Enumeration of S-Motzkin paths from left to right and from right to left: a kernel method approach

HELMUT PRODINGER

Department of Mathematics
University of Stellenbosch
7602 Stellenbosch, South Africa
email: hproding@sun.ac.za

(Received: October 23, 2019, and in revised form September 30, 2020.)

Abstract. The area of S-Motzkin paths (bijective to ternary trees) is calculated using the kernel method by enumerating these (partial) paths with fixed end-point resp. starting point.

Mathematics Subject Classification(2010). 05A15.

Keywords: ternary tree, S-Motzkin path, kernel method, generalised binomial series.

1 Introduction

The area of a lattice path $\mathcal{A}$ is defined to be $\sum_{i \geq 0} h_i(\mathcal{A})$ where $h_i(\mathcal{A})$ is the height of the path $\mathcal{A}$ at x -coordinate i . This parameter has been studied in various types of paths [8, 9, 1]. In this paper, the area will be available as a corollary of the enumeration of partial (incomplete) families of lattice paths that are bijective to ternary trees, and thus enumerated by the numbers $\frac{1}{2n+1} \binom{3n}{n}$.

We study such a family that was recently introduced by [11], and find explicit formulae for them ending after n steps at level k , both, from left to right (starting at the origin) and also from right to left (starting at the end and going backwards). The latter instance is the more challenging one.

S-Motzkin paths can be transformed into other more traditional ternary objects, like ternary trees and ternary paths. No use of this will, however, made here, to keep the discussion self-contained. This is also beneficial from a pedagogic point of view, since it shows how to deal with a system of two equations and the kernel method, in the presence of cubic equations.

The enumeration in this paper involves a lattice path called an S-Motzkin path. This is a subclass of Motzkin paths introduced by the authors in a previous publication [11]. For completeness, we provide the definition again.

DEFINITION 1.1 An *S-Motzkin path* of length $3n$ is

- a Motzkin path
- it consists of n up-steps $(1, 1)$, n down-steps $(1, -1)$, n level-steps $(1, 0)$
- ignoring the down-steps, the level-steps and the up-steps alternate, starting with a level-step.

We apply the kernel method to S-Motzkin paths as well as reverse S-Motzkin paths to obtain the area of S-Motzkin paths. As a bonus, we obtain the exact enumeration of partial S-Motzkin paths. It is particularly worthwhile to obtain results for this relatively new class of lattice paths, since, as seen in [7], this family appeared in a surprisingly unrelated context. It is perhaps of independent interest to note that S-Motzkin paths were introduced to solve a problem from a student olympiad about frog hops [10].

A state-of-the-art survey about lattice path enumeration is [6]; it does not include S-Motzkin paths, as they are new.

Our main findings are the enumerations of four classes of paths (defined later): (1), (2), (3), (4), and the area (5). Informally, these are the enumeration of (partial) S-Motzkin paths, ending on a prescribed level, both, when considering them from left-to-right and from right-to-left. The area of an S-Motzkin is the sum of all its ordinates. The sum of this is computed, when summing over all S-Motzkin paths of the same length $3n$.

2 Preliminary computations

The following computation appears frequently in this paper, and we want to do it only once: Here, $x = t(1 - t)^2$, and in all applications we will have $z^3 = x$.

$$\begin{aligned} [x^m] \frac{1}{(1-t)^j} &= \frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} \frac{1}{(1-t)^j} \\ &= \frac{1}{2\pi i} \oint \frac{dt(1-3t)(1-t)}{t^{m+1}(1-t)^{2m+2}} \frac{1}{(1-t)^j} \\ &= [t^m] \frac{1-3t}{(1-t)^{2m+j+1}} = \binom{3m+j}{m} - 3 \binom{3m+j-1}{m-1}. \end{aligned}$$

The binomial series notation as given in [5]

$$\mathcal{B}_t(x)^r = \sum_{k \geq 0} \binom{tk+r}{k} \frac{r}{tk+r} x^k$$

can often be used to express certain quantities that appear in this paper. This is elegant, but the notation using the variable t (as in $x = t(1-t)^2$) seems to be more efficient. Note that the coefficient of x^n in $\mathcal{B}_3(x)^1$ is $\frac{1}{3n+1} \binom{3n+1}{n} = \frac{1}{2n+1} \binom{3n}{n}$, which is the number of ternary trees of size n .

The generating function for ternary trees and also S-Motzkin paths and ternary paths can be rewritten as $\mathcal{B}_3(x)$ using this notation. We will frequently use the substitution $x = t(1-t)^2$.

From the Lagrange inversion formula [2, Theorem A.2] we find

$$[x^n] t^k = \frac{k}{n} [w^{n-k}] \frac{1}{(1-w)^{2n}} = \frac{k}{n} \binom{3n-k-1}{n-k} \quad \Rightarrow \quad t^k = \sum_{n \geq k} \binom{3n-k-1}{n-k} \frac{k}{n} x^n.$$

Using this we can compute an expansion that is useful in the context of ternary paths and variants:

$$\sqrt{4t-3t^2} = 2t^{1/2} \sqrt{1-\frac{3}{4}t} = 2t^{1/2} \sum_{k \geq 0} (-1)^k \binom{3}{4}^k \left(\frac{1}{2}\right)^k t^k$$

$$\begin{aligned}
&= 2t^{1/2} \sum_{k \geq 0} (-1)^k \left(\frac{3}{4}\right)^k \binom{\frac{1}{2}}{k} \sum_{n \geq k} \binom{3n-k-1}{n-k} \frac{k}{n} x^n \\
&= 2t^{1/2} \sum_{n \geq 0} x^n \sum_{1 \leq k \leq n} (-1)^k \left(\frac{3}{4}\right)^k \binom{\frac{1}{2}}{k} \binom{3n-k-1}{n-k} \frac{k}{n} \\
&= -2t^{1/2} \sum_{n \geq 0} \binom{3n-\frac{3}{2}}{2n} \frac{1}{2n-1} x^n.
\end{aligned}$$

The simplification of the inner sum was done by a computer.

This is one way to switch between expressions in the variable t and expressions in terms of the binomial series notation.

3 The enumeration of partial S-Motzkin paths

3.1 S-Motzkin paths

A partial S-Motzkin path is a Motzkin path that can be continued to be an S-Motzkin path, or we might say the first part (of length m , say) of an existing S-Motzkin path. Since S-Motzkin paths are not symmetric w. r. t. left vs. right, we will later also consider this concept from right to left (which turns out to be more difficult).

Let $a_{n,k}$ denote the number of partial S-Motzkin path of length n which ends at height k and the last step of the path from the step set $\{(1,0), (1,1)\}$ is a $(1,1)$ step. Similarly, let $b_{n,k}$ denote the number of partial S-Motzkin path of length n which ends at height k and the last step of the path from the step set $\{(1,0), (1,1)\}$ is a $(1,0)$ step. It is convenient to set $a_{0,0} = 1$ and $b_{0,0} = 0$.

The path represented by $a_{n,k}$	The path represented by $b_{n,k}$

It is easily seen that the following recurrence relations hold:

$$\begin{aligned}
a_{n,k} &= b_{n-1,k-1} + a_{n-1,k+1}, \\
b_{n,k} &= a_{n-1,k} + b_{n-1,k+1}.
\end{aligned}$$

These recurrence relations are valid for $n \geq 1$ and $k \geq 0$; the quantity $b_{n-1,-1}$ must be interpreted as 0. We let

$$A(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} a_{n,k} z^n u^k \quad \text{and} \quad B(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} b_{n,k} z^n u^k$$

and sum over n and k to obtain the following system of equations

$$\begin{aligned} A(z, u) - 1 &= zuB(z, u) + \frac{z}{u}A(z, u) - \frac{z}{u}A(z, 0), \\ B(z, u) &= zA(z, u) + \frac{z}{u}B(z, u) - \frac{z}{u}B(z, 0). \end{aligned}$$

Solving the system of equations for $A(z, u)$ and $B(z, u)$ gives

$$\begin{aligned} A(z, u) &= \frac{-u^2 + zuA(z, 0) + zu - z^2A(z, 0) + B(z, 0)z^2u^2}{z^2u^3 - u^2 + 2zu - z^2}, \\ B(z, u) &= \frac{z(-u^2 + uB(z, 0) + zuA(z, 0) - zB(z, 0))}{z^2u^3 - u^2 + 2zu - z^2}. \end{aligned}$$

The polynomial in the denominator, $z^2u^3 - u^2 + 2zu - z^2$, is of interest to us. Using the substitution $u = zw$ along with $z^3 = t(1 - t)^2$, we obtain

$$(t^2w^2 - 2tw^2 + tw + w^2 - 2w + 1)(tw - 1) = 0.$$

Therefore the three roots (expressed again in the variable u) are given by

$$v_1 = \frac{z}{t}, \quad v_2 = -z \frac{t - 2 + \sqrt{4t - 3t^2}}{2(1 - t)^2}, \quad v_3 = -z \frac{t - 2 - \sqrt{4t - 3t^2}}{2(1 - t)^2}.$$

Alternatively, the three roots can be written using $\mathcal{B}_3(z^3)$ and $\mathcal{B}_{3/2}(\pm z^{3/2})$, but this will not be used.

The roots can also be expressed as series (Puiseux series, to be exact)

$$\begin{aligned} v_1 &= z^{-2} - 2 \sum_{n \geq 0} \frac{(3n)!}{(2n)!(n+1)!} z^{3n+1}, \\ v_2 &= -6 \sum_{n \geq 0} \frac{(6n+1)!(n+1)!}{(3n)!(2n+3)!(2n)!2^{4n}} z^{3n+5/2} + \sum_{n \geq 0} \frac{(3n)!}{(2n)!(n+1)!} z^{3n+1}, \\ v_3 &= 6 \sum_{n \geq 0} \frac{(6n+1)!(n+1)!}{(3n)!(2n+3)!(2n)!2^{4n}} z^{3n+5/2} + \sum_{n \geq 0} \frac{(3n)!}{(2n)!(n+1)!} z^{3n+1}, \end{aligned}$$

and it can be shown that these series are indeed the roots by converting them to hypergeometric functions and using Clausen's identity [12], but we mention this just for interest and will not use it further. Gessel and Xin [3] used such an approach, but it is a tour de force and fortunately, here, we can avoid to go that route.

Note that

$$v_2 + v_3 = -\frac{z(t-2)}{(t-1)^2} \quad \text{and} \quad v_2v_3 = \frac{z^2}{(t-1)^2}.$$

We know that $A(z, u)$ and $B(z, u)$ have power series expansions around $(0, 0)$, so the factors $(u - v_2)$ and $(u - v_3)$ in the denominator must also be factors in the numerator. Hence we can find $A(z, 0)$ and $B(z, 0)$ by solving the system

$$\begin{aligned} 0 &= -v_2^2 + zv_2A(z, 0) + zv_2 - z^2A(z, 0) + B(z, 0)z^2v_2^2, \\ 0 &= -v_3^2 + v_3B(z, 0) + zv_3A(z, 0) - zB(z, 0), \end{aligned}$$

to obtain

$$A(z, 0) = -\frac{v_2 v_3}{z^2} + \frac{v_2}{z} + \frac{v_3}{z} \quad \text{and} \quad B(z, 0) = \frac{v_2 v_3}{z}.$$

Substituting these back into the original equations yields

$$A(z, u) = \frac{u^2 v_2 v_3 z^2 - u v_2 v_3 - u^2 z + u v_2 z + u v_3 z + v_2 v_3 z + u z^2 - v_2 z^2 - v_3 z^2}{(u - v_1)(u - v_2)(u - v_3)z^3},$$

$$B(z, u) = -\frac{1}{(u - v_1)z}.$$

Since we know that $(u - v_2)$ and $(u - v_3)$ are factors of the numerator of $A(z, u)$ we can simplify $A(z, u)$ by dividing these two factors out (we consistently use the variable t for that). After simplification,

$$A(z, u) = \frac{1}{1-t} \frac{1}{1-\frac{tu}{z}} \quad \text{and} \quad B(z, u) = \frac{t}{z^2} \frac{1}{1-\frac{tu}{z}}.$$

Extraction of coefficients is now easy:

$$[u^k]A(z, u) = \frac{1}{1-t} \frac{t^k}{z^k}, \quad [u^k]B(z, u) = \frac{t^{k+1}}{z^{k+2}}.$$

Furthermore

$$[z^n u^k]A(z, u) = [z^{n+k}]t^k$$

These coefficients are 0 unless $n + k = 3m$ for some $m \in \mathbb{N}$. Thus we will compute the coefficient of z^{3m-k} in $[u^k]A(z, u)$, and we write $x = z^3$ for convenience, as before.

$$\begin{aligned} [z^{3m-k} u^k]A(z, u) &= [z^{3m}] \frac{1}{1-t} t^k = [x^m] \frac{1}{1-t} t^k \\ &= [t^{m-k}] \frac{1-3t}{(1-t)^{2m+2}} \\ &= \binom{3m-k+1}{m-k} - 3 \binom{3m-k}{m-k-1}. \end{aligned} \tag{1}$$

Likewise,

$$[z^n u^k]B(z, u) = [z^{n+k+2}]t^{k+1}.$$

Since this only makes sense if $n + k + 2 = 3m$ for some $m \in \mathbb{N}$. Thus we read off the coefficient of z^{3m-k-2} :

$$\begin{aligned} [z^{3m-k-2} u^k]B(z, u) &= [z^{3m}]t^{k+1} = [x^m]t^{k+1} = [t^{m-k-1}] \frac{1-3t}{(1-t)^{2m+2}} \\ &= \binom{3m-k}{m-k-1} - 3 \binom{3m-k-1}{m-k-2}. \end{aligned} \tag{2}$$

3.2 Reverse S-Motzkin paths

A *reverse S-Motzkin* path is an S-Motzkin path read from right to left. Alternatively, we might say that a reverse S-Motzkin path is a Motzkin path of length $3n$ with n of each type of step such that the first step from the step set $\{(1, 0), (1, -1)\}$ is a $(1, -1)$ step. Furthermore, $(1, 0)$ and $(1, -1)$ steps alternate.

Let $c_{n,k}$ denote a partial reverse S-Motzkin path of length n which ends at height k and the last step of the path in the step set $\{(1, 0), (1, -1)\}$ is a $(1, 0)$ step. Similarly, let $d_{n,k}$ denote a partial reverse S-Motzkin path of length n which ends at height k and the last step of the path in the step set $\{(1, 0), (1, -1)\}$ is a $(1, -1)$ step.

The path represented by $c_{n,k}$	The path represented by $d_{n,k}$

possibly empty	possibly empty

It is easily seen that the following recurrence relations hold:

$$\begin{aligned} c_{n,k} &= c_{n-1,k-1} + d_{n-1,k}, \\ d_{n,k} &= d_{n-1,k-1} + c_{n-1,k+1}. \end{aligned}$$

These recurrences hold for $n \geq 1$ and $k \geq 0$; $c_{n-1,-1}$ and $d_{n-1,-1}$ must be interpreted as zero, and the initial values are $c_{0,0} = 1$ and $d_{0,0} = 0$.

Let

$$C(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} c_{n,k} z^n u^k \quad \text{and} \quad D(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} d_{n,k} z^n u^k,$$

and sum the recursion over n and k . This results in

$$\begin{aligned} C(z, u) - 1 &= zuC(z, u) + zD(z, u), \\ D(z, u) &= zuD(z, u) + \frac{z}{u}C(z, u) - \frac{z}{u}C(z, 0). \end{aligned}$$

Solving this system gives

$$C(z, u) = \frac{u - u^2 z - C(z, 0) z^2}{z^2 u^3 - 2zu^2 + u - z^2} \quad \text{and} \quad D(z, u) = \frac{C(z, 0) u z^2 - C(z, 0) z + z}{z^2 u^3 - 2zu^2 + u - z^2}.$$

Note that the denominator is given by

$$z^2 u^3 - 2zu^2 + u - z^2$$

whereas in the previous section (§3.1) the denominator was given by

$$z^2 u^3 - u^2 + 2zu - z^2 = z^2(u - v_1)(u - v_2)(u - v_3);$$

the explicit forms of the roots are repeated for convenience:

$$v_1 = \frac{z}{t}, \quad v_2 = -z \frac{t-2+\sqrt{4t-3t^2}}{2(1-t)^2}, \quad v_3 = -z \frac{t-2-\sqrt{4t-3t^2}}{2(1-t)^2}.$$

The equation $z^2u^3 - u^2 + 2zu - z^2$ along with the substitution $u = 1/u$ and multiplication by $-u^3$ gives the denominator in the current case:

$$-u^3 \left(z^2 \left(\frac{1}{u} \right)^3 - \left(\frac{1}{u} \right)^2 + 2z \left(\frac{1}{u} \right) - z^2 \right) = -z^2 + u - 2zu^2 + z^2u^3.$$

Therefore the roots of the polynomial $z^2u^3 - 2zu^2 + u - z^2$ are given by v_1^{-1} , v_2^{-1} , and v_3^{-1} , hence

$$z^2u^3 - 2zu^2 + u - z^2 = z^2 \left(u - \frac{1}{v_1} \right) \left(u - \frac{1}{v_2} \right) \left(u - \frac{1}{v_3} \right).$$

Note that

$$\frac{1}{v_1} = \frac{t}{z}, \quad \frac{1}{v_2} = \frac{-t+2+\sqrt{4t-3t^2}}{2z}, \quad \frac{1}{v_3} = \frac{-t+2-\sqrt{4t-3t^2}}{2z}.$$

In the current right-to-left enumeration, $u - v_1^{-1}$ is the factor in the denominator that is also a factor of the numerator. Plugging in $u = v_1^{-1}$ into the numerator of $D(z, u)$ (the numerator of $C(z, u)$ could also be used) gives

$$C(z, 0) = \frac{v_1}{v_1 - z}.$$

Using this value for $C(z, 0)$, it follows that

$$\frac{C(z, 0)uz^2 - C(z, 0)z + z}{u - v_1^{-1}} = \frac{z^2v_1}{v_1 - z} = z^2C(z, 0),$$

and thus

$$D(z, u) = \frac{C(z, 0)}{\left(u - \frac{1}{v_2}\right)\left(u - \frac{1}{v_3}\right)}.$$

We can further write

$$D(z, u) = \frac{1}{(1-t)\left(u - \frac{1}{v_2}\right)\left(u - \frac{1}{v_3}\right)},$$

and representing this as a partial fraction gives

$$D(z, u) = \frac{t}{z(1-t)} \frac{v_3}{(1-uv_3)(v_3-v_2)} - \frac{t}{z(1-t)} \frac{v_2}{(1-uv_2)(v_3-v_2)}.$$

Therefore we can find the coefficients of $D(z, u)$:

$$[u^k]D(z, u) = \frac{t}{z(1-t)} \frac{v_3^{k+1} - v_2^{k+1}}{v_3 - v_2}.$$

The identity [4, eq. (22)] will be useful in calculating $[z^n u^k]D(z, u)$ and $[z^n u^k]C(z, u)$, so note that

$$\begin{aligned} \frac{v_3^{k+1} - v_2^{k+1}}{v_3 - v_2} &= \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} (v_2 + v_3)^{k-2i} (v_2 v_3)^i \\ &= \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{i+k} \binom{k-i}{i} \frac{z^{k-2i} (t-2)^{k-2i}}{(t-1)^{2k-4i}} \frac{z^{2i}}{(1-t)^{2i}} \\ &= z^k \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{i+k} \binom{k-i}{i} \frac{(t-2)^{k-2i}}{(t-1)^{2k-2i}}. \end{aligned}$$

Further,

$$[u^k]D(z, u) = tz^{k-1} \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{i+k-1} \binom{k-i}{i} \frac{(t-2)^{k-2i}}{(t-1)^{2k-2i+1}}.$$

Now set $n = 3N + s - 1$, $k = 3K + s$ for $s \in \{0, 1, 2\}$. Then

$$\begin{aligned} [z^n u^k]D(z, u) &= [z^{3N+s-1} u^{3K+s}]D(z, u) \\ &= [x^{N-K}]t \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{i+k-1} \binom{k-i}{i} \frac{(t-2)^{k-2i}}{(t-1)^{2k-2i+1}} \\ &= [x^{N-K}]t \sum_{i=0}^{\lfloor k/2 \rfloor} \sum_{j=0}^{k-2i} (-1)^i \binom{k-i}{i} \binom{k-2i}{j} \frac{1}{(1-t)^{2k-2i-j+1}} \\ &= \sum_{i=0}^{\lfloor k/2 \rfloor} \sum_{j=0}^{k-2i} (-1)^i \binom{k-i}{i} \binom{k-2i}{j} \\ &\quad \times \left[\binom{n+k-2i-j+1}{(n-k+1)/3-1} - 3 \binom{n+k-2i-j}{(n-k+1)/3-2} \right]. \end{aligned} \tag{3}$$

Similarly, for $C(z, u)$ we get

$$\frac{u - u^2 z - C(z, 0)z^2}{u - v_1^{-1}} = \frac{-(uv_1 z - v_1 + z)}{v_1},$$

and thus

$$C(z, u) = \frac{1 - t - uz}{z^2 \left(u - \frac{1}{v_2}\right) \left(u - \frac{1}{v_3}\right)} = \frac{(1 - t - uz)}{(1 - t)^2 (1 - v_2 u) (1 - v_3 u)}.$$

Rewriting $C(z, u)$ using partial fractions gives

$$C(z, u) = \left[\frac{1}{1-t} - \frac{uz}{(1-t)^2} \right] \frac{1}{v_3 - v_2} \left[\frac{v_3}{1 - uv_3} - \frac{v_2}{1 - uv_2} \right],$$

which allows for coefficient extraction:

$$\begin{aligned}
[u^k]C(z, u) &= \frac{1}{1-t} \frac{v_3^{k+1} - v_2^{k+1}}{v_3 - v_2} - \frac{z}{(1-t)^2} \frac{v_3^k - v_2^k}{v_3 - v_2} \\
&= z^k \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{i+k} \binom{k-i}{i} \frac{(t-2)^{k-2i}}{(t-1)^{2k-2i+1}} \\
&\quad - z^k \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^{i+k-1} \binom{k-1-i}{i} \frac{(t-2)^{k-1-2i}}{(t-1)^{2k-2i}} \\
&= z^k \sum_{i=0}^{\lfloor k/2 \rfloor} \sum_{j=0}^{k-2i} (-1)^{i-1} \binom{k-i}{i} \binom{k-2i}{j} \frac{1}{(1-t)^{2k-2i-j+1}} \\
&\quad - z^k \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \sum_{j=0}^{k-1-i} (-1)^{i-1} \binom{k-i-1}{i} \binom{k-1-2i}{j} \frac{1}{(1-t)^{2k-2i-j}}.
\end{aligned}$$

Furthermore, with $n = 3N + s$, $k = 3K + s$,

$$\begin{aligned}
[z^n u^k]C(z, u) &= [x^{N-K}] \sum_{i=0}^{\lfloor k/2 \rfloor} \sum_{j=0}^{k-2i} (-1)^i \binom{k-i}{i} \binom{k-2i}{j} \frac{1}{(1-t)^{2k-2i-j+1}} \\
&\quad - [x^{N-K}] \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \sum_{j=0}^{k-1-i} (-1)^i \binom{k-1-i}{i} \binom{k-1-2i}{j} \frac{1}{(1-t)^{2k-2i-j}} \\
&= \sum_{i=0}^{\lfloor k/2 \rfloor} \sum_{j=0}^{k-2i} (-1)^i \binom{k-i}{i} \binom{k-2i}{j} \\
&\quad \times \left[\binom{n+k-2i-j+1}{(n-k)/3} - 3 \binom{n+k-2i-j}{(n-k)/3-1} \right] \\
&\quad - \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \sum_{j=0}^{k-1-i} (-1)^i \binom{k-1-i}{i} \binom{k-1-2i}{j} \\
&\quad \times \left[\binom{n+k-2i-j}{(n-k)/3} - 3 \binom{n+k-2i-j-1}{(n-k)/3-1} \right]. \tag{4}
\end{aligned}$$

3.3 The area of S-Motzkin paths

Consider an arbitrary S-Motzkin path of length $3n$. This path can be decomposed at any height k that the path attains into either an (1) $a_{\ell,k}$ path followed by a $c_{3n-\ell,k}$ path or (2) a $b_{\ell,k}$ path followed by a $d_{3n-\ell,k}$ path.

We see that the generating function for the total area (all the areas, summed over all S-Motzkin paths of the same length) is given by

$$\sum_{k \geq 0} k ([u^k]A(z, u) \cdot [u^k]C(z, u) + [u^k]B(z, u) \cdot [u^k]D(z, u)).$$

Fortunately, we have explicit forms for $[u^k]A(z, u)$, $[u^k]B(z, u)$, $[u^k]C(z, u)$, $[u^k]D(z, u)$, and we compute the sum, using the forms of v_2 and v_3 involving t (using a computer, of course):

Using Cauchy's integral formula we find that the area of S-Motzkin paths of length $3n$ is given by

4 Acknowledgments

The insightful comments of one reviewer are also gratefully acknowledged.

References

- [1] C. BANDERIER AND B. GITTENBERGER, *Analytic combinatorics of lattice paths: enumeration and asymptotics for the area*, Discrete Math. Theor. Comput. Sci. Proc., AG (2006) 345–355.

- [2] P. FLAJOLET AND R. SEDGEWICK, *Analytic Combinatorics*, Cambridge University Press, 2009.
- [3] I. M. GESSEL AND G. XIN, *The generating function of ternary trees and continued fractions*, Electron. J. Combin., 13 (2006) #R53, 48 pp.
- [4] H. GOULD, *The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences*, Fibonacci Quart., 37 (1999) 135–140.
- [5] R. L. GRAHAM, D. E. KNUTH AND O. PATASHNIK, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1999.
- [6] C. KRATTENTHALER, *Lattice path enumeration*, in: *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pp. 589–678. CRC Press, Boca Raton, FL, 2015.
- [7] H. MULARCZYK, *Lattice paths and pattern-avoiding uniquely sorted permutations*, at <https://arxiv.org/abs/1908.04025>.
- [8] E. PERGOLA, *Two bijections for the area of Dyck paths*, Discrete Math., 241 (2001) 435–447.
- [9] E. PERGOLA, R. PINZANI, S. RINALDI AND R. A. SULANKE, *A bijective approach to the area of generalized Motzkin paths*, Adv. in Appl. Math., 28 (2002) 580–591.
- [10] F. PETROV AND A. VERSHIK, *International mathematics competition: Day 2 problem 8*, at [/imc-math.ddns.net/pdf/imc2018-day2-questions.pdf](http://imc-math.ddns.net/pdf/imc2018-day2-questions.pdf).
- [11] H. PRODINGER, S. J. SELKIRK AND S. WAGNER, *On two subclasses of Motzkin paths and their relation to ternary trees*, in: V. Pillwein and C. Schneider (Eds.), *Algorithmic Combinatorics – Enumerative Combinatorics, Special Functions and Computer Algebra in honour of Peter Paule on his 60th birthday*, Springer, 2020, pp. 297–316.
- [12] R. VIDUNAS, *A generalization of Clausen’s identity*, Ramanujan J., 26 (2011) 133–146.