

Define

$$\mathbf{A}' = 1 - x\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -xq^2 & 1-xq & -xq & -xq & \cdots \\ 0 & -x(1+q^2) & 1-x(1+q) & -x(1+q) & \cdots \\ 0 & 0 & -x(2+q^2) & 1-x(2+q) & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

and we seek to determine the inverse matrix of $\mathbf{A}'$. In order to do so, first let $\mathbf{A}' = \mathbf{L}\mathbf{U}$ be the LU -decomposition of the matrix $\mathbf{A}'$. Let $\mathbf{A}'_n$ be the submatrix of $\mathbf{A}'$ on the first n rows and n columns and denote its determinant by d_n ; that is, $d_n = \det(\mathbf{A}'_n)$ if $n \geq 1$, with $d_0 = 1$.

LEMMA 3.2 *We have*

$$d_n = (1 + xq(q-1))d_{n-1} - x(n-2+q^2)d_{n-2}, \quad n \geq 2, \quad (7)$$

with $d_0 = d_1 = 1$. Moreover,

$$\sum_{n \geq 1} \frac{d_n}{\prod_{i=1}^{n-1} (i+q^2)} y^n = q^2 y^{1-q^2} e^{y(q^2 x - qx - xy/2 + 1)} \int_0^y (1-tx) t^{q^2-1} e^{-t(q^2 x - tx/2 - qx + 1)} dt.$$

Proof. Using repeated cofactor expansion along the last row to compute the determinant, we have

$$d_n = d_{n-1} - \sum_{j=1}^{n-1} x^{n-j} d_j (j-1+q) \prod_{i=j}^{n-2} (i+q^2), \quad n \geq 1, \quad (8)$$

with $d_0 = 1$. If $n \geq 2$, then

$$\begin{aligned} d_n - d_{n-1} &= -x d_{n-1} (n-2+q) - x(n-2+q^2) \sum_{j=1}^{n-2} x^{n-1-j} d_j (j-1+q) \prod_{i=j}^{n-3} (i+q^2) \\ &= -x d_{n-1} (n-2+q) + x(n-2+q^2) (d_{n-1} - d_{n-2}), \end{aligned}$$

which gives (7). Now define e_n by $d_n = e_n \prod_{i=1}^{n-1} (i+q^2)$. Then (8) can be rewritten as

$$e_n (n-1+q^2) = e_{n-1} - \sum_{j=1}^{n-1} x^{n-j} e_j (j-1+q), \quad n \geq 2,$$

with $e_1 = 1$. If $E(y) = \sum_{n \geq 1} e_n y^n$, then we get

$$y \frac{d}{dy} (E(y) - y) + (q^2 - 1)(E(y) - y) = yE(y) - \frac{(q-1)xy}{1-xy} E(y) - \frac{xy^2}{1-xy} \frac{d}{dy} E(y),$$

which is equivalent to

$$y \frac{d}{dy} E(y) + ((q^2 - q - y)(1 - xy) + q - 1)E(y) = q^2 y(1 - xy).$$

The solution of this differential equation with initial condition $E(0) = 0$ is given by

$$E(y) = q^2 y^{1-q^2} e^{y(q^2 x - qx - xy/2 + 1)} \int_0^y (1-tx) t^{q^2-1} e^{-t(q^2 x - tx/2 - qx + 1)} dt,$$

as required. $\square$

Remark. Note that when $q = 1$, we have $E(y) |_{q=1} = e^{y(1-xy/2)} \int_0^y (1-tx) e^{-t(1-tx/2)} dt$, which leads to $E(y) |_{q=1} = e^{y(1-xy/2)} - 1$. Hence,

$$\sum_{n \geq 0} \frac{d_n |_{q=1}}{n!} y^n = e^{y-xy^2/2}.$$

The matrix $\mathbf{A}'$ has the following LU -decomposition.

LEMMA 3.3 We have $\mathbf{A}' = \mathbf{L}\mathbf{U}$, where $\mathbf{L} = (\ell_{ij})$ and $\mathbf{U} = (u_{ij})$ are given by

$$\ell_{ij} = \begin{cases} 1, & j = i \geq 1, \\ \frac{-x(i-2+q^2)d_{i-2}}{d_{i-1}}, & j = i-1 \geq 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } u_{ij} = \begin{cases} \frac{d_i}{d_{i-1}}, & j = i \geq 1, \\ \frac{d_i}{d_{i-1}} - 1, & 2 \leq i+1 \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let a'_{ij} denote the (i, j) entry of $\mathbf{A}'$. We compute $\sum_{r \geq 1} \ell_{ir} u_{rj}$ in several cases and compare with a'_{ij} . If $j \leq i-2$, then both quantities are zero, so assume $j \geq i-1$. If $j = i-1$, then

$$\sum_{r \geq 1} \ell_{ir} u_{rj} = \ell_{ii} u_{i,i-1} + \ell_{i,i-1} u_{i-1,i-1} = 0 - \frac{x(i-2+q^2)d_{i-2}}{d_{i-1}} \cdot \frac{d_{i-1}}{d_{i-2}} = -x(i-2+q^2) = a'_{ij}.$$

If $i = j = 1$, then both entries are clearly 1, so assume $i = j \geq 2$. Then we have

$$\begin{aligned} \ell_{ii} u_{ii} + \ell_{i,i-1} u_{i-1,i} &= \frac{d_i}{d_{i-1}} + \left(\frac{-x(i-2+q^2)d_{i-2}}{d_{i-1}} \right) \left(\frac{d_{i-1}}{d_{i-2}} - 1 \right) \\ &= \frac{d_i + x(i-2+q^2)d_{i-2}}{d_{i-1}} - x(i-2+q^2), \end{aligned}$$

which equals $a'_{ii} = 1 - x(i-2+q)$, upon using (7). If $i = 1$ and $j \geq 2$, then both entries are again zero since $d_0 = d_1 = 1$. Finally, if $j \geq i+1$ with $i \geq 2$, then the (i, j) entry of the product $\mathbf{L}\mathbf{U}$ is the same quantity as in the case $i = j \geq 2$ above except that there is an extra minus 1 term. Thus, it works out to $-x(i-2+q)$, which is again the same as $a'_{i,j}$. Combining the previous cases now implies the result. $\square$

Thus we have $\mathbf{A}'^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$, where the inverse matrices $\mathbf{L}^{-1} = (\tilde{\ell}_{ij})$ and $\mathbf{U}^{-1} = (\tilde{u}_{ij})$ are given explicitly by

$$\tilde{\ell}_{ij} = \begin{cases} 1, & j = i, \\ \frac{d_{j-1}x^{i-j} \prod_{s=j+1}^i (s-2+q^2)}{d_{i-1}}, & 1 \leq j \leq i-1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \tilde{u}_{ij} = \begin{cases} \frac{d_{i-1}}{d_i}, & i = j, \\ -\frac{d_i - d_{i-1}}{d_j}, & 2 \leq i+1 \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

which may be verified by a direct computation using the formulas from Lemma 3.3.

The $(i, 1)$ term for $i \geq 2$ in the product of matrices $\mathbf{U}^{-1}\mathbf{L}^{-1}$ is given by

$$\begin{aligned} (\mathbf{U}^{-1}\mathbf{L}^{-1})_{i1} &= \sum_{k \geq 1} \tilde{u}_{ik} \tilde{\ell}_{k1} = \frac{x^{i-1} \prod_{s=0}^{i-2} (s+q^2)}{d_i} + \sum_{k \geq i+1} \tilde{u}_{ik} \tilde{\ell}_{k1} \\ &= \frac{x^{i-1} \prod_{s=0}^{i-2} (s+q^2)}{d_i} - \sum_{k \geq i+1} \frac{d_i - d_{i-1}}{d_k} \frac{x^{k-1} \prod_{s=2}^k (s-2+q^2)}{d_{k-1}} \\ &= \frac{x^{i-1} \prod_{s=0}^{i-2} (s+q^2)}{d_i} \left(1 - d_i(d_i - d_{i-1}) \sum_{k \geq i+1} \frac{\prod_{s=i-1}^{k-2} (s+q^2)}{d_{k-1}d_k} x^{k-i} \right), \end{aligned}$$

which is seen to hold also for $i = 1$ since $d_0 = d_1 = 1$. By Proposition 3.1 and Lemma 3.2, we have the following result.

THEOREM 3.4 The generating function $A(x, y) = \sum_{n \geq 0} \sum_{\ell=0}^n a(n, \ell) x^n y^\ell$ is given by

$$A(x, y) = (1, y, y^2, \dots) \mathbf{U}^{-1} \mathbf{L}^{-1} (1, 0, 0, \dots)^T \\ = \sum_{i \geq 1} \left(\frac{(xy)^{i-1} \prod_{s=0}^{i-2} (s + q^2)}{d_i} \left(1 - d_i(d_i - d_{i-1}) \sum_{k \geq i+1} \frac{\prod_{s=i-1}^{k-2} (s + q^2)}{d_{k-1} d_k} x^{k-i} \right) \right),$$

where d_n is determined by (7).

Remark. Let $d_n = d_n(x, q)$, where d_n is as above, and let $d_n(x) = d_n(x, 1)$. Note that Theorem 3.4 when $q = 1$ gives

$$A(x, y) |_{q=1} = \sum_{i \geq 1} \left(\frac{(xy)^{i-1} (i-1)!}{d_i(x)} \left(1 - \frac{d_i(x)(d_i(x) - d_{i-1}(x))}{(i-1)!} \sum_{k \geq i+1} \frac{(k-1)!}{d_{k-1}(x) d_k(x)} x^{k-i} \right) \right).$$

It would be interesting to find explicit formulas for $\sum_{i \geq 1} \frac{(i-1)! y^{i-1}}{d_i(x)}$ and $\sum_{k \geq i+1} \frac{(k-1)! x^{k-1}}{d_k(x) d_{k-1}(x)}$.

The preceding can be generalized as follows. Let $a(n, \ell; p) = a_2(n, \ell; p, q)$; note that $a(n, \ell; 1) = a(n, \ell)$. Taking $k = 2$ in (1) gives the recurrence

$$a(n, \ell; p) = p(\ell - 1 + q^2) a(n - 1, \ell - 1; p) + (\ell - 1 + q) \sum_{j=\ell}^{n-1} a(n - 1, j; p), \quad n \geq \ell \geq 1, \quad (9)$$

which reduces to (3) when $p = 1$. Similarly, let us define the generating function $A(x; \ell) = A_p(x; \ell)$ by $A(x; \ell) = \sum_{n \geq \ell} a(n, \ell; p) \frac{x^n}{n!}$. Then (9) can be written as

$$\frac{d}{dx} A(x; \ell) = p(\ell - 1 + q^2) A(x; \ell - 1) + (\ell - 1 + q) \sum_{j \geq \ell} A(x; j), \quad \ell \geq 1. \quad (10)$$

Note that (10) when $q = 1$ gives

$$\frac{d}{dx} A(x; \ell) = p\ell A(x; \ell - 1) + \ell \sum_{j \geq \ell} A(x; j),$$

with $A(x; 0) = 1$. This leads to the formula

$$A(x; \ell) |_{q=1} = \left(\frac{W(-e^{(p-1)^2 x - 1}) + p}{p - 1} \right)^\ell,$$

where $W(t)$ is the Lambert's function defined by $W(x) = e^{-W(x)}$.

One may also find $A(x; \ell)$ for arbitrary p and q by extending the prior arguments which we now describe. Let $\mathbf{v}(x)$ be the column vector $(A(x; 0), A(x; 1), \dots)^T$ and define $\mathbf{A}$ to be the infinite matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ pq^2 & q & q & q & \cdots \\ 0 & p(1 + q^2) & 1 + q & 1 + q & \cdots \\ 0 & 0 & p(2 + q^2) & 2 + q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Note that $\frac{d}{dx}\mathbf{v}(x) = \mathbf{A}\mathbf{v}(x)$. By using techniques similar to the proof of Proposition 3.1 above, one can show that the generating functions

$$A(x, y) = \sum_{n \geq 0} \sum_{\ell=0}^n a(n, \ell; p) x^n y^\ell \text{ and } E(x, y) = \sum_{n \geq 0} \sum_{\ell=0}^n a(n, \ell; p) \frac{x^n y^\ell}{n!}$$

are given by

$$A(x, y) = (1, y, y^2, \dots)(1 - \mathbf{A}x)^{-1}e_1^T$$

and

$$E(x) = (1, y, y^2, \dots)e^{\mathbf{A}x}e_1^T,$$

where $e_1 = (1, 0, 0, \dots)$.

Let

$$\mathbf{A}' = 1 - x\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -xpq^2 & 1-xq & -xq & -xq & \cdots \\ 0 & -xp(1+q^2) & 1-x(1+q) & -x(1+q) & \cdots \\ 0 & 0 & -xp(2+q^2) & 1-x(2+q) & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and let $\mathbf{A}'_n$ and d_n be defined in a comparable manner as before.

LEMMA 3.5 *We have*

$$d_n = (1 + x(p-1)(n-2) + xq(pq-1))d_{n-1} - xp(n-2+q^2)d_{n-2}, \quad n \geq 2, \quad (11)$$

with $d_0 = d_1 = 1$. Moreover,

$$\begin{aligned} & \sum_{n \geq 1} \frac{d_n}{p^{n-1} \prod_{i=1}^{n-1} (i+q^2)} y^n \\ &= pq^2 y^{1-q^2} (xy(1-p) + p)^{\frac{q^2-q}{1-p} + \frac{1}{x(1-p)^2}} e^{-\frac{y}{1-p}} \int_0^y t^{q^2-1} (1-xt)(xt(1-p) + p)^{-1 - \frac{q(q-1)}{1-p} - \frac{1}{x(1-p)^2}} e^{\frac{t}{1-p}} dt. \end{aligned}$$

Proof. Expanding the determinant repeatedly along the last row implies

$$d_n = d_{n-1} - \sum_{j=1}^{n-1} p^{n-1-j} x^{n-j} d_j (j-1+q) \prod_{i=j}^{n-2} (i+q^2),$$

from which (11) follows. Now let $d_n = p^{n-1} e_n \prod_{i=1}^{n-1} (i+q^2)$. Then

$$pe_n(n-1+q^2) = e_{n-1} - \sum_{j=1}^{n-1} x^{n-j} e_j (j-1+q), \quad n \geq 2,$$

with $e_1 = 1$. If $E(y) = \sum_{n \geq 1} e_n y^n$, then

$$py \frac{d}{dy} (E(y) - y) + p(q^2 - 1)(E(y) - y) = yE(y) - \frac{(q-1)xy}{1-xy} E(y) - \frac{xy^2}{1-xy} \frac{d}{dy} E(y).$$

The solution of this differential equation with $E(0) = 0$ and $\frac{d}{dy}E(y)|_{y=0} = pq$ is given by

$$E(y) = pq^2 y^{1-q^2} (xy(1-p) + p)^{\frac{q^2-q}{1-p} + \frac{1}{x(1-p)^2}} e^{-\frac{y}{1-p}} \\ \times \int_0^y t^{q^2-1} (1-xt)(xt(1-p) + p)^{-1-\frac{q(q-1)}{1-p} - \frac{1}{x(1-p)^2}} e^{\frac{t}{1-p}} dt,$$

as required. □

Proceeding similarly as in the proof of Theorem 3.4 above, we obtain the following result.

THEOREM 3.6 *The generating function $A(x, y) = \sum_{n \geq 0} \sum_{\ell=0}^n a(n, \ell; p) x^n y^\ell$ is given by*

$$A(x, y) = \sum_{i \geq 1} \left(\frac{(pxy)^{i-1} \prod_{s=0}^{i-2} (s + q^2)}{d_i} \left(1 - pd_i(d_i - d_{i-1}) \sum_{k \geq i+1} \frac{\prod_{s=i-1}^{k-2} (s + q^2)}{d_{k-1} d_k} (px)^{k-i} \right) \right),$$

where d_n is determined by (11).

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