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Limiting Case of the Spin Hypersurface Dirac Operator arising in the positive mass theorem for black holes

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Abstract

In this paper, we give an explicit form of the scalar curvature for the limiting case of the eigenvalue of the hypersurface Dirac operator which arises in the positive mass theorem for black holes. Then, we show that the hypersurface is an Einstein.

Keywords: Spin and Spin^c geometry, Dirac operator, Estimation of eigenvalues

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1 Introduction

On a compact Riemannian *Spin*–manifold, mathematicians and physicists have been investigated the spectrum of the Dirac operator to obtain subtle information about the topology and geometry of the manifold and its hypersurface [2–4, 6, 10, 13, 14, 17].

In 1963 A. Lichnerowicz proved with the help of the Lichnerowicz–formula that, on a compact Riemannian *Spin*–manifold (M, g) , for any eigenvalue λ of the Dirac operator the inequality

$$\lambda^2 \geq \frac{1}{4} \inf_M R \quad (1)$$

holds [16]. Here R is the scalar curvature of M . The proof of the above inequality is based on the classical spinorial Levi–Civita connection. Then, by using a modified spinorial Levi–Civita connection, T. Friedrich [5] improved the estimate given in (1) for dimension $n \geq 2$ as follows

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R. \quad (2)$$

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Later on, an interesting topological lower bound estimation is given by C. Bär in two dimensional manifold as follows

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\text{Area}(M, g)}, \quad (3)$$

where $\chi(M)$ is an Euler–Poincare characteristic of M [1].

After this point, O. Hijazi added a new geometric invariants to obtain an optimal lower bound for the square of the eigenvalue of the Dirac operator. The author improved inequality (2) by using the conformal covariance of the Dirac operator on a compact Riemannian *Spin*–manifold (M, g) of dimension $n \geq 3$,

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1, \quad (4)$$

where μ_1 is the first eigenvalue of the Yamabe operator L given by

$$L := 4\frac{n-1}{n-2}\Delta_g + R \quad (5)$$

and Δ_g is the positive Laplacian acting on functions [9]. Also, in 1995 O. Hijazi [11] modified the spinorial Levi–Civita connection in terms of the Energy–Momentum tensor Q_Ψ to obtain the following lower bound

$$\lambda^2 \geq \inf_M \left(\frac{R}{4} + |Q_\Psi|^2 \right). \quad (6)$$

In addition, in the limiting case of (6) O. Hijazi obtained the following relations

$$\begin{aligned} (\text{tr}(Q_\Psi))^2 &= \frac{1}{4}R + |Q_\Psi|^2, \\ \text{grad}(\text{tr}(Q_\Psi)) &= -\text{div}(Q_\Psi), \end{aligned} \quad (7)$$

where $\text{tr}(Q_\Psi)$ is the trace of Energy–Momentum tensor Q_Ψ .

Using the conformal covariance of the Dirac operator on a compact Riemannian *Spin*–manifold (M, g) , he proved that, any square of the eigenvalue λ of the Dirac operator D satisfies

$$\lambda^2 \geq \begin{cases} \frac{1}{4}\mu_1 + \inf_M |Q_\Psi|^2, & \text{if } n \geq 3, \\ \frac{\pi\chi(M)}{\text{Area}(M, g)} + \inf_M |Q_\Psi|^2, & \text{if } n = 2, \end{cases} \quad (8)$$

where μ_1 is the first eigenvalue of the Yamabe operator L .

After that, on a compact *Spin*–hypersurface similar studies has begun [12, 18]. Zhang obtain an estimates for the eigenvalue of the operator $\tilde{D}^*\tilde{D}$ defined in [18] in terms of the mean curvature and scalar curvature

$$\lambda \geq \sup_{\alpha} \inf_M \left(\frac{R}{n\alpha^2 - 2\alpha + 1} - \frac{(n-1)}{(1-n\alpha)^2} |H|^2 \right), \quad (9)$$

where α is any real number, $\alpha \neq \frac{1}{n}$ if $H \neq 0$. On a compact *Spin*–hypersurface $M^n \subset N^{n+1}$ of dimension $n \geq 2$, using conformal deformations of the metric, Hijazi and Zhang improved (9) for the eigenvalue of D_H (i.e. λ_H^2 is an eigenvalue of $\tilde{D}^*\tilde{D}$)

$$\lambda \geq \sup_{\alpha, u} \inf_M \left(\frac{\bar{R}e^{2u}}{n\alpha^2 - 2\alpha + 1} - \frac{(n-1)}{(1-n\alpha)^2} |H|^2 \right), \quad (10)$$

where $\bar{R}$ is the scalar curvature of M associated to a conformal deformation of metric g and for some real–valued function on N [12]. Moreover, they investigated the limiting case of the above inequality and they show that the hypersurface is

an Einstein. Also, they obtain the following estimates for the eigenvalue of the hypersurface Dirac operator D_p defined in [12],

$$\lambda^2 \geq \begin{cases} \frac{1}{4} \sup_{b,u} \inf_M \left(\frac{\bar{R}e^{2u}}{1+nb^2-2b} - \frac{(n-1)}{(1-nb)^2} |P|^2 \right); \\ \frac{1}{4} \left(\sqrt{\frac{n}{n-1}} \mu_1 - \sup_M |P|^2 \right) \text{ for } n \geq 3; \\ \frac{1}{4} \left(\sqrt{\frac{16\pi}{\text{Area}(M)}} - \sup_M |P|^2 \right) \text{ } n = 2, \text{ genus}=0, \end{cases} \quad (11)$$

where b is any real-valued function on N .

This paper is organized as follows. At first, we introduce some basic facts concerning hypersurface Dirac operator. Then, we show that the hypersurface manifold is Einstein manifold with constant Ricci curvature by considering the limiting case of the above inequality.

2 Hypersurface Dirac Operator

Let (N, g_N, p) be an $(n+1)$ -dimensional compact Riemannian *Spin*-manifold with metric tensor g_N , 2-tensor p and M be an n -dimensional *Spin*-hypersurface in N with its induced metric g . On a compact Riemannian *Spin*-manifold N , one can construct a spinor bundle denoted by $\mathbb{S}$ and globally defined along M called hypersurface spinor bundle of M [18]. Let $\tilde{\nabla}$ be the Levi-Civita connection of N , and ∇ be its induced connection on M . Accordingly, $\tilde{\nabla}$ and ∇ can be lifted to the hypersurface spinor bundle $\mathbb{S}$ and denoted by the same symbol. The Dirac operator D of M defined by ∇ on $\mathbb{S}$ and the hypersurface Dirac operator $\tilde{D}$ by $\tilde{\nabla}$ on $\mathbb{S}$.

Recall that, $\mathbb{S}$ carries a natural positive definite Hermitian metric on $\Gamma(\mathbb{S})$ denoted by $(\cdot, \cdot)$ and satisfies, for any covector field $v \in T^*N$, and any spinor fields $\Psi, \Phi \in \Gamma(\mathbb{S})$

$$(v \cdot \Psi, v \cdot \Phi) = |v|^2 (\Psi, \Phi). \quad (12)$$

This metric is globally-defined along M . Also, $\tilde{\nabla}$ and its induced connection ∇ are compatible with the Hermitian metric $(\cdot, \cdot)$ [15].

Let $x \in M$ be a fixed point and e_α be an orthonormal basis of $T_x N$ with e_0 normal to M and e_i tangent to M such that for $1 \leq i, j \leq n$,

$$(\nabla_i e_j)_x = (\tilde{\nabla}_0 e_j)_x = 0, \quad (13)$$

Moreover, let e^α be the dual coframe of e_α . Then, for $1 \leq i, j \leq n$

$$\begin{aligned} (\tilde{\nabla}_i e^j)_x &= -h_{ij} e^0, \\ (\tilde{\nabla}_i e^0)_x &= h_{ij} e^j, \end{aligned}$$

where $h_{ij} = (\tilde{\nabla}_i e_0, e_j)$ are the components of the second fundamental form at x . Let P be a function defined as [12]

$$P := g_N^{ij} p_{ij} \Big|_M. \quad (14)$$

Then, the hypersurface Dirac operator which arises in the positive mass theorem for black holes defined as follows [7, 8, 13, 19, 20].

$$D_{HP} = e^0 \cdot D - \frac{H}{2} - \frac{\sqrt{-1}}{2} P e^0 \cdot. \quad (15)$$

As in [12], in this paper we consider the hypersurface dirac operator D_p defined as

$$D_p := e^0 \cdot D - \frac{\sqrt{-1}}{2} P e^0 \cdot. \quad (16)$$

In the following, we consider limiting case of λ_p .

3 Limiting Case of the Hypersurface manifold endowed with Spin Structure

In this section, by using the modified spinorial Levi–Civita connection used to obtain the lower bound of the eigenvalue of the hypersurface Dirac operator arises in the positive mass theorem for black holes, we obtain the scalar curvature corresponding to the limiting case of the eigenvalue of the hypersurface Dirac operator. Then, we show that the hypersurface manifold (M, g) is an Einstein manifold. Finally, we give an explicit form of the eigenvalue in the limiting case.

In the following theorem, to obtain this lower bound we consider the following modified spinorial Levi–Civita connection defined as [12]:

$$\nabla_i^b \Psi = \nabla_i \Psi + \left(\frac{1-b}{2(1-nb)} \right) \sqrt{-1} P e^i \cdot \Psi - b e^i \cdot e^0 \cdot D_p \Psi. \quad (17)$$

Theorem 1. *Let $M \subset N$ be a compact Riemannian Spin–hypersurface. If λ_p achieves its minimum, M is an Einstein manifold and it has constant Ricci curvature and constant P . Also the following holds:*

$$\begin{aligned} R &= n(n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1-nb_0)^4} P, \\ \lambda_p^2 &= \frac{1}{4} \sup_{b_0} \frac{(n-1)^2}{(1-nb_0)^4} P, \\ R_{ik} &= (n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1-nb_0)^4} P \delta_{ik}, \end{aligned} \quad (18)$$

where b_0 is chosen such that the right side of (11) achieves its maximum.

Proof. If λ_p achieves its minimum, then $\nabla_i^b \Psi \equiv 0$. This implies

$$\nabla_i \Psi = - \left(\frac{1-b_0}{2(1-nb_0)} \right) \sqrt{-1} P e^i \cdot \Psi + b_0 e^i \cdot e^0 \cdot D_p \Psi, \quad (19)$$

where b is any real–valued function on M . Performing its Clifford multiplication by e^i , for any spinor field $\Psi \in \Gamma(\mathbb{S})$ and for any $i, 1 \leq i \leq n$, yields

$$e^i \cdot \nabla_i \Psi = \left(\frac{1-b_0}{2(1-nb_0)} \right) \sqrt{-1} P \Psi - b_0 e^0 \cdot D_p \Psi. \quad (20)$$

Summing over i , gives

$$\begin{aligned} D\Psi &= n \left(\frac{1-b_0}{2(1-nb_0)} \right) \sqrt{-1} P \Psi - nb_0 e^0 \cdot D_p \Psi \\ &= n \left(\frac{1-b_0}{2(1-nb_0)} \right) \sqrt{-1} P \Psi - nb_0 e^0 \cdot (e^0 \cdot D\Psi - \frac{\sqrt{-1}}{2} P e^0 \cdot \Psi) \\ &= n \left(\frac{1-b_0}{2(1-nb_0)} \right) \sqrt{-1} P \Psi + nb_0 D\Psi - nb_0 \frac{\sqrt{-1}}{2} P \Psi \\ &= \left(\frac{n^2 b_0^2 - 2b_0 n + n}{2(1-nb_0)} \right) \sqrt{-1} P \Psi + nb_0 D\Psi. \end{aligned} \quad (21)$$

As a result, we get

$$D\Psi = \left(\frac{n^2 b_0^2 - 2b_0 n + n}{2(1-nb_0)^2} \right) \sqrt{-1} P \Psi. \quad (22)$$

Again, considering (19), we obtain

$$\begin{aligned}
 \nabla_i \Psi &= -\left(\frac{1-b_0}{2(1-nb_0)}\right)\sqrt{-1}Pe^i \cdot \Psi + b_0e^i \cdot e^0 \cdot \left(e^0 \cdot D\Psi - \frac{\sqrt{-1}}{2}Pe^0 \cdot \Psi\right) \\
 &= -\left(\frac{1-b_0}{2(1-nb_0)}\right)\sqrt{-1}Pe^i \cdot \Psi + b_0e^i \cdot \left(-D\Psi + \frac{\sqrt{-1}}{2}P\Psi\right) \\
 &= -\left(\frac{1-b_0}{2(1-nb_0)}\right)\sqrt{-1}Pe^i \cdot \Psi + b_0e^i \cdot \left(-\left(\frac{n^2b_0^2-2b_0n+n}{2(1-nb_0)^2}\right)\sqrt{-1}P\Psi + \frac{\sqrt{-1}}{2}P\Psi\right) \\
 &= -\left(\frac{nb_0^2-2b_0+1}{2(1-nb_0)^2}\right)\sqrt{-1}Pe^i \cdot \Psi \\
 &= \sqrt{-1}\tilde{P}e^i \cdot \Psi,
 \end{aligned} \tag{23}$$

where $\tilde{P} = -\left(\frac{nb_0^2-2b_0+1}{2(1-nb_0)^2}\right)P$. Accordingly,

$$\begin{aligned}
 \sum_{k,l} \frac{1}{4}R_{ijkl}e^k \cdot e^l \cdot \Psi &= \left(\nabla_j \nabla_i - \nabla_i \nabla_j\right)\Psi \\
 &= \nabla_j(\sqrt{-1}\tilde{P}e^i \cdot \Psi) - \nabla_i(\sqrt{-1}\tilde{P}e^j \cdot \Psi) \\
 &= \sqrt{-1}\nabla_j\tilde{P}e^i \cdot \Psi + \sqrt{-1}\tilde{P}e^i \cdot \nabla_j\Psi - \sqrt{-1}\nabla_i\tilde{P}e^j \cdot \Psi - \sqrt{-1}\tilde{P}e^j \cdot \nabla_i\Psi.
 \end{aligned} \tag{24}$$

Clifford multiplication with e^j , we obtain

$$\begin{aligned}
 \sum_k \frac{1}{2}R_{ik}e^k \cdot \Psi &= \sum_{j,k,l} \frac{1}{4}R_{ijkl}e^j \cdot e^k \cdot e^l \cdot \Psi \\
 &= \sqrt{-1}\nabla_j\tilde{P}e^j \cdot e^i \cdot \Psi + \sqrt{-1}\tilde{P}e^j \cdot e^i \cdot \nabla_j\Psi - \sqrt{-1}\nabla_i\tilde{P}e^j \cdot e^j \cdot \Psi \\
 &\quad - \sqrt{-1}\tilde{P}e^j \cdot e^j \cdot \nabla_i\Psi \\
 &= \sqrt{-1}\nabla_j\tilde{P}e^j \cdot e^i \cdot \Psi + \sqrt{-1}\tilde{P}e^j \cdot e^i \cdot \nabla_j\Psi + n\sqrt{-1}\nabla_i\tilde{P}\Psi \\
 &\quad + n\sqrt{-1}\tilde{P}\nabla_i\Psi \\
 &= \sqrt{-1}\nabla_j\tilde{P}(-e^i \cdot e^j - 2\delta_{ij})\Psi + \sqrt{-1}\tilde{P}(-e^i \cdot e^j - 2\delta_{ij})\nabla_j\Psi \\
 &\quad + n\sqrt{-1}\nabla_i\tilde{P}\Psi + n\sqrt{-1}\tilde{P}\nabla_i\Psi.
 \end{aligned} \tag{25}$$

Again, Clifford multiplication with e^i , we have

$$\begin{aligned}
 -\frac{1}{2}R\Psi &= \sum_{i,k} \frac{1}{2}R_{ik}e^i \cdot e^k \cdot \Psi \\
 &= \sqrt{-1}\nabla_j\tilde{P}e^j \cdot (-e^i \cdot e^j - 2\delta_{ij})\Psi + \sqrt{-1}\tilde{P}e^j \cdot (-e^i \cdot e^j - 2\delta_{ij})\nabla_j\Psi \\
 &\quad + n\sqrt{-1}e^i\nabla_i\tilde{P}\Psi + n\sqrt{-1}\tilde{P}e^i \cdot \nabla_i\Psi \\
 &= n\sqrt{-1}\nabla_j\tilde{P}e^j \cdot \Psi - 2\sqrt{-1}\nabla_j\tilde{P}e^j \cdot \Psi + n\sqrt{-1}\tilde{P}e^j \cdot \nabla_j\Psi \\
 &\quad - 2\sqrt{-1}\tilde{P}e^j \cdot \nabla_j\Psi + n\sqrt{-1}e^i\nabla_i\tilde{P}\Psi + n\sqrt{-1}\tilde{P}e^i \cdot \nabla_i\Psi \\
 &= 2(n-1)\sqrt{-1}d\tilde{P} \cdot \Psi + 2(n-1)\sqrt{-1}\tilde{P}D\Psi \\
 &= 2(n-1)\sqrt{-1}d\tilde{P} \cdot \Psi - 2n(n-1)\sqrt{-1}\tilde{P}(\sqrt{-1}\tilde{P}\Psi) \\
 &= 2(n-1)\sqrt{-1}d\tilde{P} \cdot \Psi - 2n(n-1)\tilde{P}^2\Psi.
 \end{aligned} \tag{26}$$

Accordingly, $\tilde{P}$ must be constant. Therefore

$$R = 4n(n-1)\tilde{P}^2 = n(n-1)\frac{(nb_0^2-2b_0+1)^2}{(1-nb_0)^4}P. \tag{27}$$

In this case, we have

$$\lambda_p^2 = \frac{1}{4} \sup_{b_0} \frac{(n-1)^2}{(1-nb_0)^4} P.$$

On the other hand,

$$\begin{aligned} \sum_k \frac{1}{2} R_{ik} e^k \cdot \Psi &= \sqrt{-1} \tilde{P} (-e^i \cdot e^j \cdot -2\delta_{ij}) \nabla_j \Psi + n \sqrt{-1} \tilde{P} \nabla_i \Psi \\ &= -\sqrt{-1} \tilde{P} e^i \cdot D \Psi - 2\sqrt{-1} \tilde{P} \nabla_i \Psi + n \sqrt{-1} \tilde{P} \nabla_i \Psi \\ &= n \tilde{P}^2 e^i \cdot \Psi + (n-2) \tilde{P}^2 e^i \cdot \Psi \\ &= 2(n-1) \tilde{P}^2 e^i \cdot \Psi. \end{aligned} \quad (28)$$

Therefore

$$\begin{aligned} R_{ik} &= 4(n-1) \tilde{P}^2 \delta_{ik} \\ &= (n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1-nb_0)^4} P \delta_{ik}. \end{aligned} \quad (29)$$

In this part of the paper, concerning conformal change of the Riemannian metric and using the classic arguments given in [11–13], the limiting case of the square of the eigenvalue λ_p of the hypersurface dirac operator D_p is handled.

Considering the conformal metric $\bar{g}_N = e^{2u} g_N$ given with any real-valued function u on N . Accordingly, the G_u isometry defined between $SO_{g_N} N$ and $SO_{\bar{g}_N} N$ reduces an isometry reduction between the $Spin_{g_N} N$ and $Spin_{\bar{g}_N} N$ principal bundles as well as an isometry between the corresponding hypersurface spinor bundles $\mathbb{S}$ and $\bar{\mathbb{S}} (\equiv G_u \mathbb{S})$, respectively [12]. Denote by $\bar{\Psi} = G_u \Psi$ the corresponding sections of $\bar{\mathbb{S}}$, for any spinorfield Ψ of $\mathbb{S}$. Note that, $\bar{g} := \bar{g}_N|_M$ is induced metric of $\bar{g}_N$ on M . Also, by using the Clifford multiplication on $\bar{\mathbb{S}}$ is given by

$$\bar{e}^i \cdot \bar{\Psi} = \bar{e}^i \cdot \bar{\Psi}, \quad (30)$$

we obtain the following argument [12]

$$\bar{p}_{\bar{i} \bar{j}} = e^{-u} p_{ij}. \quad (31)$$

Using $\bar{p}_{\bar{i} \bar{j}} = e^{-u} p_{ij}$, we get

$$\begin{aligned} \bar{P} &= \bar{g}_N^{\bar{i} \bar{j}} \bar{p}_{\bar{i} \bar{j}}|_M \\ &= e^{2u} g_N(\bar{e}^i, \bar{e}^j) \bar{p}_{\bar{i} \bar{j}}|_M \\ &= g_N(e^i, e^j) e^{-u} p_{ij}|_M \\ &= e^{-u} P. \end{aligned} \quad (32)$$

Under the conformal change of the Riemannian metric, the modified spinorial Levi–Civita connection (17) is transformed as

$$\bar{\nabla}_{e^i}^b \bar{\Psi} = \bar{\nabla}_{e^i} \bar{\Psi} + \left(\frac{1-b}{2(1-nb)} \right) \sqrt{-1} \bar{P} \bar{e}^i \cdot \bar{\Psi} - b \bar{e}^i \cdot \bar{e}^0 \cdot \bar{D}_p \bar{\Psi}. \quad (33)$$

If λ_p achieves its minimum, then $\bar{\nabla}_{e^i}^b \bar{\Psi} = 0$. This means that

$$\bar{\nabla}_{e^i} \bar{\Psi} = - \left(\frac{1-b_0}{2(1-nb_0)} \right) \sqrt{-1} \bar{P} \bar{e}^i \cdot \bar{\Psi} + b_0 \bar{e}^i \cdot \bar{e}^0 \cdot \bar{D}_p \bar{\Psi}. \quad (34)$$

Using the method given in the proof of Theorem (1), we get

$$\bar{D} \bar{\Psi} = \left(\frac{n^2 b_0^2 - 2b_0 n + n}{2(1-nb_0)^2} \right) \sqrt{-1} \bar{P} \bar{\Psi}. \quad (35)$$

Also, (23) transform into

$$\bar{\nabla}_{e^i} \bar{\Psi} = \sqrt{-1} \tilde{P} e^i \cdot \bar{\Psi}, \quad (36)$$

where $\tilde{P} = -\left(\frac{nb_0^2 - 2b_0 + 1}{2(1 - nb_0)^2}\right) \bar{P}$. In addition, scalar curvature of $(M, \bar{g})$ is obtained as

$$\begin{aligned} -\frac{1}{2} \bar{R} \bar{\Psi} &= \sum_{i,k} \frac{1}{2} \bar{R}_{i\bar{k}} e^i \cdot e^{\bar{k}} \cdot \bar{\Psi} \\ &= 2(n-1) \sqrt{-1} d\tilde{P} \cdot \bar{\Psi} - 2n(n-1) (\tilde{P})^2 \bar{\Psi}. \end{aligned} \quad (37)$$

Accordingly, $\tilde{P}$ must be constant. Therefore

$$\begin{aligned} \bar{R} &= 4n(n-1) \tilde{P}^2 = n(n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1 - nb_0)^4} \bar{P}^2 \\ &= e^{-2u} n(n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1 - nb_0)^4} P^2. \end{aligned} \quad (38)$$

In this case, we have

$$\lambda_p^2 = \frac{1}{4} \sup_{b_0} \frac{(n-1)^2}{(1 - nb_0)^4} P.$$

On the other hand,

$$\sum_k \frac{1}{2} \bar{R}_{i\bar{k}} e^{\bar{k}} \cdot \bar{\Psi} = 2(n-1) \tilde{P} e^i \cdot \bar{\Psi}. \quad (39)$$

Therefore

$$\begin{aligned} \bar{R}_{i\bar{k}} &= 4(n-1) \tilde{P}^2 \delta_{i\bar{k}} \\ &= (n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1 - nb_0)^4} \bar{P} \delta_{i\bar{k}} \\ &= (n-1) \frac{(nb_0^2 - 2b_0 + 1)^2}{(1 - nb_0)^4} e^{-2u} P^2 \delta_{i\bar{k}}. \end{aligned} \quad (40)$$

According to above equality, $(M, \bar{g})$ is an Einstein manifold.

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