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New Principles of Non-Linear Integral Inequalities on Time Scales

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Abstract

The concept of inequalities in time scales has attracted the attention of mathematicians for a quarter century. And these studies have inspired the solution of many problems in the branches of physics, biology, mechanics and economics etc. In this article, new principles of non-linear integral inequalities are presented in time scales via diamond- α dynamic integral and the nabla integral.

Keywords: Operator theory, Time scales, Integral inequalities**AMS 2010 codes:** 47B38, 34N05, 35A23

1 Introduction

For a quarter century, the theory of time scales has played an important role in the representation of differential calculus and integral inequalities. The concept of time scales was introduced by Stefan Hilger in 1988 [1]. Later, this theory was studied by many authors. They have demonstrated various aspects of integral inequalities [2-13]. Dynamic equations and inequalities have many applications to quantum mechanics, physical problems, wave equations, heat transfer and economic problems [26, 27, 28, 29]. For example; Aly R. Seadawy et al. have done a lot of research on the applications of dynamic equations in physics. As a result of these studies, they achieved good results [30]. The most important examples of time scale studies are differential calculus and inequalities [12]. Wong et al. [6, 7] expressed some time scale integral inequalities. Yang [13] obtained a generalization of the $\diamond_\alpha$ -integral Hölder's inequality in time scales. Recently, Li Yin and Feng Qi [24] have introduced some non-linear integral inequalities under certain conditions.

Our aim of this article is to demonstrate new principles of non-linear integral inequalities in time scales via the ∇ -integral and the $\diamond_\alpha$ -integral.

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2 Auxiliary Statements and Definitions

Now, let us briefly give information about time scales and give the necessary definitions and notations for our article. For more details, we refer the reader to the articles [1-25].

Let ω be a weight function on R , i.e., ω is a non-negative, almost everywhere positive on R and $\int_R \omega(y) \nabla y < \infty$. Let $\sigma(t)$ be the forward jump operator and let $\rho(t)$ be the backward jump operator in T (T is time scale) for $t \in T$. Respectively, they are defined by

$$\sigma(t) = \inf\{s \in T : s > t\}$$

and

$$\rho(t) = \sup\{s \in T : s < t\}.$$

If $\sigma : T \rightarrow T$, $\sigma(t) > t$, then t is right-scattered. If $\rho : T \rightarrow T$, $\rho(t) < t$, then t is left-scattered. And, if $\sigma : T \rightarrow T$, $\sigma(t) = t$, then t is called right-dense, and if $\rho : T \rightarrow T$, $\rho(t) = t$, then t is called left-dense. Let two mappings $\mu, \vartheta : T \rightarrow R^+$ such that $\mu(t) = \sigma(t) - t$, $\vartheta(t) = t - \rho(t)$ are called graininess mappings. If T has a left-scattered maximum $u \in R$, then $T^k = T/u$. If not $T^k = T$. Briefly

If $\sup T < \infty$, then $T^k = [\rho \sup T, \sup T]$ and if $\sup T = \infty$, then $T^k = T$. By the same way

If $|\inf T| < \infty$, then $T^k = [\inf T, \sigma \inf T]$ and if $\inf T = -\infty$, then $T^k = T$. Let $f : T \rightarrow R$ and $f^\sigma : T \rightarrow R$ by $f^\sigma(t) = f(\sigma(t))$ for $\forall t \in T$, i.e., $f^\sigma = f \circ \sigma$. And let $f : T \rightarrow R$ and $f^\rho : T \rightarrow R$ by $f^\rho(t) = f(\rho(t))$ for $\forall t \in T$, i.e., $f^\rho = f \circ \rho$.

Assume that $h : T \rightarrow R, t \in T^k(t \neq \min T)$.

i. Let h is Δ -differentiable at point t and h is continuous at point t .

ii. Let h is left continuous at point t . t is right-scattered and h is Δ -differentiable at point t ,

$$h^\Delta(t) = (h^\sigma(t) - h(t))/\mu(t)$$

iii. Let t is right-dense, h is Δ -differentiable at point t and

$$\lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s}$$

then

$$h^\Delta(t) = \lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s}$$

iv. Let h is Δ -differentiable at point t , then

$$h^\sigma(t) = h(t) + \mu(t)h^\Delta(t).$$

Definition 2.1. [12] $H : T \rightarrow R$ is called a Δ - antiderivative of $h : T \rightarrow R$. $H^\Delta = h(t)$ holds for $\forall s, t \in T$. We define the Δ -integral of h by

$$\int_s^t h(\tau) \Delta \tau = H(t) - H(s)$$

for $s, t \in T$.

Definition 2.2. [14] Let $h : T_k \rightarrow R$ is called a ∇ -differentiable at $t \in T_k, h^\nabla(t)$, if $\varepsilon > 0$ then there exists a neighborhood V of t such that

$$|h(\rho(t)) - h(s) - h^\nabla(t)(\rho(t) - s)| \leq \varepsilon |\rho(t) - s|$$

for $\forall s \in V$.

Assume that $h : T \rightarrow R, t \in T^k (t \neq \max T)$.

i. Let h is ∇ -differentiable at point t and h is continuous at point t .

ii. Let h is right continuous at point t . t is left-scattered and h is ∇ -differentiable at point t ,

$$h^\nabla(t) = \frac{h(t) - h^\rho(t)}{v(t)}$$

iii. Let t is left-dense, h is ∇ -differentiable at point t and

$$\lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s}$$

then

$$h^\nabla(t) = \lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s}$$

iv. Let h is ∇ -differentiable at point t , then

$$h^\rho(t) = h(t) - \vartheta(t)h^\nabla(t).$$

Definition 2.3. [14] $H : T \rightarrow R$ is called a ∇ - antiderivative of $h : T \rightarrow R$. $H^\nabla = h(t)$ holds for $\forall s, t \in T$. Then, we define the ∇ -integral of h by

$$\int_s^t h(\tau) \nabla \tau = H(t) - H(s)$$

for $s, t \in T$.

Let $h(t)$ be differentiable on T . And let $b, t \in T$. Then,

$$h^{\diamond_a}(t) = bh^\Delta(t) + (1 - b)h^\nabla(t), \quad 0 \leq b \leq 1.$$

Proposition 2.4. [15] If we get $f, g : T \rightarrow R$, $\diamond_a$ -differentiable at $t \in T$, then

$$i. \quad (f + g)^{\diamond_a}(t) = f^{\diamond_a}(t) + g^{\diamond_a}(t)$$

$$ii. \quad \text{If } c \in R, \text{ then } (cf)^{\diamond_a}(t) = cf^{\diamond_a}(t).$$

$$iii. \quad (fg)^{\diamond_a}(t) = f^{\diamond_a}(t)g(t) + bf^\sigma(t)g^\Delta(t) + (1 - b)f^\rho(t)g^\nabla(t).$$

Definition 2.5. [15] If we get $b, t \in T$, $f : T \rightarrow R$, then

$$\int_b^t f(\gamma) \diamond_a \gamma = b \int_b^t f(\gamma) \Delta \gamma + (1 - b) \int_b^t f(\gamma) \nabla \gamma, \quad 0 \leq b \leq 1.$$

Proposition 2.6. [15] Let $u, v, t \in T$, $c \in R$ and if $f(\gamma), g(\gamma)$ are $\diamond_a$ -integrable functions on $[u, v]_T$, then the following statements are valid.

$$i. \quad \int_u^t [f(\gamma) + g(\gamma)] \diamond_a \gamma = \int_u^t f(\gamma) \diamond_a \gamma + \int_u^t g(\gamma) \diamond_a \gamma,$$

$$ii. \quad \int_u^t cf(\gamma) \diamond_a \gamma = c \int_u^t f(\gamma) \diamond_a \gamma,$$

- iii. $\int_u^t f(\gamma) \diamond_a \gamma = - \int_t^u f(\gamma) \diamond_a \gamma,$
- iv. $\int_u^t f(\gamma) \diamond_a \gamma = \int_u^v f(\gamma) \diamond_a \gamma + \int_v^t f(\gamma) \diamond_a \gamma,$
- v. $\int_u^u f(\gamma) \diamond_a \gamma = 0.$

Lemma 2.7. [15] Let $u, v, t \in T$ with $u < v$. Suppose that $h(\gamma)$ and $g(\gamma)$ are $\diamond_a$ -integrable functions on $[u, v]_T$, then the following statements are valid.

1. If $h(\gamma) \geq 0$ for $\forall \gamma \in [u, v]_T$, then $\int_u^v h(\gamma) \diamond_a \gamma \geq 0$.
2. If $h(\gamma) \leq g(\gamma)$ for $\forall \gamma \in [u, v]_T$, then $\int_u^v h(\gamma) \diamond_a \gamma \leq \int_u^v g(\gamma) \diamond_a \gamma$.
3. If $h(\gamma) \geq 0$ for $\forall \gamma \in [u, v]_T$, then $h(\gamma) = 0$ iff $\int_u^v h(\gamma) \diamond_a \gamma = 0$.

Lemma 2.8. (For details, Lemma 2.5 in [24]) Let $p > 1$ or $q < 0$, while $1/p + 1/q = 1$, if $g, h \in C_{rd}(T, R)$, $g(y) > 0, h(y) > 0$, while $u, v \in T$, then

$$\int_u^v \frac{[g(y)]^p}{[h(y)]^{p/q}} \Delta y \geq \frac{(\int_u^v g(y) \Delta y)^p}{(\int_u^v h(y) \Delta y)^{p/q}}.$$

Qi F. et al. [25] proved some inequalities under the condition of Δ -differentiable. In the next section, we will prove these inequalities under the conditions of the ∇ -differentiable and the $\diamond_a$ -differentiable.

3 Main Result

In this section, we will prove non-linear ∇ -differentiable weighted integral inequalities under certain conditions. Later, we will prove their $\diamond_a$ -differentiable extensions. We have listed these studies in the references of the article for the relevant readers.

Theorem 3.1. Let $h, w \in C_{rd}(T, R)$ and let w be a weight function and let $w(y), h(y) > 0, \int_u^v h(y)w(y)\nabla y < \infty$ and $p > 1$ or $q < 0$, while $1/p + 1/q = 1$. If $\int_u^v w(y)h(y) \geq (v - u)^{p-1}$, while $u, v \in T$, then

$$\int_u^v [h(y)w(y)]^p \nabla y \geq \left(\int_u^v h(y)w(y) \nabla y \right)^{p-1}. \quad (1)$$

Proof. Using Lemma 2.8, we obtain

$$\int_u^v [h(y)w(y)]^p \nabla y = \int_u^v \frac{[h(y)w(y)]^p}{1^{p-1}} \nabla y \geq \frac{[\int_u^v h(y)w(y) \nabla y]^p}{[\int_u^v 1 \nabla y]^{p-1}} \geq \left(\int_u^v h(y)w(y) \nabla y \right)^{p-1}.$$

Theorem 3.2. Let w be a weight function, $w(y), g(y) > 0, \int_u^v g(y)w(y) \nabla y < \infty$ for $y \in (u, v]$ and $g, w \in C([u, v], R)$, ∇ -differentiable in (u, v) , Let ε, ϕ be positive real numbers such that $1 < \phi < \varepsilon$. If

$$\left[(wg)^{(\varepsilon-\phi)/(\phi-1)}(y) \right]' \geq \frac{(\varepsilon - \phi)\phi^{1/(\phi-1)}}{\varepsilon - 1} \quad (2)$$

for $\forall y \in (u, v)$, then

$$\int_u^v [g(y)w(y)]^\varepsilon \nabla y \geq \left[\int_u^v g(y)w(y) \nabla y \right]^\varphi. \quad (3)$$

Proof. If we use Cauchy's Mean Value Theorem consecutively for $\delta \in (u, v)$ and $\theta \in (u, \delta)$, then we obtain

$$\begin{aligned} \frac{\left[\int_u^v g(y)w(y) \nabla y \right]^\varphi}{\int_u^v [g(y)]^\varepsilon \nabla y} &= \frac{\varphi \left[\int_u^\delta g(y)w(y) \nabla y \right]^{\varphi-1} w(y)g(\delta)}{[w(y)g(\delta)]^\varepsilon} = \left\{ \frac{\varphi^{1/(\varphi-1)} \int_u^\delta w(y)g(y) \nabla y}{[w(y)g(\delta)]^{(\varepsilon-1)/(\varphi-1)}} \right\}^{\varphi-1} \\ &= \left\{ \frac{\varphi^{1/(\varphi-1)} w(\theta)g(\theta)}{\frac{\varepsilon-1}{\varphi-1} [w(\theta)g(\theta)]^{\frac{\varepsilon-\varphi}{\varphi-1}} (wg)'(\theta)} \right\}^{\varphi-1} = \left\{ \frac{(\varepsilon-\varphi)\varphi^{1/(\varphi-1)}/(\varphi-1)}{[(wg)^{(\varepsilon-\varphi)/(\varphi-1)}(\theta)]} \right\}^{\varphi-1}. \end{aligned} \quad (4)$$

thus, (3) inequality holds.

Theorem 3.3. Let w be a weight function, $w(y), g(y) > 0, \int_u^v g(y)w(y) \nabla y < \infty$ for $y \in (u, v], g, w \in C([u, v], \mathbb{R}), \varepsilon \in \mathbb{R}$. If $\varphi = 1$ and $[w(y)g(y)]^{1-\varphi} \leq 1$ for $\forall y \in (u, v)$, then (3) holds.

Proof. For $\varphi = 1$, inequality (3) reduced to

$$\int_u^v [w(y)g(y)]^\varepsilon \nabla y \geq \int_u^v w(y)g(y) \nabla y. \quad (5)$$

If we use Cauchy's Mean Value Theorem, we obtain the following equation

$$\frac{\int_u^v [w(y)g(y)]^\varepsilon \nabla y}{\int_u^v w(y)g(y) \nabla y} = \frac{[w(\delta)g(\delta)]^\varepsilon}{w(\delta)g(\delta)} = [w(\delta)g(\delta)]^{\varepsilon-1}. \quad (6)$$

Theorem 3.4. Let w be a weight function, $\int_u^v g(y)w(y) \nabla y < \infty$ for $y \in (u, v]$, $m \in \mathbb{N}$ and $1 \leq \varphi \leq m+1$, there exist $(wg)^m(y)$ derivative of the m -th order on $[u, v]$ and $(wg)^m(y)$ is increasing, then $g^{(m)}(y) \geq 0, g^{(i)}(u) = 0$ for $0 \leq j \leq m-1$. If $w(y)g(y) \geq \left[\frac{(y-\varepsilon)^{\varphi-1}}{\varphi^{\varphi-2}} \right]^{1/(\varepsilon-\varphi)}$, then (3) holds.

Proof. If we use Cauchy's Mean Value Theorem together with the condition given in the theorem, we get the following.

$$\begin{aligned} \frac{\int_u^v [w(y)g(y)]^\varepsilon \nabla y}{\left[\int_u^v w(y)g(y) \nabla y \right]^\varphi} &= \frac{[w(c_1)g(c_1)]^{\varepsilon-1}}{\varphi \left[\int_u^{c_1} w(y)g(y) \nabla y \right]^{\varphi-1}} \quad u < c_1 < v \\ &\geq \frac{[(c_1-u)w(c_1)g(c_1)]^{\varphi-1}/\varphi^{\varphi-2}}{\varphi \left[\int_u^{c_1} w(y)g(y) \nabla y \right]^{\varphi-1}} = \left[\frac{(c_1-\varepsilon)w(c_1)g(c_1)}{\varphi \int_u^{c_1} w(y)g(y) \nabla y} \right]^{\varphi-1}. \end{aligned} \quad (7)$$

If we use Cauchy's Mean Value Theorem consecutively in (7), we obtain

$$\begin{aligned} \frac{(c_1-\varepsilon)w(c_1)g(c_1)}{\varphi \int_u^{c_1} w(y)g(y) \nabla y} &= 1 + \frac{(c_2-\varepsilon)(wg)'(c_2)}{w(c_2)g(c_2)} \quad u < c_2 < c_1 \\ &\dots \\ &= m + \frac{(c_{m+1}-\varepsilon)(wg)^{(m)}(c_{m+1})}{(wg)^{(m-1)}(c_{m+1})} \quad u < c_{m+1} < c_m. \end{aligned} \quad (8)$$

But $(wg)^{(m-1)}(k) = (wg)^{(m-1)}(k) - (wg)^{(m-1)}(u) = (k-u)(wg)^m(k_1)$ for $k_1 \in (u, k)$. If $(wg)^m(k_1) \leq g^m(k)$, then $(wg)^{(m)}(y)$ is increasing.

Hence

$$(wg)^{(m)}(k)(k-u) \geq (wg)^{(m-1)}(k) > 0. \quad (9)$$

Applying (9) to (8) yields

$$\frac{(c_1 - \varepsilon)g(c_1)}{\int_u^{c_1} g(y)\nabla y} \geq m+1. \quad (10)$$

Hence

$$\frac{\int_u^v [g(y)]^\varepsilon \nabla y}{[\int_u^v g(y)\nabla y]^\varphi} \geq \left(\frac{m+1}{\varphi}\right)^{\varphi-1}$$

for $1 \leq \varepsilon \leq m+1$.

Theorem 3.5. Suppose that w be a weight function $\int_u^v w(y)g(y)\nabla y < \infty$ for $y \in (u, v)$, $m \in \mathbb{N}$, $1 < \varphi \leq m+1$, there exist $(wg)^{(m)}(y)$ derivative of the m -th order on $[u, v]$ and $(wg)^{(m)}(y)$ is increasing, then $(wg)^{(m)}(y) \geq 0$ and $g^{(j)}(u) = 0$ for $m-1 \geq j \geq 0$.

If $w(y)g(y) \geq \left[\frac{\varphi(y-\varepsilon)^{(\varphi-1)}}{(\varphi-1)^{(\varphi-2)}}\right]^{1/(\varepsilon-\varphi)}$.

Proof. If $w(y)g(y) \geq \left[\frac{\varphi(y-\varepsilon)^{(\varphi-1)}}{(\varphi-1)^{(\varphi-2)}}\right]^{1/(\varepsilon-\varphi)}$, (6) becomes

$$\frac{\int_u^v [w(y)g(y)]^\varepsilon \nabla y}{[\int_u^v w(y)g(y)\nabla y]^\varphi} \geq \left[\frac{(c_1 - \varepsilon)w(c_1)g(c_1)}{(\varphi-1)\int_u^{c_1} w(y)g(y)\nabla y}\right]^{\varphi-1}.$$

If all terms of (8) are positive, then $\frac{(c_1 - \varepsilon)w(c_1)g(c_1)}{\int_u^{c_1} w(y)g(y)\nabla y} \geq m$.

Now let's consider the $\diamond_a$ -integral in time scales.

Theorem 3.6. Let w be a weight function, $h(y), w(y) > 0$, $\int_u^v w(y)h(y)\nabla y$ for $y \in (u, v)$, $p > 1$ or $q < 0$, while $1/p + 1/q = 1$ and $h, w \in C_{rd}(T, R)$. If $\int_u^v w(y)h(y) \diamond_a y \geq (v-u)^{p-1}$ for $u, v \in T$, then

$$\int_u^v [w(y)h(y)]^p \diamond_a y \geq \left[\int_u^v w(y)h(y) \diamond_a y\right]^{p-1}. \quad (11)$$

Proof. See proof of Theorem 3.1. Moreover, when $\alpha = 0$, (11) reduce to (1).

Theorem 3.7. Let $g, w \in C_{rd}(T, R)$, $\diamond_a$ -differentiable on (u, v) , and let w be a weight function, $\int_u^v w(y)h(y) \diamond_a y$ for $y \in (u, v]$, $w(y)g(y) > 0$, and let ε, φ be positive real numbers such that $1 < \varphi < \varepsilon$. If

$$\left[(wg)^{(\varepsilon-\varphi)/(\varphi-1)}(y)\right]' \geq \frac{(\varepsilon-\varphi)\varphi^{1/(\varphi-1)}}{\varepsilon-1}. \quad (12)$$

for $\forall y \in (u, v)$, then

$$\int_u^v [w(y)g(y)]^\varepsilon \diamond_a y \geq \left[\int_u^v w(y)g(y) \diamond_a y\right]^\varphi. \quad (13)$$

Proof. See proof of Theorem 3.2. Moreover, (13) inequality is an extension of (3) inequality. When $\alpha = 0$, (13) reduce to (3).

Theorem 3.8. Let $a \in R$, w be a weight function, $\int_u^v w(y)g(y) \diamond_y < \infty$ for $y \in (u, v)$, $g(y), w(y) > 0$, $g, w \in C([u, v], R)$ and $[w(y), g(y)]$, $\diamond_a$ -differentiable on (u, v) . If $\varphi = 1$ and $[w(y)g(y)]^{1-\varphi} \leq 1$ for $\forall y \in (u, v)$, then

$$\int_u^v [w(y)g(y)]^\varepsilon \diamond_a y \geq \left[\int_u^v w(y)g(y) \diamond_a y \right]^\varphi. \quad (14)$$

Proof. See proof of Theorem 3.3. Moreover, (14) inequality is an extension of (3) inequality. When $\alpha = 0$, (14) reduce to (3).

Theorem 3.9. Suppose that w be a weight function, $\int_u^v w(y)g(y) \diamond_y < \infty$ for $y \in (u, v)$. $m \in N$, $1 \leq \varphi \leq m+1$, there exist $(wg)^{(m)}(y)$ derivative of the m -th order on $[u, v]$, $(wg)^{(m)}(y)$ is increasing and $[w(y), g(y)]$, $\diamond_a$ -differentiable in (u, v) , then $(wg)^{(m)}(y) \geq 0$ and $g^j(u) = 0$ for $0 \leq j \leq m-1$. If

$$w(y)g(y) \geq \left[\frac{(y-\varepsilon)^{\varphi-1}}{\varphi^{\varphi-2}} \right]^{1/(\varepsilon-\varphi)},$$

then

$$\int_u^v [w(y)g(y)]^\varepsilon \diamond_a y \geq \left[\int_u^v w(y)g(y) \diamond_a y \right]^\varphi. \quad (15)$$

inequality holds.

Proof. See proof of Theorem 3.4. Moreover, (15) inequality is an extension of (3) inequality. When $\alpha = 0$, (15) reduce to (3).

Theorem 3.10. Suppose that w be a weight function, $\int_u^v w(y)g(y) \diamond_y < \infty$ for $y \in (u, v)$. $m \in N$, $1 \leq \varphi \leq m+1$, there exist $(wg)^{(m)}(y)$ derivative of the m -th order on $[u, v]$, $(wg)^{(m)}(y)$ is increasing and $[w(y), g(y)]$, $\diamond_a$ -differentiable in (u, v) , then $(wg)^{(m)}(y) \geq 0$ and $g^j(u) = 0$ for $0 \leq j \leq m-1$. If

$$w(y)g(y) \geq \left[\frac{\varphi(y-\varepsilon)^{\varphi-1}}{(\varphi-1)^{\varphi-2}} \right]^{1/(\varepsilon-\varphi)},$$

then

$$\int_u^v [w(y)g(y)]^\varepsilon \diamond_a y \geq \left[\int_u^v w(y)g(y) \diamond_a y \right]^\varphi. \quad (16)$$

inequality holds.

Proof. See proof of Theorem 3.5. Moreover, (16) inequality is an extension of (3) inequality. When $\alpha = 0$, (16) reduce to (3).

Conclusion

Integral inequalities and dynamic equations are the cornerstones of both time scales and harmonic analysis. Mathematicians proved many integral inequalities on time scales [4–9]. And they also showed generalized forms of these inequalities [10, 11, 13, 25]. Time scales theory has also been of interest in different sciences. For example, quantum mechanics, wave equations, physical problems, heat transfer, electrical engineering and economics [26–30]. In this article, we proved non-linear integral inequalities in time scales via the ∇ -integral and the $\diamond_a$ -integral. We think that the multidimensional and multivariate cases of the inequalities proved in this article are also worth examining.

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