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On Pull-Back Bundle of Tensor Bundles Defined by Projection of The Cotangent Bundle

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Abstract

Using projection (submersion) of the cotangent bundle T^*M over a manifold M , we define a semi-tensor (pull-back) bundle tM of type (p,q) . The aim of this study is to investigate complete lift of vector fields in a special class of semi-tensor bundle tM of the type (p,q) . We also have a new example for good square in this work.

Keywords: Vector field, complete lift, pull-back bundle, cotangent bundle, semi-tensor bundle.**AMS 2010 codes:** 53A45, 55R10, 57R25.

1 Introduction

Defining some structure on the tangent bundles and cotangent bundles to obtain subtle information about the topology and geometry of the manifold is the main way for mathematicians. Due to this feature, many authors have been systematically worked on the tangent bundles and cotangent bundles [1–3, 12, 13]. One of these studies is investigating complete lift of vector fields in a special class of semi-tensor (pull-back) bundle tM of type (p,q) .

Let M_n be an n -dimensional differentiable manifold of class C^∞ , and let $(T^*(M_n), \pi_1, M_n)$ be a cotangent bundle over M_n . We use the notation $(x^i) = (x^{\bar{\alpha}}, x^\alpha)$, where the indices $i, j, \dots$ run from 1 to $2n$, the indices $\bar{\alpha}, \bar{\beta}, \dots$ from 1 to n and the indices $\alpha, \beta, \dots$ from $n+1$ to $2n$, x^α are coordinates in M_n , $x^{\bar{\alpha}} = p_\alpha$ are fibre coordinates of the cotangent bundle $T^*(M_n)$.

Let now $(T_q^p(M_n), \tilde{\pi}, M_n)$ be a tensor bundle [5], [9], [10, p.118] with base space M_n , and let $T^*(M_n)$ be cotangent bundle determined by a natural projection (submersion) $\pi_1 : T^*(M_n) \rightarrow M_n$. The semi-tensor bundle (induced, pull-back [6], [7], [9], [11], [13]) of the tensor bundle $(T_q^p(M_n), \tilde{\pi}, M_n)$ is the bundle

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$(t_q^p(M_n), \pi_2, T^*(M_n))$ over cotangent bundle $T^*(M_n)$ with a total space

$$t_q^p(M_n) = \left\{ \left((x^{\bar{\alpha}}, x^{\alpha}), x^{\bar{\alpha}} \right) \in T^*(M_n) \times (T_q^p)_x(M_n) : \pi_1(x^{\bar{\alpha}}, x^{\alpha}) = \tilde{\pi}(x^{\alpha}, x^{\bar{\alpha}}) = (x^{\alpha}) \right\} \subset T^*(M_n) \times (T_q^p)_x(M_n)$$

and with the projection map $\pi_2 : t_q^p(M_n) \rightarrow T^*(M_n)$ defined by $\pi_2(x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\alpha}}) = (x^{\bar{\alpha}}, x^{\alpha})$, where

$$(T_q^p)_x(M_n) \left(x = \pi_1(\tilde{x}), \tilde{x} = (x^{\bar{\alpha}}, x^{\alpha}) \in T^*(M_n) \right)$$

is the tensor space at a point x of M_n , where $x^{\bar{\alpha}} = t_{\alpha_1 \dots \alpha_q}^{\beta_1 \dots \beta_p} (\bar{\alpha}, \bar{\beta}, \dots = 2n+1, \dots, 2n+n^{p+q})$ are fiber coordinates of the tensor bundle $T_q^p(M_n)$.

The pull-back (semi-tensor) bundle $t_q^p(M_n)$ of tensor bundle $T_q^p(M_n)$ has the natural bundle structure over M_n , its bundle projection $\pi : t_q^p(M_n) \rightarrow M_n$ being defined by $\pi : (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\alpha}}) \rightarrow (x^{\alpha})$. If we introduce a mapping $\pi_2 : t_q^p(M_n) \rightarrow T^*(M_n)$ by $\pi_2 : (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\alpha}}) \rightarrow (x^{\bar{\alpha}}, x^{\alpha})$, then $t_q^p(M_n)$ has a bundle structure over M_n . Hence $\pi = \pi_1 \circ \pi_2$.

Now, consider a diagram as

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{\pi} & D \end{array}$$

A good square of vector bundles is a diagram as above verifying

- i. α and β are fibre bundles, but not necessarily vector bundles;
- ii. γ and π are vector bundles;
- iii. the square is commutative, i.e., $\pi \circ \alpha = \beta \circ \gamma$;
- iv. the local expression

$$\begin{array}{ccccccc} A & \xrightarrow{\gamma} & B & U^n \times R^r \times G^s \times R^t & \xrightarrow{\gamma} & U^n \times G^s & (x^i, a^a, g^\lambda, b^\sigma) & \xrightarrow{\gamma} & (x^i, g^\lambda) \\ \alpha \downarrow & & \downarrow \beta & \downarrow & & \downarrow & \downarrow & & \downarrow \\ C & \xrightarrow{\pi} & D & U^n \times R^r & \xrightarrow{\pi} & U^n & (x^i, a^a) & \xrightarrow{\pi} & (x^i) \end{array}$$

where G is a manifold and superindices denote the dimension of the manifolds [3].

By means of above definition, we have

Theorem 1. Let now $\pi : t_q^p(M_n) \rightarrow M_n$ be a semi-tensor bundle and $\pi_1 : T^*(M_n) \rightarrow M_n$ be a cotangent bundle. Then, the following is a good square:

$$\begin{array}{ccccccc} t_q^p(M_n) & \xrightarrow{\pi_2} & T^*(M_n) & T^*(M_n) \times (T_q^p)_x(M_n) & \xrightarrow{\pi_2} & T^*(M_n) & (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\alpha}}) & \xrightarrow{\pi_2} & (x^{\bar{\alpha}}, x^{\alpha}) \\ \downarrow id & & \downarrow \pi_1 & id \downarrow & & \downarrow \pi_1 & id \downarrow & & \downarrow \pi_1 \\ t_q^p(M_n) & \xrightarrow{\pi} & M_n & T^*(M_n) \times (T_q^p)_x(M_n) & \xrightarrow{\pi} & M_n & (x^{\bar{\alpha}}, x^{\alpha}, x^{\bar{\alpha}}) & \xrightarrow{\pi} & (x^{\alpha}) \end{array}$$

If $(x^{i'}) = (x^{\bar{\alpha}'}, x^{\alpha'}, x^{\bar{\alpha}'})$ is another system of local adapted coordinates in the semi-tensor bundle $t_q^p(M_n)$, then we have

$$\begin{cases} x^{\bar{\alpha}'} = \frac{\partial x^{\bar{\beta}}}{\partial x^{\bar{\alpha}'}} p_{\bar{\beta}}, \\ x^{\alpha'} = x^{\alpha'}(x^{\bar{\beta}}), \\ x^{\bar{\alpha}'} = t_{\alpha'_1 \dots \alpha'_q}^{\beta'_1 \dots \beta'_p} = A_{\alpha'_1 \dots \alpha'_p}^{\beta'_1 \dots \beta'_p} A_{\alpha'_1 \dots \alpha'_q}^{\beta_1 \dots \beta_q} t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} = A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} x^{\bar{\beta}}. \end{cases} \quad (1)$$

The jacobian of (1) has components

$$\bar{A} = (A_{J'}^{I'}) = \begin{pmatrix} A_{\alpha'}^{\beta} & p_{\sigma} A_{\beta}^{\beta'} A_{\beta'}^{\sigma \alpha'} & 0 \\ 0 & A_{\beta}^{\alpha'} & 0 \\ 0 & t_{(\sigma)}^{(\alpha)} \partial_{\beta} A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\sigma)} A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)} \end{pmatrix}, \quad (2)$$

where $I = (\bar{\alpha}, \alpha, \bar{\alpha})$, $J = (\bar{\beta}, \beta, \bar{\beta})$, $I, J, \dots = 1, \dots, 2n + n^{p+q}$, $t_{(\sigma)}^{(\alpha)} = t_{\sigma_1 \dots \sigma_q}^{\alpha_1 \dots \alpha_p}$, $A_{\beta}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}}$, $A_{\alpha'}^{\beta} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}}$, $A_{\beta' \alpha'}^{\sigma} = \frac{\partial^2 x^{\sigma}}{\partial x^{\beta'} \partial x^{\alpha'}}$. It is easily verified that the condition $\text{Det} \bar{A} \neq 0$ is equivalent to the condition:

$$\text{Det}(A_{\alpha'}^{\beta}) \neq 0, \text{Det}(A_{\beta}^{\alpha'}) \neq 0, \text{Det}(A_{(\alpha)}^{(\beta')} A_{(\alpha')}^{(\beta)}) \neq 0.$$

Also, $\dim t_q^p(M_n) = 2n + n^{p+q}$.

We note that special class of semi-tensor bundle was examined in [4]. The main purpose of this paper is to study semi-tensor (pull-back) bundle $t_q^p(M_n)$ of tensor bundle $T_q^p(M_n)$ by using projection of the cotangent bundle $T^*(M_n)$.

We denote by $\mathfrak{S}_q^p(T^*(M_n))$ and $\mathfrak{S}_q^p(M_n)$ the modules over $F(T^*(M_n))$ and $F(M_n)$ of all tensor fields of type (p, q) on $T^*(M_n)$ and M_n , respectively, where $F(T^*(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ -functions on $T^*(M_n)$ and M_n , respectively.

2 Some lifts of tensor fields and γ -Operator

Let $X \in \mathfrak{S}_0^1(T^*(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. The complete lift cX of X to cotangent bundle is defined by ${}^cX = X^\alpha \partial_\alpha - p_\beta (\partial_\alpha X^\beta) \partial_{\bar{\alpha}}$ [12], p.236]. On putting

$${}^{cc}X = \begin{pmatrix} {}^{cc}X^{\bar{\beta}} \\ {}^{cc}X^{\beta} \\ {}^{cc}X^{\bar{\beta}} \end{pmatrix} = \begin{pmatrix} -p_\varepsilon (\partial_\beta X^\varepsilon) \\ X^\beta \\ \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_\varepsilon X^{\alpha_\lambda} - \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_\mu} X^\varepsilon \end{pmatrix}, \quad (3)$$

from (2), we easily see that ${}^{cc}X' = \bar{A}({}^{cc}X)$. The vector field ${}^{cc}X$ is called the complete lift of ${}^cX \in \mathfrak{S}_0^1(T^*(M_n))$ to $t_q^p(M_n)$.

Now, consider $A \in \mathfrak{S}_q^p(T^*(M_n))$ and $\varphi \in \mathfrak{S}_1^1(M_n)$, then ${}^{vv}A \in \mathfrak{S}_0^1(t_q^p(M_n))$ (vertical lift), $\gamma\varphi \in \mathfrak{S}_0^1(t_q^p(M_n))$ and $\tilde{\gamma}\varphi \in \mathfrak{S}_0^1(t_q^p(M_n))$ have respectively, components on the semi-tensor bundle $t_q^p(M_n)$ [13]

$$\begin{aligned} {}^{vv}A &= ({}^{vv}A)^I = \begin{pmatrix} {}^{vv}A^a \\ {}^{vv}A^\alpha \\ {}^{vv}A^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \end{pmatrix}, \\ \gamma\varphi &= (\gamma\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \varphi_\varepsilon^{\alpha_\lambda} \end{pmatrix}, \\ \tilde{\gamma}\varphi &= (\tilde{\gamma}\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \varphi_{\beta_\mu}^\varepsilon \end{pmatrix} \end{aligned} \quad (4)$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\alpha}})$ on $t_q^p(M_n)$, where $A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$, $\varphi_\varepsilon^{\alpha_\lambda}$ and $\varphi_{\beta_\mu}^\varepsilon$ are local components of A and φ .

On the other hand, ${}^{vv}f$ the vertical lift of function $f \in \mathfrak{S}_0^0(M_n)$ on $t_q^p(M_n)$ is defined by [9]:

$${}^{vv}f = {}^v f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \quad (5)$$

Theorem 2. For any vector fields X, Y on $T^*(M_n)$ and $f \in \mathfrak{S}_0^0(M_n)$, we have

- (i) ${}^{cc}(X + Y) = {}^{cc}X + {}^{cc}Y$,
- (ii) ${}^{cc}X {}^{vv}f = {}^{vv}(Xf)$.

Proof. (i) This immediately follows from (3).

(ii) Let $X \in \mathfrak{S}_0^1(T^*(M_n))$. Then we get by (3) and (5):

$$\begin{aligned} {}^{cc}X^{vv}f &= {}^{cc}X^I \partial_I({}^{vv}f) \\ {}^{cc}X^{vv}f &= {}^{cc}X^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}}({}^{vv}f)}_0 + {}^{cc}X^{\alpha} \partial_{\alpha}({}^{vv}f) + {}^{cc}X^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}}({}^{vv}f)}_0 \\ &= X^{\alpha} \partial_{\alpha}({}^{vv}f) \\ &= {}^{vv}(Xf), \end{aligned}$$

which gives (ii) of Theorem 2.

Theorem 3. If $\varphi \in \mathfrak{S}_1^1(M_n)$, $f \in \mathfrak{S}_0^0(M_n)$ and $A \in \mathfrak{S}_q^p(T^*(M_n))$, then

- (i) $({}^{vv}A)^{vv}f = 0$,
- (ii) $(\gamma\varphi)({}^{vv}f) = 0$,
- (iii) $(\tilde{\gamma}\varphi)({}^{vv}f) = 0$.

Proof. (i) If $A \in \mathfrak{S}_q^p(T^*(M_n))$, then, by (4) and (5), we find

$$\begin{aligned} ({}^{vv}A)^{vv}f &= ({}^{vv}A)^I \partial_I({}^{vv}f) \\ &= \underbrace{({}^{vv}A)^{\bar{\alpha}} \partial_{\bar{\alpha}}({}^{vv}f)}_0 + \underbrace{({}^{vv}A)^{\alpha} \partial_{\alpha}({}^{vv}f)}_0 + \underbrace{({}^{vv}A)^{\bar{\alpha}} \partial_{\bar{\alpha}}({}^{vv}f)}_0 \\ &= 0. \end{aligned}$$

Thus, we have (i) of Theorem 3.

(ii) If $\varphi \in \mathfrak{S}_1^1(M_n)$, then we have by (4) and (5):

$$\begin{aligned} (\gamma\varphi)({}^{vv}f) &= (\gamma\varphi)^I \partial_I({}^{vv}f) \\ &= \underbrace{(\gamma\varphi)^{\bar{\alpha}} \partial_{\bar{\alpha}}({}^{vv}f)}_0 + \underbrace{(\gamma\varphi)^{\alpha} \partial_{\alpha}({}^{vv}f)}_0 + \underbrace{(\gamma\varphi)^{\bar{\alpha}} \partial_{\bar{\alpha}}({}^{vv}f)}_0 \\ &= 0. \end{aligned}$$

Thus, we have (ii) of Theorem 3.

(iii) If $\varphi \in \mathfrak{S}_1^1(M_n)$, then we have by (4) and (5):

$$\begin{aligned} (\tilde{\gamma}\varphi)({}^{vv}f) &= (\tilde{\gamma}\varphi)^I \partial_I({}^{vv}f) \\ &= \underbrace{(\tilde{\gamma}\varphi)^{\bar{\alpha}} \partial_{\bar{\alpha}}({}^{vv}f)}_0 + \underbrace{(\tilde{\gamma}\varphi)^{\alpha} \partial_{\alpha}({}^{vv}f)}_0 + \underbrace{(\tilde{\gamma}\varphi)^{\bar{\alpha}} \partial_{\bar{\alpha}}({}^{vv}f)}_0 \\ &= 0. \end{aligned}$$

Thus, we have (iii) of Theorem 3.

Theorem 4. Let $A, B \in \mathfrak{S}_q^p(T^*(M_n))$. For the Lie product, we have

$$[{}^{vv}A, {}^{vv}B] = 0.$$

Proof. If $A, B \in \mathfrak{S}_q^p(T^*(M_n))$ and $\begin{pmatrix} [{}^{vv}A, {}^{vv}B]^b \\ [{}^{vv}A, {}^{vv}B]^{\beta} \\ [{}^{vv}A, {}^{vv}B]^{\bar{\beta}} \end{pmatrix}$ are components of $[{}^{vv}A, {}^{vv}B]^J$ with respect to the coordinates

$(x^{\bar{\beta}}, x^{\beta}, x^{\bar{\bar{\beta}}})$ on $t_q^p(M_n)$, then we have

$$\begin{aligned} [{}^{vv}A, {}^{vv}B]^J &= ({}^{vv}A)^I \partial_I ({}^{vv}B)^J - ({}^{vv}B)^I \partial_I ({}^{vv}A)^J \\ &= \underbrace{({}^{vv}A)^{\bar{\alpha}}}_{0} \partial_{\bar{\alpha}} ({}^{vv}B)^J + \underbrace{({}^{vv}A)^{\alpha}}_{0} \partial_{\alpha} ({}^{vv}B)^J \\ &\quad + \underbrace{({}^{vv}A)^{\bar{\bar{\alpha}}}}_{0} \partial_{\bar{\bar{\alpha}}} ({}^{vv}B)^J - \underbrace{({}^{vv}B)^{\bar{\alpha}}}_{0} \partial_{\bar{\alpha}} ({}^{vv}A)^J \\ &\quad - \underbrace{({}^{vv}B)^{\alpha}}_{0} \partial_{\alpha} ({}^{vv}A)^J - \underbrace{({}^{vv}B)^{\bar{\bar{\alpha}}}}_{0} \partial_{\bar{\bar{\alpha}}} ({}^{vv}A)^J \\ &= A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\alpha}} ({}^{vv}B)^J - B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\alpha}} ({}^{vv}A)^J. \end{aligned}$$

Firstly, if $J = b$, we have

$$[{}^{vv}A, {}^{vv}B]^{\bar{\beta}} = A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\alpha}} \underbrace{({}^{vv}B)^{\bar{\beta}}}_{0} - B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\alpha}} \underbrace{({}^{vv}A)^{\bar{\beta}}}_{0} = 0,$$

by virtue of (4). Secondly, if $J = \beta$, we have

$$[{}^{vv}A, {}^{vv}B]^{\beta} = A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\alpha}} \underbrace{({}^{vv}B)^{\beta}}_{0} - B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\alpha}} \underbrace{({}^{vv}A)^{\beta}}_{0} = 0,$$

by virtue of (4). Thirdly, if $J = \bar{\bar{\beta}}$, then we have

$$\begin{aligned} [{}^{vv}A, {}^{vv}B]^{\bar{\bar{\beta}}} &= A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\bar{\alpha}}} ({}^{vv}B)^{\bar{\bar{\beta}}} - B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\bar{\bar{\alpha}}} ({}^{vv}A)^{\bar{\bar{\beta}}} \\ &= A_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \underbrace{\partial_{\bar{\bar{\alpha}}} B_{\theta_1 \dots \theta_q}^{\beta_1 \dots \beta_p}}_{0} - B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \underbrace{\partial_{\bar{\bar{\alpha}}} A_{\theta_1 \dots \theta_q}^{\beta_1 \dots \beta_p}}_{0} \\ &= 0 \end{aligned}$$

by virtue of (4). Thus, we have Theorem 4.

Theorem 5. Let $X, Y \in \mathfrak{S}_0^1(T^*(M_n))$. For the Lie product, we have

$$[{}^{cc}X, {}^{cc}Y] = {}^{cc}[X, Y] (i.e. L_{ccX} {}^{cc}Y = {}^{cc}(L_X Y)).$$

Proof. If $X, Y \in \mathfrak{S}_0^1(T^*(M_n))$ and $\begin{pmatrix} [{}^{cc}X, {}^{cc}Y]^{\bar{\beta}} \\ [{}^{cc}X, {}^{cc}Y]^{\beta} \\ [{}^{cc}X, {}^{cc}Y]^{\bar{\bar{\beta}}} \end{pmatrix}$ are components of $[{}^{cc}X, {}^{cc}Y]^J$ with respect to the coordinates

$(x^{\bar{\beta}}, x^{\beta}, x^{\bar{\bar{\beta}}})$ on $t_q^p(M_n)$, then we have

$$[{}^{cc}X, {}^{cc}Y]^J = ({}^{cc}X)^I \partial_I ({}^{cc}Y)^J - ({}^{cc}Y)^I \partial_I ({}^{cc}X)^J.$$

Firstly, if $J = b$, we have

$$\begin{aligned}
 [{}^{cc}X, {}^{cc}Y]^{\bar{\beta}} &= ({}^{cc}X)^I \partial_I ({}^{cc}Y)^{\bar{\beta}} - ({}^{cc}Y)^I \partial_I ({}^{cc}X)^{\bar{\beta}} \\
 &= ({}^{cc}X)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}Y)^{\bar{\beta}}}_0 + ({}^{cc}X)^{\alpha} \partial_{\alpha} ({}^{cc}Y)^{\bar{\beta}} + ({}^{cc}X)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}Y)^{\bar{\beta}}}_0 \\
 &\quad - ({}^{cc}Y)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}X)^{\bar{\beta}}}_0 - ({}^{cc}Y)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\bar{\beta}} - ({}^{cc}Y)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}X)^{\bar{\beta}}}_0 \\
 &= p_{\varepsilon} \partial_{\alpha} X^{\varepsilon} (\partial_{\beta} Y^{\alpha}) - X^{\alpha} \partial_{\alpha} p_{\varepsilon} (\partial_{\beta} Y^{\varepsilon}) \\
 &\quad - p_{\varepsilon} \partial_{\alpha} Y^{\varepsilon} (\partial_{\beta} X^{\alpha}) + Y^{\alpha} \partial_{\alpha} p_{\varepsilon} (\partial_{\beta} X^{\varepsilon}) \\
 &= p_{\varepsilon} (\partial_{\beta} Y^{\alpha} \partial_{\alpha} X^{\varepsilon} - X^{\alpha} \partial_{\alpha} \partial_{\beta} Y^{\varepsilon} - \partial_{\beta} X^{\alpha} \partial_{\alpha} Y^{\varepsilon} + Y^{\alpha} \partial_{\alpha} \partial_{\beta} X^{\varepsilon}) \\
 &= -p_{\varepsilon} (\partial_{\beta} \underbrace{(X^{\alpha} \partial_{\alpha} Y^{\varepsilon} - Y^{\alpha} \partial_{\alpha} X^{\varepsilon})}_{[X, Y]^{\varepsilon}}) \\
 &= -p_{\varepsilon} (\partial_{\beta} [X, Y]^{\varepsilon})
 \end{aligned}$$

by virtue of (3). Secondly, if $J = \beta$, we have

$$\begin{aligned}
 [{}^{cc}X, {}^{cc}Y]^{\beta} &= ({}^{cc}X)^I \partial_I ({}^{cc}Y)^{\beta} - ({}^{cc}Y)^I \partial_I ({}^{cc}X)^{\beta} \\
 &= ({}^{cc}X)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}Y)^{\beta}}_0 + ({}^{cc}X)^{\alpha} \partial_{\alpha} ({}^{cc}Y)^{\beta} + ({}^{cc}X)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}Y)^{\beta}}_0 \\
 &\quad - ({}^{cc}Y)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}X)^{\beta}}_0 - ({}^{cc}Y)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\beta} - ({}^{cc}Y)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}X)^{\beta}}_0 \\
 &= ({}^{cc}X)^{\alpha} \partial_{\alpha} ({}^{cc}Y)^{\beta} - ({}^{cc}Y)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\beta} \\
 &= X^{\alpha} \partial_{\alpha} Y^{\beta} - Y^{\alpha} \partial_{\alpha} X^{\beta} \\
 &= [X, Y]^{\beta}
 \end{aligned}$$

by virtue of (3). Thirdly, if $J = \bar{\bar{\beta}}$, then we have

$$\begin{aligned}
 [{}^{cc}X, {}^{cc}Y]^{\bar{\bar{\beta}}} &= ({}^{cc}X)^I \partial_I ({}^{cc}Y)^{\bar{\bar{\beta}}} - ({}^{cc}Y)^I \partial_I ({}^{cc}X)^{\bar{\bar{\beta}}} \\
 &= ({}^{cc}X)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}Y)^{\bar{\bar{\beta}}}}_0 + ({}^{cc}X)^{\alpha} \partial_{\alpha} ({}^{cc}Y)^{\bar{\bar{\beta}}} + ({}^{cc}X)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}Y)^{\bar{\bar{\beta}}}}_0 \\
 &\quad - ({}^{cc}Y)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}X)^{\bar{\bar{\beta}}}}_0 - ({}^{cc}Y)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\bar{\bar{\beta}}} - ({}^{cc}Y)^{\bar{\alpha}} \underbrace{\partial_{\bar{\alpha}} ({}^{cc}X)^{\bar{\bar{\beta}}}}_0 \\
 &= X^{\alpha} \partial_{\alpha} \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_{\varepsilon} Y^{\beta_{\lambda}} - \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_{\mu}} Y^{\varepsilon} \right) \\
 &\quad + \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_{\varepsilon} X^{\alpha_{\lambda}} - \sum_{\mu=1}^q t_{\beta_1 \dots \gamma \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\alpha_{\mu}} X^{\gamma} \right) \partial_{\bar{\alpha}} \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \sigma \dots \alpha_p} \partial_{\sigma} Y^{\beta_{\lambda}} - \sum_{\mu=1}^q t_{\beta_1 \dots \gamma \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_{\mu}} Y^{\gamma} \right) \\
 &\quad - Y^{\alpha} \partial_{\alpha} \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_{\varepsilon} X^{\beta_{\lambda}} - \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_{\mu}} X^{\varepsilon} \right) \\
 &\quad - \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_{\varepsilon} Y^{\alpha_{\lambda}} - \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_{\mu}} Y^{\varepsilon} \right) \partial_{\bar{\alpha}} \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \sigma \dots \alpha_p} \partial_{\sigma} X^{\beta_{\lambda}} - \sum_{\mu=1}^q t_{\beta_1 \dots \gamma \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_{\mu}} X^{\gamma} \right)
 \end{aligned}$$

$$\begin{aligned}
&= X^\alpha \partial_\alpha \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_\varepsilon Y^{\beta_\lambda} \right) - X^\alpha \partial_\alpha \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} (\partial_{\beta_\mu} Y^\varepsilon) \\
&\quad + \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_\varepsilon X^{\alpha_\lambda} \underbrace{\partial_{\bar{\alpha}} \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \sigma \dots \alpha_p} \partial_\sigma Y^{\beta_\lambda}}_{\delta_{\alpha_\lambda}^\sigma} - Y^\alpha \partial_\alpha \left(\sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_\varepsilon X^{\beta_\lambda} \right) + Y^\alpha \partial_\alpha \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} (\partial_{\beta_\mu} X^\varepsilon) \\
&\quad - \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_\varepsilon Y^{\alpha_\lambda} \underbrace{\partial_{\bar{\alpha}} \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \sigma \dots \alpha_p} \partial_\sigma X^{\beta_\lambda}}_{\delta_{\alpha_\lambda}^\sigma} \\
&\quad + \underbrace{\sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\alpha_\mu} X^\varepsilon \partial_{\bar{\alpha}} \sum_{\mu=1}^q t_{\beta_1 \dots \gamma \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_\mu} Y^\gamma}_{\delta_\gamma^\alpha} - \underbrace{\sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\alpha_\mu} Y^\varepsilon \partial_{\bar{\alpha}} \sum_{\mu=1}^q t_{\beta_1 \dots \gamma \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_\mu} X^\gamma}_{\delta_\gamma^\alpha} \\
&\quad \quad \quad \Sigma_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\alpha_\mu} X^\varepsilon (\partial_{\beta_\mu} Y^\alpha) \quad \quad \quad \Sigma_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\alpha_\mu} Y^\varepsilon (\partial_{\beta_\mu} X^\alpha) \\
&= \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} (\partial_\varepsilon X^\sigma) \left(\partial_\sigma Y^{\beta_\lambda} \right) + \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} X^\alpha \partial_\alpha \partial_\varepsilon Y^{\beta_\lambda} \\
&\quad - \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} (\partial_\varepsilon Y^\sigma) \left(\partial_\sigma X^{\beta_\lambda} \right) - \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} Y^\alpha \partial_\alpha \partial_\varepsilon X^{\beta_\lambda} \\
&\quad + \underbrace{\sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} (-X^\alpha \partial_{\alpha_\mu} \partial_{\beta_\mu} Y^\varepsilon + \partial_{\beta_\mu} Y^\alpha \partial_{\alpha_\mu} X^\varepsilon + Y^\alpha \partial_{\alpha_\mu} \partial_{\beta_\mu} X^\varepsilon - \partial_{\beta_\mu} X^\alpha \partial_{\alpha_\mu} Y^\varepsilon)}_{-\Sigma_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} (\partial_{\beta_\mu} \underbrace{(X^\alpha \partial_{\alpha_\mu} Y^\varepsilon - Y^\alpha \partial_{\alpha_\mu} X^\varepsilon)}_{[X, Y]^\varepsilon})} \\
&= \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_\varepsilon [X, Y]^{\beta_\lambda} - \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} (\partial_{\beta_\mu} [X, Y]^\varepsilon)
\end{aligned}$$

$${}^{cc}[X, Y] = \begin{pmatrix} -p_\varepsilon(\partial_\beta[X, Y]^\varepsilon) \\ [X, Y]^\beta \\ \sum_{\lambda=1}^p t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \varepsilon \dots \alpha_p} \partial_\varepsilon[X, Y]^{\beta_\lambda} - \sum_{\mu=1}^q t_{\beta_1 \dots \varepsilon \dots \beta_q}^{\alpha_1 \dots \alpha_p} \partial_{\beta_\mu}[X, Y]^\varepsilon \end{pmatrix}$$

with respect to the coordinates $(x^{\bar{\beta}}, x^{\beta}, x^{\bar{\bar{\beta}}})$ on $t_q^p(M_n)$. Thus Theorem 5 is proved. $\square$

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