Applied Mathematics and Nonlinear Sciences

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Periodic orbits in the restricted problem of three bodies in a three-dimensional coordinate system when the smaller primary is a triaxial rigid body

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Submission Info

Communicated by Juan Luis García Guirao

Received April 6th 2020

Accepted July 17th 2020

Available online December 31st 2020

Abstract

In this paper, we have studied the equations of motion for the problem, which are regularised in the neighbourhood of one of the finite masses and the existence of periodic orbits in a three-dimensional coordinate system when $\mu = 0$. Finally, it establishes the canonical set (l, L, g, G, h, H) and forms the basic general perturbation theory for the problem.

Keywords: restricted three problem, Levi-Civita transformation, periodic orbits, triaxial rigid body

AMS 2010 codes: 70F15

1 Introduction

In this paper, we wish to study the three-dimensional generalisation of the problem studied by Bhatnagar (12–14) for the circular case. Since the Hamilton-Jacobi equation for generating a solution takes an unmanageable form for any solution, we have assumed that the third coordinate (z_3) of the infinitesimal mass is of the order $O(\mu)$. It will be interesting to observe that various equations and results worked out by Bhatnagar can be deduced from our results. In Section 2 we have determined the canonical form of the equations of motion, and in Section 3 these equations are regularised by the generalised Levi-Civita's transformation for three dimensions. Eqs (20)–(22) establish the canonical set (l, L, g, G, h, H) and Eq (32) form the basis of the general perturbation theory for the problem under consideration. During the last few years, many mathematician and astronomers have studied different types of periodic orbits in the restricted problem. Some of them are Giacaglia (7), Mayer and Schmidt (17), Markellos (19), Hadjidemetriou (10,11), Bhatnagar and Taqvi (15), Gomez and Noguera (8), Kadrnoska and Hadrava (9), Perdios et al. (21), Ahmad (1), Elipe and Lara (4), Mathlouthi (23), Scuflaire (22), Caranicolas (20), Poddar et al. (5, 6), Abouelmagd and Guirao (2) and Abouelmagd et al. (3). In this work, we have presented an analytical study of the existence of periodic orbits for $\mu = 0$ in the restricted problem of three

bodies in a three-dimensional coordinate system when the smaller primary is a triaxial rigid body.

2 Equations of Motion

The equations of motion in the canonical form of an infinitesimal mass under the gravitational field of two finite and unequal masses and moving in circles are given by

$$\dot{X}_i = \frac{\partial H}{\partial p_i}; \dot{p}_i = -\frac{\partial H}{\partial x_i} (i = 1, 2, 3) \quad (1)$$

where the Hamiltonian function H and consequently the energy integral is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + n(p_1x_2 - p_2x_1) - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} - \frac{\mu}{2r_2^3} + \frac{3\mu}{2r_2^5}x_2^2 = C \quad (2)$$

and C is a function of $\mu = C(\mu) = C_0 + \mu(C_1)$.

$$r_1^2 = (x_1 - \mu)^2 + x_2^2 + x_3^2$$

$$r_2^2 = (x_1 - \mu + 1)^2 + x_2^2 + x_3^2$$

$$p_1 = x_1 - x_2$$

$$p_2 = \dot{x}_2 + x_1$$

$$p_3 = \dot{x}_3$$

Mean motion $n = 1 + \frac{3}{4}(2\sigma_1 - \sigma_2)$

where $\sigma_1 = \frac{a^2 - c^2}{5R^2}$, $\sigma_2 = \frac{b^2 - c^2}{5R^2}$, a, b, c = semi-axes of the triaxial rigid body, R = the dimensional distance between the primaries and (x_1, x_2, x_3) are equal to the synodic rectangular dimensionless coordinates of the infinitesimal mass in a uniformly rotating system.

3 Regularisation of the Solution

We regularise the solution by Levi-Civita's (18) transformation generated by

$$S = (\mu + \xi_1^2 - \xi_2^2)p_1 + 2\xi_1\xi_2p_2 + \xi_3p_3 \quad (3)$$

Such that

$$x_i = \frac{\partial S}{\partial p_i}; \pi_i = \frac{\partial S}{\partial \xi_i} (i = 1, 2, 3) \quad (4)$$

where π_i is the momenta associated with the new coordinate ξ_i .

We have from Eqs (3) and (4)

$$\pi_1 = \frac{\partial S}{\partial \xi_1} = 2\xi_1p_1 + 2\xi_2p_2, \pi_2 = \frac{\partial S}{\partial \xi_2} = -2\xi_2p_1 + 2\xi_1p_2, \pi_3 = \frac{\partial S}{\partial \xi_3} = p_3$$

From these equations, we have

$$p_1 = \frac{\pi_1\xi_1 - \pi_2\xi_2}{2(\xi_1^2 + \xi_2^2)}, p_2 = \frac{\pi_1\xi_2 - \pi_2\xi_1}{2(\xi_1^2 + \xi_2^2)}$$

Further

$$p_3 = \pi_3, x_1 = \mu + \xi_1^2 - \xi_2^2$$

$$x_2 = 2\xi_1\xi_2, x_3 = \xi_3$$

The Hamiltonian Eq. (2) given in terms of these new variables is

$$H = \frac{\pi^2}{8\xi^2} + \frac{\pi_3^2}{2} + \frac{n(\xi_2\pi_1 - \xi_1\pi_2)}{2} - \frac{n\mu}{2\xi^2}(\xi_1\pi_2 + \xi_2\pi_1) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{\mu}{2r_2^3}(2\sigma_1 - \sigma_2) + 6\frac{\mu}{r_2^5}(\sigma_1 - \sigma_2)\xi_1^2\xi_2^2$$

$$= C = \text{const.}$$

where $r_1^2 = \xi^4 + \xi_3^2$, $r_2^2 = 1 - 2(\xi_1^2 - \xi_2^2) + \xi^4 + \xi_3^2$, $\pi^2 = \pi_1^2 + \pi_2^2$, $\xi^2 = \xi_1^2\xi_2^2$, $C = C_o + C_1(\mu)$ and $C = C_o + \mu C_1$
Now we introduce a new independent variable τ instead of t defined by

$$dt = r_1 d\tau \quad (t = 0 \text{ at } \tau = 0) \quad (5)$$

The equations of motion (1) will be transformed into

$$\frac{d_i}{d\tau} = \frac{\partial k}{\partial \pi_i}, \quad \frac{d\pi_i}{d\tau} = -\frac{\partial k}{\partial i} \quad (i = 1, 2, 3) \quad (6)$$

Where K is the new Hamiltonian given by

$$K = r_1(H - C) = \frac{\pi^2 r_1}{8\xi^2} + \frac{1}{2}\pi_3^2 r_1 + \frac{r_1 n}{2}(\xi_2\pi_1 - \pi_2\xi_1 - \frac{2c}{n})$$

$$- \frac{n\mu r_1}{2\xi^2}(\xi_1\pi_2 + \xi_2\pi_1) - (1-\mu) - \frac{r_1\mu}{r_2} - \frac{r_1\mu}{2r_2^3}(2\sigma_1 - \sigma_2) + \frac{6r_1\mu}{r_2^5}(\sigma_1 - \sigma_2)\xi_1^2\xi_2^2.$$

K can be put in the form $K_o + \mu K_1$, where

$$K_o = \frac{\pi^2 r_1}{8\xi^2} + \frac{1}{2}\pi_3^2 r_1 + \frac{r_1 n}{2}(\xi_2\pi_1 - \pi_2\xi_1 - 2c'_o) - 1 = - < o \text{ (say)} \quad (7)$$

where $c'_o = \frac{c_o}{n}$ and

$$K_1 = \frac{nr_1}{2\xi^2}(\xi_1\pi_2 + \xi_2\pi_1) - \frac{r_1}{r_2} - \frac{r_1}{2r_2^3}(2\sigma_1 - \sigma_2) + \frac{6r_1\mu}{r_2^5}(\sigma_1 - \sigma_2)\xi_1^2\xi_2^2 - \frac{(c - c_o)}{\mu}r_1 + 1 \quad (8)$$

The form given to k_0 ensures that the orbits which are analytically continued from the two-body orbits will belong to the $K = 0$ manifold. These are the solution to the regularised equation of the restricted problem. Here we have assumed that k_0 is negative (5). Thus, the corresponding two-body problem will admit bounded orbits as a solution in rotating coordinates. We can easily show that $|| < 1$.

4 Generating Solution

To write the Hamilton-Jacobi equation corresponding to the Hamilton k_0 , we take

$$\pi_i = \frac{\partial w}{\partial i} \quad (i = 1, 2, 3)$$

For generating a solution, we shall choose k_0 for our Hamiltonian function. Since τ is not involved in k explicitly, the Hamilton-Jacobi equation corresponding to k_0 may be written as

$$\frac{1}{8} \left[\left(\frac{\partial w}{\partial \xi_1} \right)^2 + \left(\frac{\partial w}{\partial \xi_2} \right)^2 \right] \frac{r_1}{\xi^2} + \frac{1}{2} \left(\frac{\partial w}{\partial \xi_3} \right)^2 r_1 + \frac{nr_1}{2} \left\{ \xi_2 \frac{\partial w}{\partial \xi_1} - \xi_1 \frac{\partial w}{\partial \xi_2} - 2c'_o \right\} = \alpha. \quad (9)$$

where $\alpha = 1 - \varepsilon$.

We take ξ_3 of the order of μ , then we have

$$r_1 = \xi^2 + 0(\mu)$$

Putting

$$\xi_1 = \xi \cos \phi, \quad \xi_2 = \xi \sin \phi$$

Equation (9) may be written as

$$\frac{1}{8} \left[\left(\frac{\partial w}{\partial \xi} \right)^2 + \frac{1}{\xi^2} \left(\frac{\partial w}{\partial \phi} \right)^2 \right] + \frac{1}{2} \xi^2 \left(\frac{\partial w}{\partial \xi_3} \right)^2 + \frac{1}{2} n \xi^2 \left[-\frac{\partial w}{\partial \phi} - 2c'_0 \right] = \alpha \quad (10)$$

Whose solution of Eq. (10) may be written as

$$W = u(\xi) + G\phi + \bar{H}\xi_3 \quad (11)$$

where G is an arbitrary parameter and taking $\xi^2 = z$ we have

$$\left(\frac{\partial u}{\partial z} \right)^2 = \frac{\bar{H}^2 - 2n(G + c'_0)}{z^2} f(z) \quad (12)$$

where

$$f(z) = \frac{G^2}{2n(G + c'_0) - \bar{H}^2} - \frac{2\alpha z}{2n(G + c'_0) - \bar{H}^2} - z^2 \quad (13)$$

We suppose that $G + c'_0 < 0$ then the equation $f(z) = 0$ has two positive roots z_1 and z_2 and is positive between them. Also

$$z_1 + z_2 = -\frac{2\alpha}{2n(G + c'_0) - \bar{H}^2} > 0$$

$$z_1 z_2 = -\frac{G^2}{2n(G + c'_0) - \bar{H}^2} > 0$$

The solution of Eq. (12) is

$$u(Z, G, \alpha) [\bar{H}^2 - 2n(G + c'_0)]^{1/2} \int_{Z_1}^{Z_2} \frac{\sqrt{f(z)}}{Z} dz \quad (14)$$

Let us introduce the parameter a, e, l using the relation

$$Z_1 = a(1 - e), Z_2 = a(1 + e)$$

$$Z = Z_1 \cos^2 \frac{l}{2} + Z_1 \sin^2 \frac{l}{2} = a(1 - e \cos l) \quad (15)$$

where $0 \leq e \leq 1$. It may be noted that $Z = Z_1$ when $l = 0$.

The equations of motion to K_0 are

$$\xi'_i = \frac{\partial K_0}{\partial \pi_i}, (i = 1, 2, 3)$$

$$\xi'_1 = \frac{\partial K_0}{\partial \pi_1} = \frac{\pi_1}{4} \frac{r_1}{\xi^2} + \frac{1}{2} n r_1 \xi_2$$

$$\xi'_2 = \frac{\partial K_0}{\partial \pi_2} = \frac{\pi_1}{4} \frac{r_1}{\xi^2} + \frac{1}{2} n r_1 \xi_2$$

$$\xi_3' = \frac{\partial K_0}{\partial 3} = \pi_3 r_1 \quad (16)$$

Here ' denotes differentiation with respect to τ

Now $\frac{1}{4}(\xi_1 \pi_1 + \xi_2 \pi_2) = \xi \xi'$

Therefore $\frac{dz}{d\tau} = \sqrt{\bar{H}^2 + 2(G + c_0')} \cdot \sqrt{f(z)}$

Integrating, we have

$$\int_{z_1}^z \frac{dz}{\sqrt{f(z)}} = (\tau - \tau_0) [\bar{H}^2 + 2n(G + c_0')]^{1/2} \quad (17)$$

where $z = z_1$ at $\tau = \tau_0$.

Introducing L by relation

$$\alpha = L [\bar{H}^2 + 2n(G + c_0')]^{1/2} > 0, L > 0$$

We have $a = \frac{L}{[\bar{H}^2 + 2n(G + c_0')]^{1/2}}$

$$e = \left[1 - \frac{G^2}{L^2}\right]^{1/2} \leq 1 \quad (18)$$

$$\sqrt{f(z)} = ae \sin l$$

$$l = (\tau - \tau_0) [\bar{H}^2 + 2n(G + c_0')]^{1/2} \quad (19)$$

Now taking L and G for the arbitrary constants instead of α and G , the solution may be given by the relation

$$\frac{\partial w}{\partial L} = \frac{\partial u}{\partial L} = l \quad (20)$$

$$\frac{\partial w}{\partial G} = 2 + \frac{\partial u}{\partial G} = 2 + \frac{n\sqrt{L^2 + G^2}}{[\bar{H}^2 + 2n(G + c_0')]^{1/2}} \sin l - f = g \text{ (say)}. \quad (21)$$

where $f = \sqrt{1 - e^2} \int_0^l \frac{dl}{1 - e \cos l}$

$$\frac{\partial w}{\partial H} = 3 + \frac{n\sqrt{L^2 + G^2}}{[\bar{H}^2 + 2n(G + c_0')]^{1/2}} \sin l = h \text{ (say)} \quad (22)$$

and for $e = 1$, we have $G = 0, f = 0$. Eqs (20)–(22) establish the canonical set $(l, L, g, G, h, \bar{H})$ since $k_0 = \alpha - 1$.

It follows that

$$K_0 = L [\bar{H}^2 + 2n(G + c_0')]^{1/2} - 1 > 0$$

and therefore, for the problem generated by this Hamiltonian (regularised two-body problems in Rotating coordinates), we have

$$\frac{dL}{d\tau} = -\frac{\partial k_0}{\partial l} = 0, L = \text{constant} = L_0 \text{ (say)}$$

$$\frac{dG}{d\tau} = -\frac{\partial k_0}{\partial g} = 0, G = \text{constant} = G_0 \text{ (say)}$$

$$\frac{d\bar{H}}{dt} = -\frac{\partial k_0}{\partial h} = 0, \bar{H} = \text{constant} = H_0 \text{ (say)}$$

$$\frac{dl}{d} = \frac{\partial k_0}{\partial L} = [H^2 - 2N(G + c_0')]^{\frac{1}{2}} = \text{const} = n_l \therefore l = n_l \tau + l_0$$

$$\begin{aligned}\frac{dg}{d} &= \frac{\partial k_0}{\partial G} = \frac{-nL}{[\bar{H}^2 + 2n(G + c'_0)]^{1/2}} = \text{const} = n_g \therefore g = n_g \tau + g_o \\ \frac{dh}{d} &= \frac{\partial k_0}{\partial H} = \frac{LH}{[\bar{H}^2 + 2n(G + c'_0)]^{1/2}} = \text{const} = n_h \therefore h = n_h \tau + h_o\end{aligned}\quad (23)$$

where l_0, g_0, h_0 are the values of l, g, h respectively at $\tau = 0$.

The angle θ is obtained from the equation

$$\begin{aligned}\phi &= \frac{1}{2}g + \frac{1n[L^2 + G^2]}{2[\bar{H}^2 - 2n(G + c'_0)]} \sin l, \quad \text{when } e \neq 1 \\ \phi &= \frac{1}{2}g + \frac{1LN}{2[\bar{H}^2 - 2nc'_0]} \sin l, \quad \text{when } e = 1.\end{aligned}\quad (24)$$

The variables ξ_i, π_i ($i = 1, 2, 3$) can be expressed by the canonical elements we have

$$\begin{aligned}\xi_1 &= \pm \sqrt{z} \cos \phi = \pm \sqrt{a(1 - e \cos l)} \cos \phi \\ \xi_2 &= \pm \sqrt{z} \sin \phi = \pm \sqrt{a(1 - e \cos l)} \sin \phi \\ \xi_3 &= \frac{h - \bar{H}(L^2 - G^2)^{\frac{1}{2}}}{[\bar{H}^2 - 2n(G + c'_0)]^{\frac{1}{2}}} \sin l \\ \pi_1 &= \frac{\partial w}{\partial \xi} \cos \phi - \frac{\partial w}{\partial \phi} \frac{\sin \phi}{\xi} \\ \pi_2 &= \frac{\partial w}{\partial \xi} \sin \phi - \frac{\partial w}{\partial \phi} \frac{\cos \phi}{\xi} \\ \pi_3 &= \bar{H} \\ \frac{\partial w}{\partial} &= \frac{du}{d} = \frac{du}{dz} \cdot \frac{dz}{d} = 2\xi \frac{du}{dz} = \frac{\left\{ \bar{H}^2 - 2n(G + c'_0) f(z) \right\}^{\frac{1}{2}}}{z^2} \\ &= \pm \frac{2eL \sin l}{\sqrt{a(1 - e \cos l)}}\end{aligned}\quad (25)$$

and $\frac{\partial w}{\partial} = 2G$

Therefore,

$$\pi_1 = \frac{2el \sin l \cos - 2G \sin}{\pm [a_1(1 - e \cos l)]^{\frac{1}{2}}}, \pi_2 = \frac{2el \sin l \sin + 2G \cos}{\pm [a_1(1 - e \cos l)]^{\frac{1}{2}}}$$

where (is given by the first of Eq. (24). When $e = 1$ ($G = 0$),

$$\begin{aligned}\xi_1 &= \pm \sqrt{2a} \sin \frac{l}{2} \cos \phi \\ \xi_2 &= \pm \sqrt{2a} \sin \frac{l}{2} \sin \phi \\ \xi_3 &= h - \frac{\bar{H}L}{\bar{H}^2 - 2c'_0} \sin l \\ \pi_1 &= \frac{4L}{\sqrt{2a}} \cos \frac{l}{2} \cos \phi\end{aligned}$$

$$\begin{aligned}\pi_2 &= \frac{4L}{\sqrt{2a}} \cos \frac{l}{2} \sin \phi \\ \pi_3 &= \bar{H}\end{aligned}\quad (26)$$

where ϕ is given by the second of the Eq. (24)

The original synodic Cartesian coordinates are obtained from equations ($\mu = 0$), i.e.

$$\begin{aligned}x_1 &= \xi_1^2 - \xi_2^2 \\ x_2 &= 2 \xi_1 \xi_2 \\ x_3 &= \xi_3 \\ p_1 &= \frac{1}{2z} \{ \pi_1 \xi_1 - \pi_2 \xi_2 \} \\ p_2 &= \frac{1}{2z} \{ \xi_1 \pi_2 + \xi_2 \pi_1 \} \\ p_3 &= \pi_3\end{aligned}\quad (27)$$

where $z = a(1 - e \cos l)$.

The sidereal Cartesian coordinates are given by

$$\begin{aligned}X_1 &= x_1 \cos t - x_2 \sin t, \quad X_2 = x_1 \sin t + x_2 \cos t \\ X_3 &= x_3 \\ \dot{X}_1 &= p_1 \cos t - p_2 \sin t \\ \dot{X}_2 &= p_1 \sin t + p_2 \cos t \\ \dot{X}_3 &= p_3\end{aligned}\quad (28)$$

where

$$dt = r_1 d\tau \quad (29)$$

or

$$t = \int_0^\tau \xi^2 d\tau + o(\mu)$$

Therefore

$$\begin{aligned}t - t_0 &= \int_0^\tau z \frac{d\tau}{dz} dz \\ &= \frac{a}{[\bar{H}^2 - 2n(G + c'_0)]^{\frac{1}{2}}} \int_0^l (1 - e \cos l) \\ &= \frac{a(1 - e \sin l)}{[\bar{H}^2 - 2n(G + c'_0)]^{\frac{1}{2}}}\end{aligned}\quad (30)$$

where t_0 is a constant. It is seen that l is the eccentric anomaly of the problem of two-body.

In terms of the canonical variables, the complete Hamiltonian may be written as

$$\begin{aligned}K &= K_0 + \mu K_1 \\ &= L [\bar{H}^2 - 2n(G + c'_0)]^{1/2} - 1 +\end{aligned}$$

$$\mu \left[-\frac{1}{2} \left\{ \frac{r_1}{2} (\xi_1 \pi_1 - \xi_2 \pi_2) \right\} - \frac{r_1}{r_2} - \frac{r_1}{2r_2^3} (2\sigma_1 - \sigma_2) + 6 \frac{r_1}{r_2^5} (\sigma_1 - \sigma_2) \xi_1^2 \xi_2^2 - \frac{(c - c_0) r_1}{\mu} + 1 \right] \quad (31)$$

where

$$\begin{aligned} r_1^2 &= \xi^4 + \xi_3^2 \\ r_2^2 &= 1 + \xi^4 + 2(\xi_1^2 - \xi_2^2) + \xi_3^2 \\ \xi^2 &= \xi_1^2 + \xi_2^2 \end{aligned}$$

and $\xi_1, \xi_2, \xi_3, \pi_1, \pi_2, \pi_3$ are given by Eq. (25).

The equations of motion for the complete Hamiltonian are

$$\begin{aligned} \frac{dl}{d\tau} &= \frac{\partial K}{\partial L} = \left[\bar{H}^2 - 2n(G + c'_0) \right]^{\frac{1}{2}} + \mu \frac{\partial R}{\partial L} \\ \frac{dg}{d\tau} &= \frac{\partial K}{\partial G} = \frac{-nL}{[\bar{H}^2 - 2n(G + c'_0)]^{1/2}} + \mu \frac{\partial R}{\partial L} \\ \frac{dh}{d\tau} &= \frac{\partial K}{\partial H} = \frac{LH}{[\bar{H}^2 - 2n(G + c'_0)]^{1/2}} + \mu \frac{\partial R}{\partial H} \\ \frac{dL}{d\tau} &= -\frac{\partial K}{\partial l} = -\mu \frac{\partial R}{\partial l} \\ \frac{dG}{d\tau} &= -\frac{\partial K}{\partial g} = -\mu \frac{\partial R}{\partial g} \\ \frac{d\bar{H}}{d\tau} &= -\frac{\partial K}{\partial h} = -\mu \frac{\partial R}{\partial h} \end{aligned} \quad (32)$$

These equations form the basis of the general perturbation theory for the problem in question.

The solution described by Eqs (25) or (26) is periodic if l and g have commensurable frequencies, i.e. if

$$\frac{n_l}{n_g} = \frac{2n(G + C'_0)}{L} = \frac{p}{q}$$

where p and q are integers.

The period of $\xi_i \pi_i$ is $\frac{4\pi}{n_l}$ and $\frac{4\pi}{n_g}$, and therefore, in the case of commensurability the period of solution is $\frac{4\pi p}{n_l}$ or $\frac{4\pi q}{n_g}$.

5 Conclusion

We have shown that the equations of motion for the problem are regularised by the generalised Levi-Civita's transformation for three dimensions in the neighbourhood of one of the finite masses and the existence of periodic orbits for $\mu = 0$ in the three-dimensional coordinate systems.

Equations (20)–(22) establish the canonical set (l, L, g, G, h, H) and Eq. (32) form the basis of the general perturbation theory for the problem in question. The solution described by Eq. (25) or (26) is periodic if l and g have commensurable frequencies, that is, if

$$\frac{n_l}{n_g} = \frac{2n(G + C'_0)}{L} = \frac{p}{q} \text{ (say)}$$

where p and q are integers.

The period of $\xi_i \pi_i$ is $\frac{4\pi}{n_l}$ and $\frac{4\pi}{n_g}$, so that in case of commensurability the period of solution is $\frac{4\pi p}{n_l}$ or $\frac{4\pi q}{n_g}$.

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