Applied Mathematics and Nonlinear Sciences

<https://www.sciendo.com>

A Modified Iterative Method for Solving Nonlinear Functional Equation

Arya Mohit[†], Ujlayan Amit

Department of Mathematics, School of Vocational Studies and Applied Sciences, Gautam Buddha University,
Greater Noida, UP, 201310 India

Submission Info

Communicated by Jagdev Singh

Received October 18th 2019

Accepted April 14th 2020

Available online November 20th 2020

Abstract

An iterative method (AIM) is one of the numerical method, which is easy to apply and very time convenient for solving nonlinear differential equations. However, if we want to work in a large interval, sometimes it may be difficult to apply AIM. Therefore, a multistage AIM named Multistage Modified Iterative Method (MMIM) is introduced in this article to work in a large computational interval. The applicability of MMIM for increasing the solution domain of the given problems is construed in this article. Some problems are solved numerically using MMIM, which provides a better result in the extended interval as compared to AIM. Comparison tables and some graphs are included to demonstrate the results.

Keywords: an iterative method, multistage modified iterative method, nonlinear integral equations, ordinary differential equations
AMS 2010 codes: 34K05

1 Introduction

Nonlinear differential and integral equations have been a key in the various fields of Engineering, Physics, Life Sciences, and Mathematics. There is no general method to solve the nonlinear differential equations. Also, it is not possible to convert a nonlinear differential equation into an appropriate form using variable transformations and sometimes not possible to get the close form analytical solution of the nonlinear differential equations. George Adomian developed a numerical method from 1970 to 1990. The important feature of the method is the deputation of ‘Adomian polynomials’, which acknowledge the approximation of nonlinear part without using perturbation and linearisation techniques. In articles [1–11], a number of renowned nonlinear problems have been solved by some other efficient numerical methods.

However, numerical and analytical methods are not limited in the fields of ODE(s) and PDE(s), but may have a crucial role in the field of nuclear magnetic resonance, astrophysics, cold plasma, porous media, and physics through fractional differential equations [12–23]. In spite of these abovementioned methods, AIM [24] is getting

[†]Corresponding author.

Email address: mohitarya818@gmail.com

a lot of attention in getting the solution for nonlinear differential equations. However, this method works for a small interval only. But to find a numerical solution for a large interval using AIM, we have to use large number of components. So, in most of the problems, after some iterations, either mathematical software declines to give the solution for a given problem or the iterations become so much complicated to move to successive steps of the solution. In Multistage Adomian Method [25], a comparatively large interval is split into a number of small subintervals and these small subintervals are used to find the solution.

The objective of this article is to solve a nonlinear problem for a large interval using AIM. In the present work, we introduce MMIM, developed by using AIM in each of subinterval separately. Therefore, MMIM is helpful to find the solution in a comparatively large interval in comparison to AIM.

The procedure given in this article can be extended to solve the various problems, like Riccati differential equation, Abel's differential equation, integral equations, partial differential equation and fractional differential equations.

In Section 2, the outline of AIM is provided. In Section 3, two-step modified iterative method is discussed. In Section 4, an analysis of the two-step modified iterative method has been discussed. MMIM is explained in Section 5. An absolute error formula is discussed in Section 6. Finally, some examples are solved to establish the competency of MMIM, in Section 7.

2 An Iterative Method [24]

Consider a general functional equation:

$$u = \psi + \xi(u), \quad (1)$$

where $\xi(u)$ is a nonlinear operator from a Banach space $B \rightarrow B$ and called as known function or source term. To apply AIM, the solution of Eq. (1) has been assumed to be in convergent series form as

$$u = \sum_{n=0}^{\infty} u_n. \quad (2)$$

The nonlinear terms can be decomposed as

$$\begin{aligned} \xi(u) &= \xi\left(\sum_{i=0}^{\infty} u_i\right), \\ &= \xi(u_0) + \sum_{i=1}^{\infty} \left\{ \xi\left(\sum_{j=0}^i u_j\right) - \xi\left(\sum_{j=0}^{i-1} u_j\right) \right\}, \end{aligned} \quad (3)$$

and using Eqs (2) and (3), Eq. (1) can be written as

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n, \\ &= \psi + \xi(u_0) + \sum_{i=1}^{\infty} \left\{ \xi\left(\sum_{j=0}^i u_j\right) - \xi\left(\sum_{j=0}^{i-1} u_j\right) \right\}, \end{aligned} \quad (4)$$

therefore a recurrence relation is defined as follows:

$$\begin{cases} u_0 = \psi \\ u_1 = \xi(u_0) \\ u_{m+1} = \xi(u_0 + \dots + u_m) - \xi(u_0 + \dots + u_{m-1}) \\ m = 1, 2, \dots \end{cases} \quad (5)$$

This implies

$$u_1 + u_2 + \cdots + u_{m+1} = \xi(u_0 + u_1 + \cdots + u_m) \quad (6)$$

Therefore, using Eq. (6) in Eq. (1),

$$u = \psi + \sum_{n=1}^{\infty} u_n. \quad (7)$$

If ξ is contraction, i.e.,

$$\|\xi(x) - \xi(y)\| \leq K\|x - y\|, 0 < K < 1,$$

then using Eq. (5), we have

$$\begin{aligned} \|u_{m+1}\| &= \|\xi(u_0 + u_1 + \cdots + u_m) - \xi(u_0 + u_1 + \cdots + u_{m-1})\| \\ &\leq K\|u_m\| \leq K^{m+1}\|u_0\|, \text{ where } m = 0, 1, 2, \dots \end{aligned}$$

Therefore using Banach fixed point theorem [26] a solution for Eq. (1) is obtained.

3 Two-Step Modified Iterative Method (TSMIM)

Consider, the general functional Eq. (1). In TSMIM, source term ψ has been divided into two parts ψ_1 and ψ_2 (ψ_1 and ψ_2 may contain more than one term). Under this assumption, we set

$$\psi = \psi_1 + \psi_2 \quad (8)$$

and on applying a slight variation to the components u_0 and u_1 , a modified recursive scheme defined as follows,

$$\begin{cases} u_0 = \psi_1 \\ u_1 = \psi_2 + \xi(u_0) \\ u_{m+1} = \xi(u_0 + \cdots + u_m) - \xi(u_0 + \cdots + u_{m-1}) \\ m = 1, 2, 3, \dots \end{cases} \quad (9)$$

When we analyse Eqs (5) and (9), it is observed that zeroth component u_0 is defined by the function ψ_1 as a part of ψ , the remaining part ψ_2 is added to the definition of the component u_1 in Eq. (9). This change in u_0 may result as a lection in the computational work and accelerate the convergence of the solution. This slight variation in the definition of the components u_0 and u_1 may provide the solution in only two iterations.

It should be noted that the success of this method depends on proper choice of the part ψ_1 or ψ_2 , but it cannot be said that which part of ψ , either ψ_1 or ψ_2 , would be appropriate to take as a zeroth component.

4 Analysis of the Two-Step Modified Iterative Method

For the sake of convenience, let us consider Eq. (1)

In the study of TSMIM, the source term has been split into two parts ψ_1 and ψ_2 . Now, modified Eq. (1) can be written as

$$u = \psi_1 + \psi_2 + \xi(u). \quad (10)$$

Using the recursive scheme Eq. (9) and choosing u_0 such that $u_0 = \psi_1 = u$, we get

$$u_1 = \psi_2 + \xi(u_0), \quad (11)$$

using Eq. (10), Eq. (11) implies

$$u_1 = u - \psi_1 = 0 \quad (12)$$

i.e., using the TSMIM, the solution can be obtained in two iterations. However, the drawback of this method is that, the zeroth component should satisfy the given problem.

If zeroth component does not satisfy the given problem, then the method discussed in article [24] can be used.

5 Multistage Modified Iterative Method

In most of the problems, the computational interval $I = [a, J], J \in R$ cannot be always small. Now to accelerate the rate of convergence and to improve the accuracy of the solutions in a large interval using AIM, the entire interval I is divided into “ n ” subintervals. The main advantage of splitting the interval I into n subintervals (only a few components) gives a better approximation of the solution in the J_i^{th} subinterval, where

$$I = \sum_{i=1}^n J_i.$$

Let $[a, J]$ is the interval, in which, the given problem to be solved. Here, the interval $I = [a, J], J \in R$ is split into n subintervals, such that $t_i < t_{i+1}$ and for $i = 0, 1, 2, \dots, n-1$.

Therefore, to solve the problem of type Eq. (1), initial guess of the solution should be known in the interval and the solution can be considered as

$$\phi_i = \sum_{n=0}^{\infty} \phi_{in}, \quad (13)$$

where $\phi_1(t)$ is the approximate numerical solution for the first sub-interval $[a, t_1)$. This first subinterval will be used to obtain an initial condition at $t = t_1$, i.e., $y(t_1) = a_2$ for the solution $\phi_2(t)$ in the second subinterval $[t_1, t_2)$. Similarly the initial condition for the third interval at $t = t_2$ is $y(t_2) = a_3$ and get $\phi_3(t)$. This procedure continues over n subintervals and we obtain a numeric solution $\phi_i(t)$ over the i th subinterval $[t_{i-1}, t_i)$. Using this multistage method, the function $\phi_i(t)$ can be obtained throughout the entire interval for all i .

Note 1: Before applying this method, it should be noted that

- (i) The length of the subinterval should be small to get the higher accuracy.
- (ii) It is not necessary that the length of each subinterval to be equal, it can also be changed to achieve better results.

6 Absolute Error Formula

Since MMIM is a modification of AIM, and in each subinterval AIM will be used separately, therefore the absolute error formula for the solution ϕ_i in i th subinterval $[t_{i-1}, t_i]$ will be

$$\text{Absolute Error} = |u - \phi_i| = \left| u - \sum_{n=0}^{\infty} \phi_{in} \right|, \text{ for } i = 1, 2, 3, \dots$$

where n is the number of components taken for the numerical solution in that particular subinterval.

7 Illustrative Examples

In this section, some examples have been included for the verification of the MMIM. AIM and MMIM are applied separately to find the solution in a given interval. Further, solutions of the both methods are compared graphically and the comparative study is summarised in Tables 1 and 2, respectively.

Example 1. Consider the nonlinear integral equation

$$y = \frac{1}{5}x^5 + x^2 - \int_0^x y^2(t)dt, \quad (14)$$

such that exact solution is $y = x^2$ in $[0, 2]$.

Table 1 Comparative study of AIM and MMIM

t	Exact value	Absolute error (AIM)	Absolute error (MMIM)
0.4	0.16	3.49×10^{-9}	3.496×10^{-9}
0.8	0.64	5.833×10^{-5}	5.833×10^{-5}
1.2	1.44	1.761×10^{-2}	1.671×10^{-5}
1.6	2.56	0.989	5.644×10^{-4}
2.0	4.0	5.71	1.009×10^{-3}

Table 2 Comparative study of AIM and MMIM

t	Exact value	Absolute error (AIM)	Absolute error (MMIM)
0.41.4		2.878×10^{-3}	4.475×10^{-6}
0.81.8		0.147	1.260×10^{-5}
1.22.2		1.203	2.865×10^{-5}
1.62.6		3.393	5.664×10^{-5}
2.03.0		5.401	1.012×10^{-4}

Solution: Here, it can be observed that the solution of the given problem is already inbuilt in the given problem. Therefore, we can apply TSMIM.

Let the solution of Eq. (14) exists in a converging series form,

$$y = \sum_{i=1}^{\infty} y_i, \quad (15)$$

using Eq. (9), we can define the following recurrence relation:

$$\begin{aligned} y_0 &= x^2 \\ y_1 &= \frac{1}{5}x^5 - \int_0^x y_0^2(t)dt = 0 \\ y_2 &= y_3 = \dots = y_{k+1} = 0, \end{aligned}$$

where $k \geq 0$.

Hence, the required solution would be

$$y = y_0 + y_1 + y_2 + \dots = x^2.$$

It should be noted that, if we take the 0th component as the solution of the given problem, the next component becomes zero. But if we do not take 0th component as the solution of the given problem, we shall apply AIM.

Using AIM: Suppose the problem Eq. (14) has the series solution of the form Eq. (15). Therefore, the recurrence relation can be defined as,

$$\begin{aligned} y_0 &= \frac{1}{5}x^5, \\ y_1 &= x^2 - \int_0^x y_0^2(t)dt = x^2 - \frac{x^{11}}{275}, \\ y_2 &= - \int_0^x (y_0(t) + y_1(t))^2 dt + \int_0^x y_0^2 dt = -\frac{x^5}{5} - \frac{x^8}{20} + \dots, \\ y_3 &= - \int_0^x (y_0(t) + y_1(t) + y_2(t))^2 dt + \int_0^x (y_0(t) + y_1(t))^2 dt = \frac{x^8}{20} + \frac{7x^{11}}{550} + \dots \end{aligned}$$

Fig. 1 A comparison graph between AIM and exact solution.

⋮

The graph of the solution Eq. (15) at $i = 1, 2, 3, 4$ and the comparison with the exact solution $y = x^2$ are shown in Figure 1.

Figure 1 shows a comparison between exact solution ‘_’ and computed solution ‘_’.

It can be illustrated from Figure 1 that the computed solution coincide with the exact solution in the interval $[0, 1]$. The behaviour of the exact solution changes rapidly for $x \geq 1.5$, i.e., the approximated solution does not give a better result with respect to the exact solution in the interval $[0, 2]$. Therefore to increase the efficiency of the solution in the computational interval, one may use MMIM.

Using MMIM: As the given problem explain above, the solution of the problem Eq. (14) lies in the interval $[0, 2]$, but to evaluate the solution using AIM in the interval $[0, 2]$ may be comparatively difficult. Therefore, we can apply MMIM by dividing the given interval such that $P = \{0, 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2\}$. For the success of MMIM, AIM will be applied in these intervals separately.

First, we find solution in the interval $[0, 1]$ by taking sum of the five components, $y_1(x) = y_{10} + y_{11} + y_{12} + y_{13} + y_{14}$, where

$$\begin{aligned} y_{10} &= 0, \\ y_{12} &= \frac{1}{5}x^5 + x^2 - \int_0^x y_{10}^2(t)dt = x^2 + \frac{x^5}{5}, \\ y_{11} &= -\int_0^x (y_{10}(t) + y_{11}(t))^2 dt + \int_0^x y_{10}^2(t)dt = -\frac{x^5}{5} - \frac{x^8}{20} - \frac{x^{11}}{275} + \dots, \\ y_{13} &= -\int_0^x (y_{10}(t) + y_{11}(t) + y_{12}(t))^2 dt + \int_0^x (y_{10}(t) + y_{11}(t))^2 dt \\ &= \frac{x^8}{20} + \frac{7x^{11}}{550} + \frac{x^{14}}{1925} - \frac{x^{17}}{6800} - \frac{x^{20}}{55000} - \frac{x^{23}}{1739375} + \dots, \end{aligned}$$

and

$$y_{14} = -\int_0^x (y_{10}(t) + y_{11}(t) + y_{12}(t) + y_{13}(t))^2 dt + \int_0^x (y_{10}(t) + y_{11}(t) + y_{12}(t))^2 dt = -\frac{x^{11}}{110} - \frac{x^{14}}{1925} - \dots.$$

The graph of the solution $y_1(x) = y_{10} + y_{11} + y_{12} + y_{13} + y_{14}$ in the interval $[0, 1]$ is shown in Figure 2

The end point of the first interval $[0, 1]$ will work as the initial point for the second interval $[1, 1.1]$, such that

$$y_1(1) = 0.9986 \approx 1 = y_{20}.$$

Fig. 2 Graph of the solution using AIM for the interval $[0, 1)$.

Fig. 3 Graph of the solution using AIM for the interval $[1, 1.1)$.

Therefore, solution in the interval $[1, 1.1)$ is given by,

$$\begin{aligned} y_{20} &= 1, \\ y_{21} &= -x + x^2, \\ y_{22} &= -0.5435064935 + 0.35x + 0.8x^2 + \dots, \\ y_{23} &= -0.2295528059 + 0.5742110811 + \dots, \\ y_{24} &= -0.02507479237 + 0.06506311753x + \dots. \end{aligned}$$

The graph of the solution $y_2(x) = y_{20} + y_{21} + y_{22} + y_{23} + y_{24}$ for the interval is shown in Figure 3.

Take end point of the interval $[1, 1.1)$ as the initial point for the interval $[1.1, 1.2)$, such that

$$y_2(1.1) = 1.2099 \approx 1.21 = y_{30}.$$

Since, solution for the interval $[1.1, 1.2)$ will be given by,

$$\begin{aligned} y_{30} &= 1.21, \\ y_{31} &= 0.40051 - 1.461x + \dots, \\ y_{32} &= -0.6157294163 + 1.4641x + \dots, \end{aligned}$$

$$y_{33} = -0.5432305405 + 1.207498697x + \dots,$$

$$y_{34} = -0.1195253165 + 0.4357396811x + \dots.$$

The graph of the solution $y_3 = y_{30} + y_{31} + y_{32} + y_{33} + y_{34}$ for the interval $[1.1, 1.2)$ is shown in Figure 4.

Fig. 4 Graph of the solution using AIM for the interval $[1.1, 1.2)$.

Take end point of the interval $[1.1, 1.2)$, as the initial point for the interval $[1.2, 1.3)$, such that

$$y_3(1.2) = 1.4399 \approx 1.44 = y_{40}.$$

Then, the solution in the interval $[1.2, 1.3)$ will be given by,

$$y_{40} = 1.44,$$

$$y_{41} = 1.04832 + 2.0736x + \dots,$$

$$y_{42} = -0.34573339 + 2.0736x + \dots,$$

$$y_{43} = -1.005231724 + 1.256973647x + \dots,$$

$$y_{44} = -0.5589816161 + 2.296545961x + \dots.$$

Therefore, the graph of the solution $y_4 = y_{40} + y_{41} + y_{42} + y_{43} + y_{44}$ for the interval $[1.2, 1.3)$ is shown in Figure 5.

Fig. 5 Graph of the solution using AIM in the interval $[1.2, 1.3)$.

Proceeding in similar way up to interval $[1.9, 2]$, with length of the subinterval 0.1 and after combining or merging all subintervals, we get solution for the interval $[0, 2]$ in figure 6, which is quite similar to the graph of exact solution, therefore for more clarity we observe table 1. It is clear from table 1 that the solution using MMIM is more promising than AIM.

Fig. 6 Graph of the solution using MMIM for the interval $[0, 2]$.

Example 2. Consider the differential equation

$$y'(x) = -2x - x^2 + y^2(x), \quad y(0) = 1, \quad (16)$$

such that the exact solution is $y(x) = 1 + x$ in $[0, 2]$.

Solution: The solution of Eq. (16) can be written in an integral form as

$$y(x) = 1 - x^2 - \frac{x^3}{3} + \int_0^x y^2(t) dt \quad (17)$$

Here, we see that the solution of the problem Eq. (17) is not inbuilt under the given problem. Therefore, we use AIM.

Using AIM: Let the solution of Eq. (17) is

$$y = \sum_{i=1}^{\infty} y_i$$

and the recurrence relation can be defined as,

$$\begin{aligned} y_0 &= 1, \\ y_1 &= x - x^2 - \frac{x^3}{3}, \\ y_2 &= x^2 - \frac{x^3}{3} - \dots, \\ y_3 &= \frac{2x^3}{3} + \frac{x^4}{3} - \dots, \\ &\vdots \end{aligned}$$

The graph of the computed solution using AIM, $y = \sum_{i=0}^4 y_i$ and the exact solution are shown in Figure 7.

Fig. 7 Comparison graph of the solution for the interval $[0, 2]$ using AIM and exact solution.

Using MMIM: Here, we can see that the solution of the problem Eq. (16) given by AIM works only for the interval $[0, 0.3]$. If we want to find out the solution of the given problem for the interval $[0, 2]$, then we can apply MMIM to solve this problem. We shall apply this method by dividing the given interval $[0, 2]$ such that $P = \{0, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2\}$.

First, we find solution for the interval $[0, 0.3)$ by taking sum of the five components $y_1 = y_{10} + y_{11} + y_{12} + y_{13} + y_{14}$. Using Eq. (5), the graph of the solution in the interval $[0, 0.3)$ is shown in Figure 8.

Fig. 8 Graph of the solution using AIM for the interval $[0, 0.3)$.

The end point of the interval $[0, 0.3)$ will work as the initial point for second interval $[0.3, 0.4)$ such that

$$y_1(0.3) = 1.299 \approx 1.3 = y_{20}.$$

Solution for the interval $[0.3, 0.4)$ is given by,

$$y_{20} = 1.3,$$

$$y_{21} = -0.417 + 1.9x + \dots,$$

$$y_{22} = -0.3110044886 - 1.69x + \dots,$$

$$y_{23} = -0.02693861645 + 0.2508741836x - 0.6935121566x^2 + \dots,$$

$$y_{24} = 0.004469301741 - 0.05439096203x + 0.2284531252x^2 \dots.$$

The graph of the solution $y_2 = y_{20} + y_{21} + y_{22} + y_{23} + y_{24}$ for the interval $[0.3, 0.4)$ is shown in Figure 9.

Fig. 9 Graph of the solution using AIM in the interval $[0.3, 0.4]$.

Fig. 10 Graph of the solution using MMIM for the interval $[0, 2]$.

In the similar manner, the solution of the given problem Eq. (16) can be calculated up to interval $[1.9, 2]$ with length of the subinterval 0.1. Combining and merging all the solution graphs, the graph of the solution for the interval $[0, 2]$ is shown in Figure 10. In figure 10, which is quite similar to the graph of exact solution, therefore for more clarity we observe table 2. It is clear from table 1 that the solution using MMIM is more promising than AIM.

8 Conclusion

In our work, a multistage iterative method as a modification in AIM is introduced, which is known as MMIM. MMIM is the extension of AIM, although the solution given by AIM is limited for a very small interval. Therefore MMIM is useful to solve the functional equations for a comparatively large interval by integrating AIM. The numerical examples and associated graphs and tables have been defined and constructed in this article. Figures 2–5 are the graphs of the solution of Eq. (14) for the subintervals $[0, 1]$ to $[1.2, 1.3]$. These graphs show the behaviour of the numerical solution calculated using AIM separately in their respective subintervals and the procedure is continued up to the subinterval $[1.9, 2]$. Then all the solution graphs are merged and the final graph is drawn as in Figure 6 with the concept of MMIM. We can observe with the help of Figure 1 that for the interval $[0, 2]$, MMIM converges to the exact solution more accurately than the AIM, and further the efficiency of the method is shown in Table 1.

In Table 1, a comparison between absolute error between AIM and MMIM in the computational interval

[0, 2] at different points are shown when used them separately. Therefore with the help of solution graphs and table of Eq. (14), we can state that MMIM is more accurate than AIM.

Similar results are followed in all the figures and table in Example 2. Therefore, MMIM is simple and easy to apply, requires less computational complexity and also provides more reliable results.

References

- [1] Harendra Singh. An efficient computational method for the approximate solution of nonlinear Lane-Emden type equations arising in astrophysics. *Astrophysics and Space Science*, 363: 71, 2018.
- [2] Harendra Singh, Hari M Srivastava, and Devendra Kumar. A reliable algorithm for the approximate solution of the nonlinear Lane-Emden type equations arising in astrophysics. *Numerical Methods for Partial Differential Equations*, 34: 1524-1555, 2018.
- [3] Randhir Singh, Jitendra Kumar. The Adomian decomposition method with Green's function for solving nonlinear singular boundary value problems. *Journal of Applied Mathematics and Computing*, 44: 397-416, 2013.
- [4] Jing Niu, Minqiang Xu, Yingzhen Lin, and Qing Xue. Numerical solution of nonlinear singular boundary value problems. *Journal of Computational and Applied Mathematics*, 331: 42-51, 2018.
- [5] Mohit Arya and Amit Ujlayan. Solution of Riccati differential equation with Green's function and Pade' approximation technique. *Advances in Differential Equations and Control Processes*, 21: 31-52, 2019.
- [6] Esmaeil Babolian, Jafar Biazar, and Alireza R Vahidi. Solution of a system of nonlinear equations by Adomian decomposition method. *Applied Mathematics and Computation*, 150: 847-854, 2004.
- [7] Salah M El-Sayed. The modified decomposition method for solving nonlinear algebraic equations. *Applied Mathematics and Computation*, 132: 589-597, 2002.
- [8] George Adomian. Solving frontier problems of physics: The decomposition method. *Kluwer Academic Publishers, London*, 1994.
- [9] Jafar Biazar, Esmaeil Babolian, and Rafiqul Islam. Solution of a system of Volterra integral equations of the first kind by Adomian method. *Applied Mathematics and Computation*, 139: 249-258, 2003.
- [10] Jafar Biazar, Esmaeil Babolian, and Rafiqul Islam. Solution of the system of ordinary differential equations by Adomian decomposition method. *Applied Mathematics and Computation*, 147: 713-719, 2004.
- [11] Abdul M Wazwaz. A first course in integral equations. *World Scientific Publishing Co, Pte. Ltd, USA*, 1997.
- [12] Harendra Singh. A new numerical algorithm for fractional model of Bloch equation in nuclear magnetic resonance. *Alexandria Engineering Journal*, 55: 2863-2869, 2016.
- [13] Chandra S Singh, Harendra Singh, Vineet K Singh, and Om P Singh. Fractional order operational matrix methods for fractional singular integro-differential. *Applied Mathematics Modelling*, 40: 10705-10718, 2016.
- [14] Harendra Singh. Operational matrix approach for approximate solution of fractional model of Bloch equation, operational matrix for solution of fractional model of Bloch equation. *Journal of King Saud University Science*, 29: 235-240, 2017.
- [15] Harendra Singh and Chandra S Singh. Stable numerical solutions of fractional partial differential equations using Legendre scaling functions operational matrix. *Ain Shams Engineering Journal*, 9: 717-725, 2018.
- [16] Harendra Singh, Hari M Srivastava, and Devendra Kumar. A reliable numerical algorithm for the fractional vibration equation. *Chaos Solitons & Fractals*, 103: 131-138, 2017.
- [17] Harendra Singh, Devendra Kumar, Jagdev Singh, and Chandra S Singh. A reliable numerical algorithm for the fractional Klein-Gordon equation. *Engineering Transactions*, 67: 21-34, 2019.
- [18] Harendra Singh, Devendra Kumar, and Dumitru Baleanu. Methods of mathematical modelling: fractional differential equations. *CRC Press: Taylor & Francis Group, New York*, 2019.
- [19] Harendra Singh. Approximate solution of fractional vibration equation using Jacobi Polynomials. *Applied Mathematics and Computation*, 317: 85-100, 2018.
- [20] Devendra Kumar, Jagdev Singh, Sunil D Purohit, and Ram Swroop. A hybrid analytical algorithm for nonlinear fractional wave-like equations. *Mathematical Modelling of Natural Phenomena*, 14: 304, 2019.
- [21] Amit Goswami, Jagdev Singh, Devendra Kumar, and Sushila. An efficient analytical approach for fractional equal width equations describing hydro-magnetic waves in cold plasma. *Physics A: Statistical Mechanics and its Applications*, 524: 563-575, 2019.
- [22] Devendra Kumar, Jagdev Singh, Kumud Tanwar, and Dumitru Baleanu. A new fractional exothermic reactions model having constant heat source in porous media with power, exponential and Mittag-Leffler laws. *International Journal of Heat and Mass Transfer*, 138:1333-1327, 2019.
- [23] Sanjay Bhattar, Amit Mathur, Devendra Kumar, and Jagdev Singh. A new analysis of fractional Drinfeld-Sokolov-Wilson model with exponential memory. *Physics A: Statistical Mechanics and its Applications*, 537:122578, 2020.
- [24] Varsha D Gejji and Hossein Jafari. An iterative method for solving nonlinear functional equations. *Journal of Mathe-*

matical Analysis and Applications, 316: 753-763, 2006.

- [25] Gilberto G Parra, Abrahm J Arenas, and Lucas Jódar. Piecewise finite series solutions of seasonal diseases models using multistage Adomian method. *Communications in Nonlinear Science and Numerical Simulation*, 14: 3967-3977, 2009.
- [26] Erwin Kreyszig, Introductory functional analysis with applications. *Wiley India Pvt. Ltd.*, 2014.

This page is intentionally left blank