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Some Properties of Diagonal Lifts in Semi-Cotangent Bundles

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Abstract

We analyze some properties of diagonal lift of tensor fields of type (1,1) in semi-cotangent bundles with the help of adapted frames.

Keywords: Vector field, complete lift, diagonal lift, pull-back bundle, cross-section, semi-cotangent bundle

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1 Lifts of Vector Fields on a Cross-Section in the Semi-Cotangent Bundle

Defining some structure on the tangent bundles and cotangent bundles to obtain subtle information about the topology and geometry of the manifold is the main way for mathematicians. Due to this feature, many authors have been systematically worked on them [1, 2, 9, 13, 14]. One of these studies is analyzing some properties of diagonal lift of tensor fields of type (1,1) in semi-cotangent bundles with the help of adapted frames

Let M_n be an n -dimensional differentiable manifold of class C^∞ and $T(M_n)$ the tangent bundle determined by a natural projection (submersion) $\pi_1 : T(M_n) \rightarrow M_n$. We use the notation $(x^i) = (x^{\bar{\alpha}}, x^\alpha)$, where the indices $i, j, \dots$ run from 1 to $2n$, the indices $\alpha, \beta, \dots$ from 1 to n and the indices $\bar{\alpha}, \bar{\beta}, \dots$ from $n+1$ to $2n$, x^α are coordinates in M_n , $x^{\bar{\alpha}} = y^\alpha$ are fibre coordinates of the tangent bundle $T(M_n)$. If $(x^{i'}) = (x^{\bar{\alpha}'}, x^{\alpha'})$ is another system of local adapted coordinates in the tangent bundle $T(M_n)$, then we have

$$\begin{cases} x^{\bar{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^{\bar{\beta}}} y^{\bar{\beta}}, \\ x^{\alpha'} = x^{\alpha'}(x^{\bar{\beta}}). \end{cases} \quad (1)$$

The Jacobian of (1) has components

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$$(A_j^{\alpha'}) = \left(\frac{\partial x^{\alpha'}}{\partial x^j} \right) = \begin{pmatrix} A_\beta^{\alpha'} & A_{\beta\epsilon}^{\alpha'} y^\epsilon \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$, $A_{\beta\epsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\epsilon}$. Let $T_x^*(M_n)(x = \pi_1(\tilde{x}), \tilde{x} = (x^\alpha, x^\alpha) \in T(M_n))$ be the cotangent space at a point x of M_n . If p_α are components of $p \in T_x^*(M_n)$ with respect to the natural coframe $\{dx^\alpha\}$, i.e. $p = p_i dx^i$, then by definition the set $t^*(M_n)$ of all points $(x^j) = (x^\alpha, x^\alpha, x^\alpha)$, $x^\alpha = p_\alpha$; $I, J, \dots = 1, \dots, 3n$ with projection $\pi_2 : t^*(M_n) \rightarrow T(M_n)$ (i.e. $\pi_2 : (x^\alpha, x^\alpha, x^\alpha) \rightarrow (x^\alpha, x^\alpha)$) is a semi-cotangent (pull-back [12]) bundle of the cotangent bundle by submersion $\pi_1 : T(M_n) \rightarrow M_n$ (For definition of the pull-back bundle, see for example [3], [5], [6], [7]). It is remarkable fact that the semi-cotangent (pull-back) bundle has a degenerate symplectic structure [12]

$$\omega = (\omega_{AB}) = dp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta_\beta^\alpha \\ 0 & \delta_\alpha^\beta & 0 \end{pmatrix}.$$

It is clear that the pull-back bundle $t^*(M_n)$ of the cotangent bundle $T^*(M_n)$ also has the natural bundle structure over M_n , its bundle projection $\pi : t^*(M_n) \rightarrow M_n$ being defined by $\pi : (x^\alpha, x^\alpha, x^\alpha) \rightarrow (x^\alpha)$, and hence $\pi = \pi_1 \circ \pi_2$. Thus $(t^*(M_n), \pi_1 \circ \pi_2)$ is the composite bundle [14], p.9) or step-like bundle [15].

The main purpose of the present paper is to study complete lift of vector fields and tensor fields of type (1,1) from tangent bundle $T(M_n)$ to semi-cotangent (pull-back) bundle $(t^*(M_n), \pi_2)$.

We denote by $\mathfrak{S}_q^p(T(M_n))$ and $\mathfrak{S}_q^p(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type (p, q) on $T(M_n)$ and M_n respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ -functions on $T(M_n)$ and M_n , respectively.

To a transformation (1) of local coordinates of $T(M_n)$, there corresponds on $t^*(M_n)$ the coordinate transformation [10]:

$$\begin{cases} x^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta, \\ x^{\alpha'} = x^{\alpha'}(x^\beta), \\ x^{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} p_\beta. \end{cases} \quad (2)$$

The Jacobian of (2) has components [10]:

$$\bar{A} = (A_j^{\alpha'}) = \begin{pmatrix} A_\beta^{\alpha'} & A_{\beta\epsilon}^{\alpha'} y^\epsilon & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & p_\sigma A_\beta^{\sigma'} A_{\beta'\alpha'}^\sigma & A_{\alpha'}^\beta \end{pmatrix}, \quad (3)$$

where

$$A_{\beta\epsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\epsilon}, \quad A_{\beta'\alpha'}^\alpha = \frac{\partial^2 x^\alpha}{\partial x^{\beta'} \partial x^{\alpha'}}.$$

We denote by $\mathfrak{S}_q^p(T(M_n))$ and $\mathfrak{S}_q^p(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type (p, q) on $T(M_n)$ and M_n , respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ -functions on $T(M_n)$ and M_n , respectively.

Let θ be a covector field on $T(M_n)$. Then the transformation $p \rightarrow \theta_p$, θ_p being the value of θ at $p \in T(M_n)$, determines a cross-section β_θ of semi-cotangent bundle. Thus if $\sigma : M_n \rightarrow T^*(M_n)$ is a cross-section of $(T^*(M_n), \tilde{\pi}, M_n)$, such that $\tilde{\pi} \circ \sigma = I_{(M_n)}$, an associated cross-section $\beta_\theta : T(M_n) \rightarrow t^*(M_n)$ of semi-cotangent (pull-back) bundle $(t^*(M_n), \pi_2, T(M_n))$ of cotangent bundle by using projection (submersion) of the tangent bundle $T(M_n)$ defined by [4], p. 217-218], [9], p. 301]:

$$\beta_\theta(x^\alpha, x^\alpha) = (x^\alpha, x^\alpha, \sigma \circ \pi_1(x^\alpha, x^\alpha)) = (x^\alpha, x^\alpha, \sigma(x^\alpha)) = (x^\alpha, x^\alpha, \theta_\alpha(x^\beta)).$$

If the covector field θ has the local components $\theta_\alpha(x^\beta)$, the cross-section $\beta_\theta(T(M_n))$ of $t^*(M_n)$ is locally expressed by

$$x^{\bar{\alpha}} = y^\alpha = V^\alpha(x^\beta), \quad x^\alpha = x^\alpha, \quad x^{\bar{\alpha}} = p_\alpha = \theta_\alpha(x^\beta) \quad (4)$$

with respect to the coordinates $x^A = (x^{\bar{\alpha}}, x^\alpha, x^{\bar{\alpha}})$ in $t^*(M_n)$. $x^{\bar{\alpha}} = y^\alpha$ being considered as parameters. Differentiating (4) by $x^{\bar{\alpha}} = y^\alpha$, we have vector fields $B_{(\bar{\beta})}$ ($\bar{\beta} = 1, \dots, n$) with components

$$B_{(\bar{\beta})} = \frac{\partial x^A}{\partial x^{\bar{\beta}}} = \partial_{\bar{\beta}} x^A = \begin{pmatrix} \frac{\partial V^\alpha}{\partial x^{\bar{\beta}}} \\ \frac{\partial x^\alpha}{\partial x^{\bar{\beta}}} \\ \frac{\partial \theta_\alpha}{\partial x^{\bar{\beta}}} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta(T(M_n))$ [10].

Thus $B_{(\bar{\beta})}$ have components

$$B_{(\bar{\beta})} : \begin{pmatrix} B_{(\bar{\beta})}^A \end{pmatrix} = \begin{pmatrix} \delta_{\bar{\beta}}^\alpha \\ 0 \\ 0 \end{pmatrix}$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\alpha}})$ in $t^*(M_n)$, where

$$\delta_{\bar{\beta}}^\alpha = A_{\bar{\beta}}^\alpha = \frac{\partial x^\alpha}{\partial x^{\bar{\beta}}}.$$

Let $X \in \mathfrak{Z}_0^1(T(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. We denote by BX the vector field with local components

$$BX : \begin{pmatrix} B_{(\bar{\beta})}^A X^{\bar{\beta}} \end{pmatrix} = \begin{pmatrix} \delta_{\bar{\beta}}^\alpha X^{\bar{\beta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{\bar{\beta}}^\alpha X^{\bar{\beta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X^\alpha \\ 0 \\ 0 \end{pmatrix} \quad (5)$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\alpha}})$ in $t^*(M_n)$, which is defined globally along $\beta_\theta(T(M_n))$. Then a mapping

$$B : \mathfrak{Z}_0^1(T(M_n)) \rightarrow \mathfrak{Z}_0^1(\beta_\theta(T(M_n)))$$

is defined by (5). The mapping B is the differential of $\beta_\theta : T(M_n) \rightarrow t^*(M_n)$ and so an isomorphism of $\mathfrak{Z}_0^1(T(M_n))$ onto $\mathfrak{Z}_0^1(\beta_\theta(T(M_n)))$ [10].

Since a cross-section is locally expressed by $x^{\bar{\alpha}} = y^\alpha = \text{const.}$, $x^{\bar{\alpha}} = p_\alpha = \text{const.}$, $x^\alpha = x^\alpha$, x^α being considered as parameters. Differentiating (4) by x^α , we have vector fields $C_{(\beta)}$ ($\beta = n+1, \dots, 2n$) with components

$$C_{(\beta)} = \frac{\partial x^A}{\partial x^\beta} = \partial_\beta x^A = \begin{pmatrix} \frac{\partial V^\alpha}{\partial x^\beta} \\ \frac{\partial x^\alpha}{\partial x^\beta} \\ \frac{\partial \theta_\alpha}{\partial x^\beta} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta(T(M_n))$.

Thus $C_{(\beta)}$ have components

$$C_{(\beta)} : \begin{pmatrix} C_{(\beta)}^A \end{pmatrix} = \begin{pmatrix} \frac{\partial V^\alpha}{\partial x^\beta} \\ \delta_\beta^\alpha \\ \frac{\partial \theta_\alpha}{\partial x^\beta} \end{pmatrix}$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\alpha}})$ in $t^*(M_n)$, where

$$\delta_\beta^\alpha = A_\beta^\alpha = \frac{\partial x^\alpha}{\partial x^\beta}.$$

Let $X \in \mathfrak{S}_0^1(T(M_n))$. Then we denote by CX the vector field with local components

$$CX : \left(C_{(\beta)}^A X^\beta \right) = \begin{pmatrix} X^\beta \partial_\beta V^\alpha \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix} \quad (6)$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\bar{\alpha}}})$ in $t^*(M_n)$, which is defined globally along $\beta_\theta(T(M_n))$. Then a mapping

$$C : \mathfrak{S}_0^1(T(M_n)) \rightarrow \mathfrak{S}_0^1(\beta_\theta(T(M_n)))$$

is defined by (6). The mapping C is the differential of $\beta_\theta : T(M_n) \rightarrow t^*(M_n)$ and so an isomorphism of $\mathfrak{S}_0^1(T(M_n))$ onto $\mathfrak{S}_0^1(\beta_\theta(T(M_n)))$ [10].

Now, consider $\omega \in \mathfrak{S}_1^0(M_n)$ and vector field $X \in \mathfrak{S}_0^1(T(M_n))$, then ${}^{vv}\omega$ (vertical lift), ${}^{cc}X$ (complete lift) and ${}^{HH}X$ (horizontal lift) have respectively, components on the semi-cotangent bundle $t^*(M_n)$ [11]:

$${}^{vv}\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \quad {}^{cc}X = \begin{pmatrix} y^\varepsilon \partial_\varepsilon X^\alpha \\ X^\alpha \\ -p_\sigma (\partial_\alpha X^\sigma) \end{pmatrix}, \quad {}^{HH}X = \begin{pmatrix} -\Gamma_\beta^\alpha X^\beta \\ X^\alpha \\ X^\beta \Gamma_{\beta\alpha} \end{pmatrix} \quad (7)$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\bar{\alpha}}})$, where

$$\Gamma_\beta^\alpha = V^\varepsilon \Gamma_\varepsilon^\alpha{}_\beta, \quad \Gamma_{\beta\alpha} = \theta_\varepsilon \Gamma_\beta^\varepsilon{}_\alpha.$$

On the other hand, the fibre is locally represented by

$$x^{\bar{\alpha}} = y^\alpha = \text{const.}, \quad x^\alpha = \text{const.}, \quad x^{\bar{\bar{\alpha}}} = p_\alpha = p_\alpha,$$

p_α being considered as parameters. Thus, on differentiating with respect to p_α , we easily see that the vector fields $E_{(\bar{\beta})} = {}^{vv}(dx^{\bar{\beta}})$ ($\bar{\beta} = 2n+1, \dots, 3n$) with components

$$E_{(\bar{\beta})} : \left(E_{(\bar{\beta})}^A \right) = \partial_{(\bar{\beta})} x^A = \begin{pmatrix} \partial_{\bar{\beta}} y^\alpha \\ \partial_{\bar{\beta}} x^\alpha \\ \partial_{\bar{\beta}} p_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta_\alpha^{\bar{\beta}} \end{pmatrix}$$

is tangent to the fibre, where

$$\delta_\alpha^{\bar{\beta}} = A_\alpha^{\bar{\beta}} = \frac{\partial x^{\bar{\beta}}}{\partial x^\alpha}.$$

Let ω be an 1-form with local components ω_α on M_n , so that ω is a 1-form with local expression $\omega = \omega_\alpha dx^\alpha$. We denote by $E\omega$ the vector field with local components

$$E\omega : \left(E_{(\bar{\beta})}^A \omega_\beta \right) = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \quad (8)$$

which is tangent to the fibre. Then a mapping

$$E : \mathfrak{S}_1^0(M_n) \rightarrow \mathfrak{S}_0^1(t^*(M_n))$$

is defined by (8) and so an isomorphism of $\mathfrak{S}_1^0(M_n)$ in to $\mathfrak{S}_0^1(t^*(M_n))$ [10].

We consider in $\pi^{-1}(U)$ $3n$ local vector fields $B_{(\bar{\beta})}$, $C_{(\beta)}$ and $E_{(\bar{\beta})}$ along $\beta_\theta(T(M_n))$, which are respectively represented by

$$B_{(\bar{\beta})} = B \frac{\partial}{\partial x^{\bar{\beta}}}, \quad C_{(\beta)} = C \frac{\partial}{\partial x^\beta}, \quad E_{(\bar{\beta})} = E dx^{\bar{\beta}}.$$

Theorem 1. Let X be a vector field on $T(M_n)$. We have along $\beta_\theta(T(M_n))$ the formula

$${}^{cc}X = CX + B(L_V X) + E(-L_X \theta),$$

where $L_V X$ denotes the Lie derivative of X with respect to V , and $L_X \theta$ denotes the Lie derivative of θ with respect to X [10].

On the other hand, on putting $C_{(\bar{\beta})} = E_{(\bar{\beta})}$, we write the adapted frame of $\beta_\theta(T(M_n))$ as $\{B_{(\bar{\beta})}, C_{(\beta)}, C_{(\bar{\beta})}\}$. The adapted frame $\{B_{(\bar{\beta})}, C_{(\beta)}, C_{(\bar{\beta})}\}$ of $\beta_\theta(T(M_n))$ is given by the matrix

$$\tilde{A} = (\tilde{A}_B^A) = \begin{pmatrix} \delta_\beta^\alpha & \partial_\beta V^\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \partial_\beta \theta_\alpha & \delta_\alpha^\beta \end{pmatrix}. \quad (9)$$

Since the matrix $\tilde{A}$ in (9) is non-singular, it has the inverse. Denoting this inverse by $(\tilde{A})^{-1}$, we have

$$(\tilde{A})^{-1} = (\tilde{A}_C^B)^{-1} = \begin{pmatrix} \delta_\theta^\beta & -\partial_\theta V^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta \theta_\beta & \delta_\beta^\theta \end{pmatrix}, \quad (10)$$

where $\tilde{A}(\tilde{A})^{-1} = (\tilde{A}_B^A)(\tilde{A}_C^B)^{-1} = \delta_C^A = \tilde{I}$, where $A = (\bar{\alpha}, \alpha, \bar{\alpha})$, $B = (\bar{\beta}, \beta, \bar{\beta})$, $C = (\bar{\theta}, \theta, \bar{\theta})$.

Then we see from Theorem 1 that the complete lift ${}^{cc}X$ of a vector field $X \in \mathfrak{S}_0^1(T(M_n))$ has along $\beta_\theta(T(M_n))$ components of the form

$${}^{cc}X : \begin{pmatrix} L_V X^\alpha \\ X^\alpha \\ -L_X \theta_\alpha \end{pmatrix}$$

with respect to the adapted frame $\{B_{(\bar{\beta})}, C_{(\beta)}, C_{(\bar{\beta})}\}$ [10].

Theorem 2. The complete lift ${}^{cc}X$ of a vector field X in M_n to $t^*(M_n)$ is tangent to the cross-section $\beta_\theta(T(M_n))$ determined by a 1-form θ and vector field V in M_n if and only if

$$L_X \theta = 0, L_V X = 0,$$

where $L_V X$ denotes the Lie derivative of X with respect to V , and $L_X \theta$ denotes the Lie derivative of θ with respect to X .

BX , CX and $E\omega$ also have components:

$$BX = \begin{pmatrix} X^\alpha \\ 0 \\ 0 \end{pmatrix}, \quad CX = \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, \quad E\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix} \quad (11)$$

respectively, with respect to the adapted frame $\{B_{(\bar{\beta})}, C_{(\beta)}, C_{(\bar{\beta})}\}$ of the cross-section $\beta_\theta(T(M_n))$ determined by a 1-form θ on $T(M_n)$ [10].

2 Complete Lift of Tensor Fields of Type (1,1) on a Cross-Section in Semi-Cotangent Bundle

Suppose now that $F \in \mathfrak{S}_1^1(T(M_n))$ and F has local components F_β^α in a neighborhood U of M_n , $F = F_\beta^\alpha \partial_\alpha \otimes dx^\beta$. Then the semi-cotangent (pull-back) bundle $t^*(M_n)$ of cotangent bundle $T^*(M_n)$ by using projection of the tangent bundle $T(M_n)$ admits the complete lift ${}^{cc}F$ of F with components [11]:

$${}^{cc}F = ({}^{cc}F_J^I) = \begin{pmatrix} F_\beta^\alpha & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix}, \quad (12)$$

with respect to the coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\bar{\alpha}}})$ on $t^*(M_n)$. Then ${}^{cc}F$ has components F_B^A given by

$${}^{cc}F = ({}^{cc}F_B^A) = \begin{pmatrix} F_\beta^\alpha & L_V F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \phi_F \theta & F_\alpha^\beta \end{pmatrix} \quad (13)$$

with respect to the adapted frame $\{B_{(\bar{\beta})}, C_{(\beta)}, C_{(\bar{\bar{\beta}})}\}$ of the cross-section $\beta_\theta(T(M_n))$ determined by a 1-form θ in $T(M_n)$, where $A = (\bar{\alpha}, \alpha, \bar{\bar{\alpha}})$, $B = (\bar{\beta}, \beta, \bar{\bar{\beta}})$ [10]. Also, the component ${}^{cc}F_\beta^{\bar{\bar{\alpha}}}$ of ${}^{cc}F_B^A$ is defined as Tachibana operator $\phi_F \theta$ of F , i.e.,

$${}^{cc}F_\beta^{\bar{\bar{\alpha}}} = \phi_F \theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) \theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma,$$

and $L_V F_\beta^\alpha$ denotes the Lie derivative of F_β^α with respect to V , i.e.,

$$L_V F_\beta^\alpha = V^\gamma \partial_\gamma F_\beta^\alpha + F_\gamma^\alpha \partial_\beta V^\gamma - F_\beta^\gamma \partial_\gamma V^\alpha.$$

3 Adapted Frames and Diagonal Lifts of Affinor Fields

Let ∇ be a symmetric affine connection in M_n . In each coordinate neighborhood $\{U, x^\alpha\}$ of M_n , we put

$$X_{(\alpha)} = \frac{\partial}{\partial x^\alpha}, \quad \theta^{(\alpha)} = dx^\alpha.$$

Then $3n$ local vector fields $Y_{(\alpha)}$, ${}^{HH}X_{(\alpha)}$ and ${}^{vv}\theta^{(\alpha)}$ have respectively components of the form

$$Y_{(\alpha)} : \begin{pmatrix} \delta_\alpha^\beta \\ 0 \\ 0 \end{pmatrix}, \quad {}^{HH}X_{(\alpha)} : \begin{pmatrix} -\Gamma_\beta^\alpha \\ \delta_\alpha^\beta \\ \Gamma_{\beta\alpha} \end{pmatrix}, \quad {}^{vv}\theta^{(\alpha)} : \begin{pmatrix} 0 \\ 0 \\ \delta_\beta^\alpha \end{pmatrix} \quad (14)$$

with respect to the induced coordinates $(x^{\bar{\alpha}}, x^\alpha, x^{\bar{\bar{\alpha}}})$ in $\pi^{-1}(U)$, where we have used (7). We call the set $\{Y_{(\alpha)}, {}^{HH}X_{(\alpha)}, {}^{vv}\theta^{(\alpha)}\}$ the frame adapted to the symmetric affine connection ∇ in $\pi^{-1}(U)$. On putting

$$\widehat{e}_{(\bar{\alpha})} = Y_{(\alpha)}, \quad \widehat{e}_{(\alpha)} = {}^{HH}X_{(\alpha)}, \quad \widehat{e}_{(\bar{\bar{\alpha}})} = {}^{vv}\theta^{(\alpha)} \quad (15)$$

we write the adapted frame as

$$\{\widehat{e}_{(B)}\} = \{\widehat{e}_{(\bar{\alpha})}, \widehat{e}_{(\alpha)}, \widehat{e}_{(\bar{\bar{\alpha}})}\}. \quad (16)$$

The adapted frame $\{\widehat{e}_{(B)}\} = \{\widehat{e}_{(\overline{\alpha})}, \widehat{e}_{(\alpha)}, \widehat{e}_{(\overline{\alpha})}\}$ is given by the matrix

$$\widehat{A} = (\widehat{A}_B^A) = \begin{pmatrix} \delta_\beta^\alpha - \Gamma_\beta^\alpha & 0 \\ 0 & \delta_\beta^\alpha \\ 0 & \Gamma_{\beta\alpha} & \delta_\alpha^\beta \end{pmatrix}. \quad (17)$$

Since the matrix $\widehat{A}$ in (17) is non-singular, it has the inverse. Denoting this inverse by $(\widehat{A})^{-1}$, we have

$$(\widehat{A})^{-1} = (\widehat{A}_C^B)^{-1} = \begin{pmatrix} \delta_\theta^\beta & \Gamma_\theta^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\Gamma_{\theta\beta} & \delta_\beta^\theta \end{pmatrix}, \quad (18)$$

where $\widehat{A}(\widehat{A})^{-1} = (\widehat{A}_B^A)(\widehat{A}_C^B)^{-1} = \delta_C^A = \widetilde{I}$, where $A = (\overline{\alpha}, \alpha, \overline{\alpha})$, $B = (\overline{\beta}, \beta, \overline{\beta})$, $C = (\overline{\theta}, \theta, \overline{\theta})$.

If we take account of (16), we see that the diagonal lift ${}^{DD}F$ of $F \in \mathfrak{S}_1^1(T(M_n))$ has components [10]:

$${}^{DD}F = ({}^{DD}F_J^I) = \begin{pmatrix} -F_\beta^\alpha & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon - \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \Gamma_{\beta\sigma} F_\alpha^\sigma + \Gamma_{\alpha\sigma} F_\beta^\sigma - F_\alpha^\beta \end{pmatrix}, \quad (19)$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\alpha}})$ on $t^*(M_n)$, where

$$\Gamma_\varepsilon^\alpha = y^\gamma \Gamma_{\gamma\varepsilon}^\alpha, \quad \Gamma_{\alpha\sigma} = p_\gamma \Gamma_{\alpha\sigma}^\gamma.$$

which proves (19).

We now see, from (16), that the diagonal lift ${}^{DD}F$ of $F \in \mathfrak{S}_1^1(T(M_n))$ has components of the form

$${}^{DD}F = ({}^{DD}F_B^A) = \begin{pmatrix} -F_\beta^\alpha & 0 & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & 0 & -F_\alpha^\beta \end{pmatrix}$$

with respect to the adapted frame $\{\widehat{e}_{(B)}\}$ in $t^*(M_n)$.

We now obtain from (19) that the diagonal lift ${}^{DD}F$ of an affinor field $F \in \mathfrak{S}_1^1(T(M_n))$ has along $\beta_\theta(T(M_n))$ components of the form [10]:

$${}^{DD}F : \begin{pmatrix} -F_\beta^\alpha - (\nabla_\varepsilon V^\alpha) F_\beta^\varepsilon - (\nabla_\beta V^\varepsilon) F_\varepsilon^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -(\nabla_\beta \theta_\sigma) F_\alpha^\sigma - (\nabla_\alpha \theta_\sigma) F_\beta^\sigma - F_\alpha^\beta \end{pmatrix}, \quad (20)$$

with respect to the adapted frame $\{B_{(\overline{\beta})}, C_{(\beta)}, C_{(\overline{\beta})}\}$.

Then we see from (7) that the horizontal lift ${}^{HH}X$ of a vector field $X \in \mathfrak{S}_0^1(T(M_n))$ has along $\beta_\theta(T(M_n))$ components of the form

$${}^{HH}X : \begin{pmatrix} -X^\beta (\nabla_\beta V^\alpha) \\ X^\alpha \\ -(\nabla_\beta \theta_\alpha) X^\beta \end{pmatrix} \quad (21)$$

with respect to the adapted frame $\{B_{(\overline{\beta})}, C_{(\beta)}, C_{(\overline{\beta})}\}$ [10].

Using (7), (20) and (21), we have along $\beta_\theta(T(M_n))$:

Theorem 3. If F and X are affnor and vector fields on $T(M_n)$, and $\omega \in \mathfrak{S}_1^0(M_n)$, then with respect to a symmetric affine connection ∇ in M_n , we have

- (i) ${}^{DD}F({}^{HH}X) = {}^{HH}(FX)$,
- (ii) ${}^{DD}F({}^{vv}\omega) = -{}^{vv}(\omega \circ F)$ [10].

Theorem 4. If $F, G \in \mathfrak{S}_1^1(M_n)$, then with respect to a symmetric affine connection ∇ in M_n , we have

$${}^{DD}F{}^{DD}G + {}^{DD}G{}^{DD}F = {}^{HH}(FG + GF).$$

Proof. If $\omega \in \mathfrak{S}_1^0(M_n)$ and $X \in \mathfrak{S}_0^1(T(M_n))$, then by Theorem 3 and ([11], ${}^{HH}F{}^{HH}X = {}^{HH}(F \circ X)$), we have

$$\begin{aligned} ({}^{DD}F{}^{DD}G + {}^{DD}G{}^{DD}F){}^{HH}X &= {}^{DD}F({}^{HH}GX) + {}^{DD}G({}^{HH}FX) \\ &= {}^{HH}(FGX) + {}^{HH}(GFX) \\ &= {}^{HH}((FG + GF)X) \\ &= {}^{HH}(FG + GF){}^{HH}X. \end{aligned}$$

Thus, we have ${}^{DD}F{}^{DD}G + {}^{DD}G{}^{DD}F = {}^{HH}(FG + GF)$.

Theorem 5. If $F, G \in \mathfrak{S}_1^1(M_n)$, then with respect to a symmetric affine connection ∇ in M_n , we have

$${}^{DD}F{}^{HH}G + {}^{DD}G{}^{HH}F = {}^{HH}F{}^{DD}G + {}^{HH}G{}^{DD}F = {}^{DD}(FG + GF).$$

Proof. If $\omega \in \mathfrak{S}_1^0(M_n)$ and $F, G \in \mathfrak{S}_1^1(T(M_n))$, then by Theorem 3 and ([11], ${}^{HH}F{}^{vv}\omega = {}^{vv}(\omega \circ F)$), we have

$$\begin{aligned} ({}^{DD}F{}^{DD}G + {}^{DD}G{}^{DD}F){}^{vv}\omega &= -{}^{DD}F({}^{vv}(\omega \circ G)) - {}^{DD}G({}^{vv}(\omega \circ F)) \\ &= {}^{vv}(\omega \circ GF) - {}^{vv}(\omega \circ FG) \\ &= {}^{vv}(\omega \circ (GF + FG)) \\ &= {}^{HH}(GF + FG){}^{vv}\omega. \end{aligned}$$

Thus, we have Theorem 5.

Putting $F = G$ in Theorem 4 and Theorem 5, we have

$$\begin{aligned} {}^{HH}F{}^{DD}F &= {}^{DD}F{}^{HH}F = {}^{DD}(F^2) \\ ({}^{DD}F)^{2p} &= {}^{HH}(F^{2p}), ({}^{DD}F)^{2p+1} = {}^{DD}(F^{2p+1}), (p = 1, 2, \dots) \end{aligned}$$

for any $F \in \mathfrak{S}_1^1(T(M_n))$.

Theorem 6. The diagonal lift $\hat{J}$ of the identity tensor field I of type $(1, 1)$ has the components

$$\hat{J}: \begin{pmatrix} -\delta_{\beta}^{\alpha} & 2\Gamma_{\beta}^{\alpha} & 0 \\ 0 & \delta_{\beta}^{\alpha} & 0 \\ 0 & 2\Gamma_{\beta\alpha} & -\delta_{\alpha}^{\beta} \end{pmatrix}. \quad (22)$$

Proof. Since the matrix $\hat{J}$ in (22) is non-singular, it has the inverse. Denoting this inverse by $(\hat{J})^{-1}$, we have

$$(\hat{J})^{-1} = (\hat{J}_C^B)^{-1} = \begin{pmatrix} -\delta_{\theta}^{\beta} & 2\Gamma_{\theta}^{\beta} & 0 \\ 0 & \delta_{\theta}^{\beta} & 0 \\ 0 & 2\Gamma_{\theta\beta} & -\delta_{\beta}^{\theta} \end{pmatrix}, \quad (23)$$

where $\hat{J}(\hat{J})^{-1} = (\hat{J}_B^A)(\hat{J}_C^B)^{-1} = \delta_C^A = \tilde{I}$, where $A = (\bar{\alpha}, \alpha, \bar{\alpha})$, $B = (\bar{\beta}, \beta, \bar{\beta})$, $C = (\bar{\theta}, \theta, \bar{\theta})$. In fact, from (22) and (23), we easily see that

$$\begin{aligned}\hat{J}(\hat{J})^{-1} &= (\hat{J}_B^A)(\hat{J}_C^B)^{-1} \\ &= \begin{pmatrix} -\delta_\beta^\alpha & 2\Gamma_\beta^\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & 2\Gamma_{\beta\alpha} & -\delta_\alpha^\beta \end{pmatrix} \begin{pmatrix} -\delta_\theta^\beta & 2\Gamma_\theta^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & 2\Gamma_{\theta\beta} & -\delta_\beta^\theta \end{pmatrix} \\ &= \begin{pmatrix} \delta_\theta^\alpha & 2\Gamma_\theta^\alpha - 2\Gamma_\theta^\alpha & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & 2\Gamma_{\theta\alpha} - 2\Gamma_{\theta\alpha} & \delta_\alpha^\theta \end{pmatrix} \\ &= \begin{pmatrix} \delta_\theta^\alpha & 0 & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & 0 & \delta_\alpha^\theta \end{pmatrix} \\ &= \delta_C^A \\ &= \hat{I}.\end{aligned}$$

References

- [1] S. Eker, Seiberg–Witten equations on the Strictly–Pseudoconvex CR-7 manifold, *Miskolc Mathematical Notes*, 20, No1, (2019), 233–243.
- [2] S. Eker, The Bochner vanishing theorems on the conformal killing vector fields, *TWMS Journal of Applied and Engineering Mathematics*, 9, No. 1, (2019), 114–120.
- [3] D. Husemoller, *Fibre Bundles*, Springer, New York, 1994.
- [4] C.J. Isham, "Modern differential geometry for physicists", World Scientific, 1999.
- [5] H.B. Lawson and M.L. Michelsohn, *Spin Geometry*, Princeton University Press., Princeton, 1989.
- [6] L.S. Pontryagin, *Characteristic classes of differentiable manifolds*, Transl. Amer. Math. Soc., 7 (1962) 279–331.
- [7] N. Steenrod, *The Topology of Fibre Bundles*. Princeton University Press., Princeton, 1951.
- [8] V. V. Vishnevskii, *Integrable affinor structures and their plural interpretations*, Geometry, 7.J. Math. Sci., (New York) 108, no. 2 (2002) 151–187.
- [9] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker, Inc., New York, 1973.
- [10] F. Yıldırım, *Diagonal lift in the semi-cotangent bundle and its applications*. Turk J Math, 42, no.3 (2018) 1312–1327.
- [11] F. Yıldırım, *On a special class of semi-cotangent bundle*, Proceedings of the Institute of Mathematics and Mechanics, (ANAS) 41 no. 1 (2015) 25–38.
- [12] F. Yıldırım and A. Salimov, *Semi-cotangent bundle and problems of lifts*, Turk J. Math, 38 (2014) 325–339.
- [13] F. Yıldırım and A. Salimov, *Horizontal Lift Problems in a Special Class of Semi-Cotangent Bundle*, Conference on Mathematical Sciences and Applications (IECMSA), Wien, 324, 2014.
- [14] W.A. Poor, *Differential Geometric Structures*, New York, McGraw-Hill, 1981.
- [15] N.M. Ostianu, *Step-fibred spaces*, Tr Geom Sem 5, Moscow. (VINITI), 259–309, 1974.

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