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On Some Integral Inequalities via Conformable Fractional Integrals

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Abstract

In the present note, we have given a new integral identity via Conformable fractional integrals and some further properties. We have proved some integral inequalities for different kinds of convexity via Conformable fractional integrals. We have also showed that special cases of our findings gave some new inequalities involving Riemann-Liouville fractional integrals.

Keywords: Conformable fractional integrals, m -convexity, s -convexity, quasi-convexity, Euler Beta function

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1 Introduction

We will recall some preliminaries concepts to refresh our memories:

Definition 1. [See [5]] A mapping $f : I \rightarrow [0, \infty)$ is said to be log-convex or multiplicatively convex if $\log f$ is convex or equivalently for all $\theta, \vartheta \in I$ and $\varsigma \in [0, 1]$, one has the inequality:

$$f(\varsigma\theta + (1-\varsigma)\vartheta) \leq [f(\theta)]^\varsigma [f(\vartheta)]^{1-\varsigma} \quad (1.1)$$

We note that a log-convex function satisfy the condition of convexity, but the converse may not necessarily be true.

Definition 2. [See [6]] Let $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (1.2)$$

for all $x, y \in \mathbb{R}_+$ and $t \in [0, 1]$.

In [7], s -convexity introduced by Breckner as a generalization of convex functions. Also, Breckner proved the fact that the set valued map is s -convex only if the associated support function is s -convex function in [8]. Several properties of s -convexity in the first sense are discussed in the paper [6]. Obviously, s -convexity means just convexity when $s = 1$.

Definition 3. (See [5]) A function $f : [a, b] \rightarrow \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(\lambda x + (1 - \lambda)y) \leq \max \{f(x), f(y)\}, \quad (QC)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Clearly, any convex function is quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex.

Let us recall the definition of Riemann-Liouville fractional integrals:

Definition 4. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx, \quad t > a$$

and

$$J_{b-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (x-t)^{\alpha-1} f(x) dx, \quad t < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. Here $J_{a+}^0 f(t) = J_{b-}^0 f(t) = f(t)$

In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

We will mention the Beta function (See [4]):

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is Gamma function.

Incomplete Beta function is defined as:

$$B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0.$$

In spite of its valuable contributions to mathematical analysis, the Riemann-Liouville Fractional integrals have deficiencies. For example the solution of the differential equation that is given as;

$$y^{(\frac{1}{2})} + y = x^{(\frac{1}{2})} + \frac{2}{\Gamma(2.5)} x^{(\frac{3}{2})}, \quad y(0) = 0$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$.

The solution of the above differential equation have caused to imagine on a new and simple representation of the definition of fractional derivative. In [2], Khalil et al. gave a new definition that is called "conformable fractional derivative". They not only proved further properties of this definitions but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. In [1], Abdeljawad gave the following definitions of Right-Left conformable fractional integrals:

Definition 5. Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, \dots$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by

$$(I_{\alpha}^a f)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x) dx$$

Definition 6. Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^b I_{\alpha} f)(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x) dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1 - n = 1$, hence $(I_{n+1}^a f)(t) = (J_{a+}^{n+1} f)(t)$ and $({}^b I_{n+1} f)(t) = (J_{b-}^{n+1} f)(t)$.

In [1] and [2], authors have pointed that the Riemann-Liouville derivatives are not valid for product of two functions. In this case, the inequalities that have been proved by Riemann-Liouville integrals are not valid. The results which are obtained by using the conformable fractional integrals have a wide range of validity. (Let us consider the function f defined as $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $f = x^2 e^x$ which is convex.). The interested readers can find several new integral inequalities via different fractional integral operators in the papers [9–14].

In this paper, some new integral inequalities have been proved by using conformable fractional integrals for functions whose derivatives of absolute values are quasi-convex, s -convex and log-convex functions.

2 Main Results

In order to prove our main theorems, we need the following lemma.

Lemma 1. (See [3]) Let $f: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. Then for all $x \in [a, b]$ and $\alpha \in (n, n+1]$, the following equality holds:

$$\begin{aligned} & \frac{(x-a)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) dt \right. \\ & \quad \left. - \int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) dt \right] \\ & - \frac{(b-x)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) dt \right. \\ & \quad \left. + \int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) dt \right] \\ & = \frac{\Gamma(\alpha-n)(x-a)^{\alpha}}{\Gamma(\alpha+1)(b-a)} [f(x) + f(a)] - \frac{2^{\alpha}}{b-a} \left[{}^x I_{\alpha} f \left(\frac{x+a}{2} \right) + I_{\alpha}^a f \left(\frac{x+a}{2} \right) \right] \\ & \quad + \frac{\Gamma(\alpha-n)(b-x)^{\alpha}}{\Gamma(\alpha+1)(b-a)} [f(x) + f(b)] - \frac{2^{\alpha}}{b-a} \left[{}^x I_{\alpha} f \left(\frac{x+b}{2} \right) + I_{\alpha}^b f \left(\frac{x+b}{2} \right) \right] \end{aligned}$$

where $B_t(a, b)$ is incompleated beta function.

Proof. By using the integration by parts formula in the left hand side of the above equality, the right hand side can be obtained. The details of the proof are left to the interested reader. $\square$

For the simplicity, in the sequel of the paper, we will use following notation

$$\begin{aligned} & F_f(\alpha, n; x) \\ &= \frac{\Gamma(\alpha - n)(x - a)^\alpha}{\Gamma(\alpha + 1)} \frac{[f(x) + f(a)]}{(b - a)} - \frac{2^\alpha}{b - a} \left[{}^x I_\alpha f\left(\frac{x + a}{2}\right) + I_\alpha^a f\left(\frac{x + a}{2}\right) \right] \\ &+ \frac{\Gamma(\alpha - n)(b - x)^\alpha}{\Gamma(\alpha + 1)} \frac{[f(x) + f(b)]}{(b - a)} - \frac{2^\alpha}{b - a} \left[{}^x I_\alpha f\left(\frac{x + b}{2}\right) + I_\alpha^b f\left(\frac{x + b}{2}\right) \right] \end{aligned}$$

The next theorems give new results of conformable fractional integrals by means of Lemma 1 for quasi-convex, s -convex, m -convex and log-convex functions.

Theorem 1. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds for conformable fractional integrals:

$$\begin{aligned} & |F_f(\alpha, n; x)| \\ & \leq \frac{\Gamma(\alpha - n + 1)}{\Gamma(\alpha + 2)(b - a)} \left((x - a)^{\alpha + 1} \sup \{|f'(x)|, |f'(a)|\} + (b - x)^{\alpha + 1} \sup \{|f'(x)|, |f'(b)|\} \right) \end{aligned}$$

with $\alpha \in (n, n + 1]$, $n = 0, 1, 2, \dots$

Proof. Since $|f'|$ is quasi-convex on $[a, b]$ and by Lemma 1, we can write

$$\begin{aligned} I_1 &= \left| \frac{(x - a)^{\alpha + 1}}{n!(b - a)} \left[\int_0^1 \frac{B_t(n + 1, \alpha - n)}{2} f'\left(\frac{1 + t}{2}x + \frac{1 - t}{2}a\right) dt \right. \right. \\ &\quad \left. \left. - \int_0^1 \frac{B_t(n + 1, \alpha - n)}{2} f'\left(\frac{1 - t}{2}x + \frac{1 + t}{2}a\right) dt \right] \right| \\ &\leq \frac{(x - a)^{\alpha + 1}}{n!(b - a)} \left[\int_0^1 \left| \frac{B_t(n + 1, \alpha - n)}{2} \right| \left| f'\left(\frac{1 + t}{2}x + \frac{1 - t}{2}a\right) \right| dt \right. \\ &\quad \left. + \int_0^1 \left| \frac{B_t(n + 1, \alpha - n)}{2} \right| \left| f'\left(\frac{1 - t}{2}x + \frac{1 + t}{2}a\right) \right| dt \right] \\ &\leq \frac{(x - a)^{\alpha + 1}}{n!(b - a)} \sup \{|f'(x)|, |f'(a)|\} \int_0^1 |B_t(n + 1, \alpha - n)| dt \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} I_2 &= \left| \frac{(b - x)^{\alpha + 1}}{n!(b - a)} \left[\int_0^1 \frac{B_t(n + 1, \alpha - n)}{2} f'\left(\frac{1 + t}{2}x + \frac{1 - t}{2}b\right) dt \right. \right. \\ &\quad \left. \left. - \int_0^1 \frac{B_t(n + 1, \alpha - n)}{2} f'\left(\frac{1 - t}{2}x + \frac{1 + t}{2}b\right) dt \right] \right| \\ &\leq \frac{(b - x)^{\alpha + 1}}{n!(b - a)} \sup \{|f'(x)|, |f'(b)|\} \int_0^1 |B_t(n + 1, \alpha - n)| dt \end{aligned}$$

We apply the integration by parts formula to incomplete beta function

$$\begin{aligned} \int_0^1 |B_t(n + 1, \alpha - n)| dt &= B(n + 1, \alpha - n) - B(n + 2, \alpha - n) \\ &= \frac{n! \Gamma(\alpha - n + 1)}{\Gamma(\alpha + 2)} \end{aligned}$$

Adding the above quantities lead to Theorem 1 and the proof is completed. $\square$

Remark 1. Under conditions of Theorem 1, if $\alpha = n + 1$, then

$$\begin{aligned} & \left| \frac{(x-a)^{n+1}}{\Gamma(n+2)(b-a)} [f'(x) + f'(a)] - \frac{2^{n+1}}{(b-a)} [J_{x-}^{n+1} + J_{a+}^{n+1}] \right. \\ & \quad \left. + \frac{(b-x)^{n+1}}{\Gamma(n+2)(b-a)} [f'(x) + f'(b)] - \frac{2^{n+1}}{(b-a)} [J_{x-}^{n+1} + J_{b+}^{n+1}] \right| \\ & \leq \frac{(x-a)^{n+2} \sup\{|f'(x)|, |f'(a)|\} + (b-x)^{n+2} \sup\{|f'(x)|, |f'(b)|\}}{(n+2)!(b-a)} \end{aligned}$$

Theorem 2. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. If $|f'|$ is s -convex in the second sense with $s \in (0, 1]$, then the following inequality holds for conformable fractional integrals:

$$\begin{aligned} & |F_f(\alpha, n; x)| \\ & \leq \frac{(\mathcal{A}_1 + \mathcal{A}_2)}{2^{s+1}n!(b-a)} [(x-a)^{\alpha+1} (|f'(x)| + |f'(a)|) + (b-x)^{\alpha+1} (|f'(x)| + |f'(b)|)] \end{aligned}$$

with $\alpha \in (n, n+1]$, $n = 0, 1, 2, \dots$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given as:

$$\begin{aligned} \mathcal{A}_1 &= \int_0^1 |B_t(n+1, \alpha-n)| (1+t)^s dt = \Gamma(n+1) \Gamma(\alpha-n) \\ & \quad \times \left[\frac{2^s - \Gamma(\alpha+1) {}_2F_1(n+1, -1-s; \alpha+1; -1)}{\Gamma(\alpha+1)(s+1)} \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_2 &= \int_0^1 |B_t(n+1, \alpha-n)| (1-t)^s dt \\ &= \frac{B(n+1, \alpha-n+s+1)}{(s+1)}. \end{aligned}$$

Proof. Since $|f'|$ is s -convex on $[a, b]$ and by Lemma 1, we can write

$$\begin{aligned} Q_1 &= \left| \frac{(x-a)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) dt \right. \right. \\ & \quad \left. \left. - \int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) dt \right] \right| \\ & \leq \frac{(x-a)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \left| \frac{B_t(n+1, \alpha-n)}{2} \right| \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \right| dt \right. \\ & \quad \left. + \int_0^1 \left| \frac{B_t(n+1, \alpha-n)}{2} \right| \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) \right| dt \right] \\ & \leq \frac{(x-a)^{\alpha+1}}{2n!(b-a)} \left[\int_0^1 |B_t(n+1, \alpha-n)| \left\{ \left(\frac{1+t}{2} \right)^s |f'(x)| + \left(\frac{1-t}{2} \right)^s |f'(a)| \right\} dt \right. \\ & \quad \left. + \int_0^1 |B_t(n+1, \alpha-n)| \left\{ \left(\frac{1-t}{2} \right)^s |f'(x)| + \left(\frac{1+t}{2} \right)^s |f'(a)| \right\} dt \right] \\ & = \frac{(x-a)^{\alpha+1}}{2^{s+1}n!(b-a)} \left[|f'(x)| \int_0^1 |B_t(n+1, \alpha-n)| (1+t)^s dt + |f'(a)| \int_0^1 |B_t(n+1, \alpha-n)| (1-t)^s dt \right. \\ & \quad \left. + |f'(x)| \int_0^1 |B_t(n+1, \alpha-n)| (1-t)^s dt + |f'(a)| \int_0^1 |B_t(n+1, \alpha-n)| (1+t)^s dt \right] \end{aligned}$$

Similarly, we have

$$\begin{aligned} Q_2 &= \left| \frac{(b-x)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) dt \right. \right. \\ &\quad \left. \left. - \int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) dt \right] \right| \\ &\leq \frac{(b-x)^{\alpha+1}}{2^{s+1}n!(b-a)} \left[|f'(x)| \int_0^1 |B_t(n+1, \alpha-n)| (1+t)^s dt \right. \\ &\quad \left. + |f'(b)| \int_0^1 |B_t(n+1, \alpha-n)| (1-t)^s dt \right. \\ &\quad \left. + |f'(x)| \int_0^1 |B_t(n+1, \alpha-n)| (1-t)^s dt + |f'(b)| \int_0^1 |B_t(n+1, \alpha-n)| (1+t)^s dt \right] \end{aligned}$$

Substituting $\mathcal{A}_1$ and $\mathcal{A}_2$ into Q_1 and Q_2 inequalities and simplifying lead to the required inequality. The proof of Theorem 2 is completed. $\square$

Remark 2. Under conditions of Theorem 2,

1. if $s = 1$, then

$$\begin{aligned} |F_f(\alpha, n; x)| &\leq \frac{1}{4n!(b-a)} \left[2B(n+1, \alpha-n) - B(n+1, \alpha-n-1) - B(n+2, \alpha-n) - \frac{1}{2}B(n+3, \alpha-n) \right] \\ &\times [(x-a)^{\alpha+1} (|f'(x)| + |f'(a)|) + (b-x)^{\alpha+1} (|f'(x)| + |f'(b)|)] \end{aligned}$$

2. if $\alpha = n+1$, then

$$\begin{aligned} &\left| \frac{(x-a)^{n+1}}{\Gamma(n+2)(b-a)} [f'(x) + f'(a)] - \frac{2^{n+1}}{(b-a)} [J_{x-}^{n+1} + J_{a+}^{n+1}] \right. \\ &\quad \left. + \frac{(b-x)^{n+1}}{\Gamma(n+2)(b-a)} [f'(x) + f'(b)] - \frac{2^{n+1}}{(b-a)} [J_{x-}^{n+1} + J_{b+}^{n+1}] \right| \\ &\leq \frac{1}{2^{s+1}(s+1)(b-a)} \left[\left(\frac{2^s - \Gamma(n+2)_2 F_1(n+1, -1-s; n+2; -1)}{\Gamma(n+2)} + \frac{\Gamma(s+2)}{\Gamma(n+s+3)} \right) \right] \\ &\quad \times [(x-a)^{n+2} (|f'(x)| + |f'(a)|) + (b-x)^{n+2} (|f'(x)| + |f'(b)|)] \end{aligned}$$

3. if $\alpha = n+1, s = 1$, we have

$$\begin{aligned} &\left| \frac{(x-a)^{n+1}}{\Gamma(n+2)(b-a)} [f'(x) + f'(a)] - \frac{2^{n+1}}{(b-a)} [J_{x-}^{n+1} + J_{a+}^{n+1}] \right. \\ &\quad \left. + \frac{(b-x)^{n+1}}{\Gamma(n+2)(b-a)} [f'(x) + f'(b)] - \frac{2^{n+1}}{(b-a)} [J_{x-}^{n+1} + J_{b+}^{n+1}] \right| \\ &\leq \frac{(x-a)^{n+2}}{4n!(b-a)} (|f'(x)| + |f'(a)|) \\ &\quad \times \left(\frac{3(n+2)(n+3) - 2(n+1)(n+3) - (n+1)(n+2) + n(n+3)}{2(n+1)(n+2)(n+3)} \right) \\ &\quad + \frac{(b-x)^{n+2}}{4n!(b-a)} (|f'(x)| + |f'(b)|) \left(\frac{3(n+2)(n+3) - 2(n+1)(n+3) - (n+1)(n+2) + n(n+3)}{2(n+1)(n+2)(n+3)} \right) \end{aligned}$$

Theorem 3. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. If $|f'|$ is log-convex on $[a, b]$, then the following inequality holds for conformable fractional integrals:

$$|F_f(\alpha, n; x)| \leq \frac{B(n+1, \alpha-n) |f'(x)|^{\frac{1}{2}}}{2n!(b-a)} \left[(x-a)^{\alpha+1} |f'(a)|^{\frac{1}{2}} (\Omega_1 + \Omega_2) + (b-x)^{\alpha+1} |f'(b)|^{\frac{1}{2}} (\Omega_3 + \Omega_4) \right]$$

where $B(a, b)$ Euler beta function with $\alpha \in (n, n+1]$, $n = 0, 1, 2, \dots$

Proof. Since $|f'|$ is log-convex on $[a, b]$ and by Lemma 1, we can write

$$\begin{aligned} \Psi_1 &= \left| \frac{(x-a)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) dt \right. \right. \\ &\quad \left. \left. - \int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) dt \right] \right| \\ &\leq \frac{(x-a)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \left| \frac{B_t(n+1, \alpha-n)}{2} \right| \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \right| dt \right. \\ &\quad \left. + \int_0^1 \left| \frac{B_t(n+1, \alpha-n)}{2} \right| \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) \right| dt \right] \\ &\leq \frac{(x-a)^{\alpha+1}}{2n!(b-a)} \left[\int_0^1 |B_t(n+1, \alpha-n)| |f'(x)|^{\left(\frac{1+t}{2}\right)} |f'(a)|^{\left(\frac{1-t}{2}\right)} dt \right. \\ &\quad \left. + \int_0^1 |B_t(n+1, \alpha-n)| |f'(x)|^{\left(\frac{1-t}{2}\right)} |f'(a)|^{\left(\frac{1+t}{2}\right)} dt \right] \\ &= \frac{(x-a)^{\alpha+1}}{2n!(b-a)} |f'(x)|^{\frac{1}{2}} |f'(a)|^{\frac{1}{2}} \left[\int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(x)|}{|f'(a)|} \right]^{\frac{t}{2}} dt \right. \\ &\quad \left. + \int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(a)|}{|f'(x)|} \right]^{\frac{t}{2}} dt \right] \end{aligned}$$

Similarly, we have

$$\begin{aligned} \Psi_2 &= \left| \frac{(b-x)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) dt \right. \right. \\ &\quad \left. \left. - \int_0^1 \frac{B_t(n+1, \alpha-n)}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) dt \right] \right| \\ &\leq \frac{(b-x)^{\alpha+1}}{n!(b-a)} \left[\int_0^1 \left| \frac{B_t(n+1, \alpha-n)}{2} \right| \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) \right| dt \right. \\ &\quad \left. + \int_0^1 \left| \frac{B_t(n+1, \alpha-n)}{2} \right| \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) \right| dt \right] \\ &\leq \frac{(b-x)^{\alpha+1}}{2n!(b-a)} |f'(x)|^{\frac{1}{2}} |f'(b)|^{\frac{1}{2}} \left[\int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(x)|}{|f'(b)|} \right]^{\frac{t}{2}} dt \right. \\ &\quad \left. + \int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(b)|}{|f'(x)|} \right]^{\frac{t}{2}} dt \right] \end{aligned}$$

Substituting

$$\begin{aligned}
 \Omega_1 &= \int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(x)|}{|f'(a)|} \right]^{\frac{t}{2}} dt \\
 &= \frac{\sqrt{\frac{|f'(x)|}{|f'(a)|}}}{\ln \frac{|f'(x)|}{|f'(a)|}} B(n+1, \alpha-n) - \frac{1}{\ln \sqrt{\frac{|f'(x)|}{|f'(a)|}}} \int_0^1 t^n (1-t)^{\alpha-n-1} \left(\sqrt{\frac{|f'(x)|}{|f'(a)|}} \right)^t dt \\
 \Omega_2 &= \int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(a)|}{|f'(x)|} \right]^{\frac{t}{2}} dt \\
 &= \frac{\sqrt{\frac{|f'(a)|}{|f'(x)|}}}{\ln \frac{|f'(a)|}{|f'(x)|}} B(n+1, \alpha-n) - \frac{1}{\ln \sqrt{\frac{|f'(a)|}{|f'(x)|}}} \int_0^1 t^n (1-t)^{\alpha-n-1} \left(\sqrt{\frac{|f'(a)|}{|f'(x)|}} \right)^t dt \\
 \Omega_3 &= \int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(x)|}{|f'(b)|} \right]^{\frac{t}{2}} dt \\
 &= \frac{\sqrt{\frac{|f'(x)|}{|f'(b)|}}}{\ln \frac{|f'(x)|}{|f'(b)|}} B(n+1, \alpha-n) - \frac{1}{\ln \sqrt{\frac{|f'(x)|}{|f'(b)|}}} \int_0^1 t^n (1-t)^{\alpha-n-1} \left(\sqrt{\frac{|f'(x)|}{|f'(b)|}} \right)^t dt \\
 \Omega_4 &= \int_0^1 |B_t(n+1, \alpha-n)| \left[\frac{|f'(b)|}{|f'(x)|} \right]^{\frac{t}{2}} dt \\
 &= \frac{\sqrt{\frac{|f'(b)|}{|f'(x)|}}}{\ln \frac{|f'(b)|}{|f'(x)|}} B(n+1, \alpha-n) - \frac{1}{\ln \sqrt{\frac{|f'(b)|}{|f'(x)|}}} \int_0^1 t^n (1-t)^{\alpha-n-1} \left(\sqrt{\frac{|f'(b)|}{|f'(x)|}} \right)^t dt
 \end{aligned}$$

into Ψ_1 and Ψ_2 inequality and simplifying lead to the required inequality. The proof of Theorem 3 is completed. $\square$

References

- [1] T. Abdeljawad, On conformable fractional calculus, *Journal of Computational and Applied Mathematics*, 279 (2015), 57-66.
- [2] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, *Journal of Computational and Applied Mathematics*, 264 (2014), 65-70.
- [3] A. Yalçın, Integral inequalities for different kinds of convex functions via conformable fractional integrals, Master Thesis, Ağrı İbrahim Çeçen University, 2016.
- [4] E.D. Rainville, *Special Functions*, The Mcmillan Company, New York, 1960.
- [5] J. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press (1992), Inc.
- [6] H. Hudzik, L. Maligranda, Some remarks on s -convex functions, *Aequationes Math.*, 48 (1994) 100–111.
- [7] W.W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, *Publ. Inst. Math.*, 23 (1978) 13–20.
- [8] W.W. Breckner, Continuity of generalized convex and generalized concave set-valued functions, *Rev Anal. Numér. Thkor. Approx.*, 22 (1993) 39–51.
- [9] A. Ekinici and M. E. Ozdemir, *Some New Integral Inequalities Via Riemann Liouville Integral Operators*, *Applied and Computational Mathematics*, 3 (2019), 288–295.
- [10] A. Ekinici and M.E. Özdemir, *Some New Integral Inequalities Via Riemann-Liouville Integral Operators*, *Applied and Computational Mathematics*, Vol. 18, No:3, 2019, pages: 288-295.
- [11] D. Nie, S. Rashid, A.O. Akdemir, D. Baleanu and J.-B. Liu, On Some New Weighted Inequalities for Differentiable Exponentially Convex and Exponentially Quasi-Convex Functions with Applications, *Mathematics* 2019, 7(8), 727.

- [12] A.O.Akdemir, A. Ekinici and E. Set, Conformable Fractional Integrals And Related New Integral Inequalities, Journal of Nonlinear and Convex Analysis, Volume 18, Number 4, 2017, 661-674.
- [13] A. Kashuri and R. Liko, Ostrowski Type Conformable Fractional Integrals For Generalized (g, s, m, ϕ) -Preinvex Functions, Turkish Journal of Inequalities, 2 (2) (2018), Pages 54–70.
- [14] M. Kunt and İ. İşcan, Fractional Hermite-Hadamard-Fejer Type Inequalities For Ga- Convex Functions, Turkish Journal of Inequalities, 2 (1) (2018), Pages 1-20.

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