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Regarding new wave distributions of the non-linear integro-partial Ito differential and fifth-order integrable equations

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Abstract

This paper applies a powerful scheme, namely Bernoulli sub-equation function method, to some partial differential equations with high non-linearity. Many new travelling wave solutions, such as mixed dark-bright soliton, exponential and complex domain, are reported. Under a suitable choice of the values of parameters, wave behaviours of the results obtained in the paper – in terms of 2D, 3D and contour surfaces – are observed.

Keywords: Integro-partial differential equation, Fifth-Order integrable model, Analytical method, Rational function solution, Complex Solution, Contour surface, Travelling wave solutions, Mixed dark-bright soliton.

1 Introduction

Mathematical models have been used to explain many real-world problems, in the past decade. In this sense, Qi et al. [1] have investigated some important models describing certain waves in physics. Colucci et al. [2] have introduced a new partial differential equation to define the ice crystal size delivery. Another novel model considered to explain the nucleation of spherical agglomerates using the immersion mechanism has been developed by Tash et al. [3]. Baleanu et al. [4] have presented a new study about people's liver using Caputo–Fabrizio fractional model. Pignotti et al. [5] have given another novel differential model related to the project of extraction in mines. Pompa et al. [6] proposed some important models about gastrointestinal absorption for availability of drugs to biological mechanisms. Compression of main electrocardiography signals using a new genetic programming-based mathematical modelling algorithm has been studied by Feli and Abdali-Mohammadi [7]. With the aim of assessing the Bang-Bang model related to hysteresis influences on heat and mass transmit in spongy building material, another important article has been proposed by Berger et al. [8].

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Tsur et al. [9] studied the reaction of melanoma patients to the immune checkpoint surrounding (including understandings collected) in an assessment of a new mathematical mechanistic sample. Camaraza-Medina et al. [10] presented a new study on the mathematical inference of computation of heat transmission by thickenings inside tubes. Another powerful model involving chemical reaction systems has been proposed by Amin et al. [11]. Kortcheva et al. [12] explored new ways and differential equations related to peripheral risk administration in harbours. Aiming to get data using rubrics, Sahin and Baki have developed a new model for measuring mathematical success [13]. Meena et al. [14] composed a new mathematical model about the influencing agent in biofilms under toxic situations to discuss the values of parameters. Hamzehlou et al. have explored a unique way to model and predict the active progress of particle morphology mathematically [15]. There are many other such studies [16–20, 29–44].

The remainder of this current paper is constructed in the following parts. In Section 2, we introduce the Bernoulli sub-equation function method (BSEFM) in detail. In Section 3, as a first application, we apply BSEFM to the (1+1)-dimensional integro-differential Ito equation (ITOE) defined as follows [21]:

$$u_{tt} + u_{xxx} + 3(2u_x u_t + uu_{xt}) + 3u_{xx} \partial_x^{-1}(u_t) = 0. \quad (1)$$

Gepreel et al. [21] have applied the modified simple equation method to Eq. (1) for getting some important properties. Wazwaz has investigated the physical meaning of Eq. (1) [22]. Further, Eq. (1) has been investigated by using meshless discrete collocation method, numerically in another paper [23].

As a second application, we consider the (2+1)-dimensional fifth-order integrable equation (FOIE) given as follows [24]:

$$u_{ttt} + u_{tyyyy} - u_{txx} - \alpha(u_y u_{yt})_y = 0, \quad (2)$$

in which α is a real constant and non-zero. Thus, Eq. (2) was first presented by Wazwaz in 2014, along with some analytical solutions for $\alpha = 4$ by using Hirota's direct method. In Section 4, we introduce some important properties of the results obtained in this paper as the Conclusion.

2 Basic Characteristics of BSEFM

In this sub-section of the paper, the scheme considered herein is introduced [25–27].

Step 1. Let us take the following non-linear partial model, in a general form:

$$P(u, u_x, u_t, u_{xt}, u_{xx}, u^2, \dots) = 0, \quad (3)$$

with the wave transformation given as follows:

$$u(x, t) = V(\xi), \quad \xi = kx - ct, \quad (4)$$

in which α and k are real constants and non-zero. Substituting Eq. (4) into Eq. (3) yields a non-linear ordinary differential equation (NLODE) as follows:

$$N(V, V', V'', V^2, \dots) = 0, \quad (5)$$

where $V = V(\xi)$, $V' = \frac{dV}{d\xi}$, $V'' = \frac{d^2V}{d\xi^2}$, $\dots$.

Step 2. Take the trial equation of solution for Eq. (5) below:

$$V(\xi) = \sum_{i=0}^n a_i F^i = a_0 + a_1 F + a_2 F^2 + \dots + a_n F^n, \quad (6)$$

and

$$F' = bF + dF^M, \quad b \neq 0, \quad d \neq 0, \quad M \in \mathbb{R} - \{0, 1, 2\}, \quad (7)$$

where $F(\xi)$ is the Bernoulli differential polynomial. Changing Eq. (6) with Eq. (7) in Eq. (5), we obtain an equation of polynomial $\sum_{i=0}^s (F(\xi))^i$ of $F(\xi)$ below:

$$\sum_{i=0}^s (F(\xi))^i = \rho_s F(\xi)^s + \cdots + \rho_1 F(\xi) + \rho_0 = 0. \quad (8)$$

According to the balance principle, we can obtain a relationship between n and M .

Step 3. Let the factors of $\sum_{i=0}^s (F(\xi))^i$ all be zero. It will give the following algebraic equations system:

$$\rho_i = 0, \quad i = 0, \dots, s. \quad (9)$$

Solving this system, the values of $a_0, a_1, a_2, \dots, a_n$ will be determined later.

Step 4. When we solve Eq. (7), we get two different situations as below according to b and d ;

$$F(\xi) = \left[\frac{-d}{b} + \frac{E}{e^{b(M-1)\xi}} \right]^{\frac{1}{1-M}}, \quad b \neq d, \quad (10)$$

$$F(\xi) = \left[\frac{(E-1) + (E+1) \tanh\left(\frac{b(1-M)\xi}{2}\right)}{1 - \tanh\left(\frac{b(1-M)\xi}{2}\right)} \right]^{\frac{1}{1-M}}, \quad b = d, \quad E \in \mathbb{R}.$$

When we use a complete discrimination system for polynomial, we get the solutions to Eq. (5) through computational programs and classify certain solutions to Eq. (5). For a better understanding of the results obtained in this manner, we can draw two- and three-dimensional surfaces of solutions by taking into consideration appropriate values of parameters.

3 Implementations of the BSEFM

This section applies BSEFM to the governing models, such as ITOE and FOIE models, to find new travelling wave solutions.

3.1 BSEFM for ITOE

If we take $u(x, t) = v_x(x, t)$ in Eq. (1) for simplicity, we can rewrite it again in the following manner:

$$v_{xtt} + v_{xxxxt} + 6v_{xx}v_{xt} + 3v_xv_{xxt} + 3v_{xxx}v_t = 0. \quad (11)$$

If we consider the travelling wave transformation as

$$v(x, t) = U(\xi), \quad \xi = kx - ct, \quad (12)$$

we obtain the following:

$$k^3 U''' - cU' + 3k^2 (U')^2 = 0. \quad (13)$$

When $U' = w$, Eq. (13) may be rewritten as follows:

$$k^3 w'' - cw + 3k^2 w^2 = 0. \quad (14)$$

Balancing, n and M can be found as follows:

$$2M = n + 2. \quad (15)$$

From Eq. (15), we can get many entirely new travelling wave solutions to Eq. (1).

Case 1: If $n = 4$ and $M = 3$, we can set the trial solution form as follows:

$$w = a_0 + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4, \quad (16)$$

$$w' = a_1 b F + a_1 d F^3 + 2a_2 b F^2 + 2a_2 d F^4 + 3a_3 b F^3 + 3a_3 d F^5 + 4a_4 b F^4 + 4a_4 d F^6, \quad (17)$$

and

$$w'' = a_1 d^2 F + 4a_1 b d F^3 + 3a_1 b^2 F^5 + 4a_2 d^2 F^2 + 12a_2 b d F^4 + 8a_2 b^2 F^6 + 9a_3 d^2 F^3 + 24a_3 b d F^5 + 15a_3 b^2 F^7 + 16a_4 d^2 F^4 + 40a_4 b d F^6 + 24a_4 b^2 F^8, \quad (18)$$

where $a_4 \neq 0$, $b \neq 0$, $d \neq 0$. Substituting Eqs. (16, 18) into Eq. (14), we obtain a system of algebraic equations. Solving this system, we find the following variables and solutions.

Case 1.1. For $b \neq d$, we can consider the following coefficients:

$$a_0 = 0, a_1 = 0, a_2 = -8bdk, a_3 = 0, a_4 = -8b^2k, c = 4d^2k^3 \quad (19)$$

Putting these into Eq. (16) by considering Eq. (10), we get the following new exponential function solution for Eq. (1):

$$u(x, t) = -8bd^3 e^{2d(-4d^2k^3t+kx)} k^2 E \left(b e^{2d(-4d^2k^3t+kx)} - dE \right)^{-2} \quad (20)$$

Here, b, d, k, E are non-zero real constants. With the appropriate values of variables, we can plot various singular wave surfaces of Eq. (20) in Figures 1 and 2.

Fig. 1 The 3D and contour surfaces of Eq. (20) when the values are $E = 0.1, b = 0.2, d = 0.3, k = 0.4$.

Case 1.2. For $b \neq d$, we can consider the following coefficients:

$$a_0 = 0, a_1 = 0, a_2 = \frac{4b\sqrt{c}}{\sqrt{k}}, a_3 = 0, a_4 = -8b^2k, d = -\frac{\sqrt{c}}{2k^{3/2}} \quad (21)$$

Fig. 2 The 2D graph of Eq. (20) when the values are $E = 0.1, b = 0.2, d = 0.3, k = 0.4, t = 0.5$.

Fig. 3 The 3D and contour surfaces of Eq. (22) when the values are $E = 0.1, b = 0.2, c = 0.3, k = 0.4$.

Putting these variables into Eq. (16) by taking into account Eq. (10), we get the following new exponential function solution for Eq. (1):

$$u(x, t) = 4bc^{3/2}e^{\sqrt{c}(-ct+kx)}k^{-3/2}\sqrt{k}E\left(2bk^{3/2} + \sqrt{c}e^{\sqrt{c}(-ct+kx)}k^{-3/2}E\right)^{-2}, \quad (22)$$

where b, c, k, E are non-zero real constants. Considering some values of the parameters, a singular wave simulation of Eq. (22) can be presented as in Figures 3 and 4.

Case 1.3. For $b \neq d$, if we consider the following coefficients

$$a_0 = 0, a_1 = 0, a_2 = -2^{7/3}bc^{1/3}d^{1/3}, a_3 = 0, a_4 = -\frac{2^{7/3}b^2c^{1/3}}{d^{2/3}}, k = \frac{c^{1/3}}{2^{2/3}d^{2/3}}, \quad (23)$$

we get the following different solution for Eq. (1);

$$u(x, t) = -2^{5/3}bc^{2/3}d^{5/3}e^{2d\left(-ct + \frac{c^{1/3}x}{2^{2/3}d^{2/3}}\right)}E\left(be^{-2dct + \frac{2dc^{1/3}}{2^{2/3}d^{2/3}}x} - dE\right)^{-2}, \quad (24)$$

where b, c, k, E are non-zero real constants. Surfaces of Eq. (24) can be observed in Figures 5 and 6.

Case 1.4. For $b \neq d$, when

$$a_0 = -\frac{4d^2k}{3}, a_1 = 0, a_2 = -8bdk, a_3 = 0, a_4 = -8b^2k, c = -4d^2k^3, \quad (25)$$

Fig. 4 The 2D graph of Eq. (22) when the values are $E = 0.1, b = 0.2, c = 0.3, k = 0.4, t = 0.5$.

Fig. 5 The 3D and contour surfaces of Eq. (24) when the values are $E = 0.1, b = 0.2, c = 0.3, d = 0.4$.

Fig. 6 The 2D graph of Eq. (24) when the values are $E = 0.1, b = 0.2, c = 0.3, d = 0.4, t = 0.5$.

the following exponential results are obtained for Eq. (11):

$$v(x, t) = -\frac{16}{3}d^4k^4t - \frac{4}{3}d^2k^2x + 4d^2kE \left(be^{2d(4d^2k^3t + kx)} - dE \right)^{-1}, \quad (26)$$

where b, d, k, E are non-zero real constants.

Fig. 7 The 3D and contour surfaces of Eq. (26) when the values are $E = 1, b = 0.3, d = 0.4, k = 0.8$.

Fig. 8 The 2D graph of Eq. (26) when the values are $E = 1, b = 0.3, d = 0.4, k = 0.8, t = 0.9$.

Case 1.5. When

$$a_0 = \frac{c}{3k^2}, a_1 = 0, a_2 = \frac{4ib\sqrt{c}}{\sqrt{k}}, a_3 = 0, a_4 = -8b^2k, d = -\frac{i\sqrt{c}}{2k^{3/2}}, b \neq d, \quad (27)$$

the following new complex periodic solution for the governing model of Eq. (11) is obtained:

$$v(x, t) = -\frac{1}{3}c^2tk^{-2} + \frac{1}{3}cxk^{-1} + 4bck \left(2ib\sqrt{c}k^{3/2} - ce^{i\sqrt{c}(-ct+kx)k^{-3/2}}E \right)^{-1}, \quad (28)$$

where b, c, k, E are non-zero real constants. The wave simulations of Eq. (28) may be observed in Figures 9 and 10 with some suitable values of parameters.

Case 1.6. Once

$$a_0 = \frac{2}{3}(-1)^{2/3}2^{1/3}c^{1/3}d^{4/3}, a_1 = 0, a_2 = 4(-1)^{2/3}2^{1/3}bc^{1/3}d^{1/3}, a_3 = 0, b \neq d, \\ a_4 = \frac{4(-1)^{2/3}2^{1/3}b^2c^{1/3}}{d^{2/3}}, k = -\left(-\frac{1}{2}\right)^{2/3}\frac{c^{1/3}}{d^{2/3}}, \quad (29)$$

we find the following complex travelling wave solution for Eq. (11);

$$v = \frac{2}{3}c^{1/3}(-1)^{2/3}2^{1/3}d^{4/3}\tau - 2cE(\psi)^{-1/2} \left(2b\psi^{3/2}e^{i\sqrt{c}\tau\psi^{-3/2}} - i\sqrt{c}E \right)^{-1}, \quad (30)$$

Fig. 9 The 3D and contour surfaces of Eq. (28) when the values are $E = 1, b = 0.8, c = 0.4, k = 0.5$.

Fig. 10 The 2D graph of Eq. (28) when the values are $E = 1, b = 0.8, c = 0.4, k = 0.5, t = 0.01$.

where $\tau = -ct - (-1/2)^{2/3} c^{1/3} x d^{-2/3}$, $\psi = -(-1/2)^{2/3} c^{1/3} d^{-2/3}$, and b, c, d, E are non-zero real constants. By considering some suitable values of parameters, one can observe the simulations in Figures 11 and 12.

3.2 BSEFM for FOIE Model

This sub-section applies BSEFM to the FOIE model for finding some new travelling wave solutions. First of all, considering the travelling wave transformation as

$$u(x, y, t) = U(\xi), \xi = kx + wy - ct \quad (31)$$

Fig. 11 The 3D and contour surfaces of Eq. (30) when the values are $E = 0.1, b = 0.2, c = 0.3, d = 0.4, k = 0.5$.

Fig. 12 The 2D graph of Eq. (30) when the values are $E = 0.1, b = 0.2, c = 0.3, d = 0.4, k = 0.5, t = 0.6$.

where k, w, c are real stables and non-zero, we get the following non-linear ordinary differential equation:

$$2c^2U' + 2w^4U''' - 2k^2U' - \alpha w^3(U')^2 = 0. \quad (32)$$

For simplicity, if we take $V = U'$, then, we can rewrite Eq. (32) as follows:

$$2w^4V'' - 2(k^2 + c^2)V - \alpha w^3V^2 = 0. \quad (33)$$

With the help of the balance principle, we obtain the following:

$$2M = n + 2. \quad (34)$$

This gives many new travelling wave solutions to Eq. (2).

Case 1: If $n = 4$ and $M = 3$, we obtain the following:

$$V = a_0 + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4, \quad (35)$$

$$V' = a_1 b F + a_1 d F^3 + 2a_2 b F^2 + 2a_2 d F^4 + 3a_3 b F^3 + 3a_3 d F^5 + 4a_4 b F^4 + 4a_4 d F^6, \quad (36)$$

and

$$V'' = a_1 d^2 F + 4a_1 b d F^3 + 3a_1 b^2 F^5 + 4a_2 d^2 F^2 + 12a_2 b d F^4 + 8a_2 b^2 F^6 + 9a_3 d^2 F^3 + 24a_3 b d F^5 + 15a_3 b^2 F^7 + 16a_4 d^2 F^4 + 40a_4 b d F^6 + 24a_4 b^2 F^8, \quad (37)$$

where $a_4 \neq 0$, $b \neq 0$, $d \neq 0$. Putting Eqs. (35, 37) into Eq. (33), we get the following results.

Case 1.1. When

$$a_0 = 8d^2 w \alpha^{-1}, a_1 = 0, a_2 = 48bdw \alpha^{-1}, a_3 = 0, b \neq d, k = \sqrt{c^2 - 4d^2 w^4}, a_4 = 48b^2 w \alpha^{-1}, \quad (38)$$

we get the following solution for the governing model of Eq. (2):

$$u(x, y, t) = 8d^2 w \alpha^{-1} \left(-ct + \tau x + wy + 3E \left(-be^{2d(-ct + \tau x + wy)} + dE \right)^{-1} \right), \quad (39)$$

where $\tau = \sqrt{c^2 - 4d^2 w^4}$; α, b, c, d, w, E are non-zero real constants here. It is possible to plot the surfaces of Eq. (39) using appropriate values of variables, as in Figures 13 and 14.

Fig. 13 The 3D and contour surfaces of Eq. (39) when the values are $E = 1, w = 0.2, b = 0.3, c = 0.8, d = 0.04, \alpha = 0.6, y = 0.01$.

Case 2.2. When we consider the following coefficients,

$$a_0 = \frac{2(c-k)(c+k)}{w^3 \alpha}, a_1 = 0, a_3 = 0, a_4 = \frac{w^3 \alpha (a_2)^2}{12(c^2 - k^2)}, d = -\frac{i\sqrt{-c^2 + k^2}}{2w^2}, b \neq d, \quad (40)$$

$$b = \frac{iw\alpha a_2}{24\sqrt{-c^2 + k^2}}.$$

we obtain the following complex travelling wave solution for Eq. (2):

$$u = 2\psi \tau w^3 \alpha - 12i\sqrt{-\psi} w^2 a_2 \left(12e^{i\sqrt{-\psi}(-ct + kx + wy)} w^{-2} \psi E - w^3 \alpha a_2 \right)^{-1} \quad (41)$$

in which $\tau = -ct + kx + wy$, $\psi = c^2 - k^2$ and $k^2 - c^2 > 0$; α, a_2, c, k, w, E are non-zero real constants. Figures 15 and 16 show how to plot the surfaces of Eq. (41) with some values of parameters.

Fig. 14 The 2D graph of Eq. (39) when the values are $E = 1, w = 0.2, b = 0.3, c = 0.8, d = 0.04, \alpha = 0.6, y = 0.01, t = 0.02$.

Fig. 15 The 3D and contour surfaces of Eq. (41) when the values are $E = 0.1, w = 0.2, c = 0.3, a_2 = 0.4, \alpha = 0.6, k = 0.8, y = 0.03$.

Case 2.3. For $b \neq d$, we select the following coefficients:

$$a_0 = 0, a_1 = 0, a_3 = 0, a_4 = -\frac{w^3 \alpha (a_2)^2}{12(c^2 - k^2)}, b = -\frac{w \alpha a_2}{24\sqrt{-c^2 + k^2}}, d = -\frac{\sqrt{-c^2 + k^2}}{2w^2}. \quad (42)$$

Putting these variables into Eq. (35) by taking into account Eq. (10), we get the following exponential function solution for Eq. (2):

$$u(x, y, t) = 12\psi^2 w^2 a_2 (-\psi)^{-3/2} \left(12e^{\frac{\sqrt{-\psi}(-ct + kx + wy)}{w^2}} \psi E + w^3 \alpha a_2 \right)^{-1}, \quad (43)$$

Fig. 16 The 2D graph of Eq. (41) for $E = 0.1, w = 0.2, c = 0.3, a_2 = 0.4, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

where the strain conditions are $\psi = c^2 - k^2$ and $k^2 - c^2 > 0$; α, a_2, c, k, w, E are non-zero real constants. In Figures 17 and 18, the 2D and 3D graphics can be seen easily.

Fig. 17 The 3D and contour surfaces of Eq. (43) when the values are $E = 0.1, w = 0.2, c = 0.3, a_2 = 0.4, \alpha = 0.6, k = 0.8, y = 0.03$.

Fig. 18 The 2D graph of Eq. (43) for $E = 0.1, w = 0.2, c = 0.3, a_2 = 0.4, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

Case 2.4. For $b \neq d$, when we take the following values,

$$\begin{aligned} a_0 = 0, a_1 = 0, a_2 = -\frac{24i\sqrt{2}b\sqrt{d}(-c^2+k^2)^{1/4}}{\alpha}, a_3 = 0, w = -\frac{i(-c^2+k^2)^{1/4}}{\sqrt{2}\sqrt{d}}, \\ a_4 = -\frac{24i\sqrt{2}b^2(-c^2+k^2)^{1/4}}{\sqrt{d}\alpha}, \end{aligned} \quad (44)$$

we find the following complex travelling wave solution for the governing model of Eq. (2):

$$u = -4i\sqrt{2}d^{3/2} \left(c^2\tau - k^2\tau - 3i\psi^{3/2}E \left(ibe^{-2id\tau}\sqrt{\psi}d^{-1} + \sqrt{\psi}E \right) d^{-1} \left(b^2e^{-4id\tau}\psi d^{-1} - c^2E^2 + k^2E^2 \right)^{-1} \right) \psi^{-3/4}\alpha^{-1}, \quad (45)$$

for the strain conditions $\tau = -ct + kx - i\psi^{1/4}y2^{-1/2}d^{-1/2}$, $\psi = -c^2 + k^2$ and $k^2 - c^2 > 0$; here, α, b, c, d, k, E are non-zero real constants. With the appropriate values of variables, we can plot the various surfaces of Eq. (45) as in Figures 19 and 20.

Fig. 19 The 3D and contour surfaces of Eq. (45) when the values are $E = 0.1, w = 0.2, c = 0.3, b = 0.4, d = 0.5, \alpha = 0.6, k = 0.8, y = 0.03$.

Case 2.5. When the parameters are chosen as follows,

$$\begin{aligned} a_0 = -\frac{(4+4i)d^{3/2}(-c^2+k^2)^{1/4}}{\alpha}, a_1 = 0, a_2 = -\frac{(24+24i)b\sqrt{d}(-c^2+k^2)^{1/4}}{\alpha}, a_3 = 0, \\ a_4 = -\frac{(24+24i)b^2(-c^2+k^2)^{1/4}}{\sqrt{d}\alpha}, w = -\frac{(\frac{1}{2}+\frac{i}{2})(-c^2+k^2)^{1/4}}{\sqrt{d}}, \end{aligned} \quad (46)$$

Fig. 20 The 2D graph of Eq. (45) for $E = 0.1, w = 0.2, c = 0.3, b = 0.4, d = 0.5, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

Fig. 21 The 3D and contour surfaces of Eq. (47) for $E = 0.1, w = 0.2, c = 0.3, b = 0.4, d = 0.5, \alpha = 0.6, k = 0.8, y = 0.03$.

the following another complex new exponential function solution for Eq. (2) is obtained:

$$u(x, y, t) = -(4 + 4i)d^{3/2}(-c^2 + k^2)^{1/4} \left(\tau - \frac{3E}{be^{2d\tau} - dE} \right) \alpha^{-1}, \quad (47)$$

in which the strain conditions are $\tau = -ct + kx - \left(\frac{1}{2} + \frac{i}{2}\right)(-c^2 + k^2)^{1/4}y d^{-1/2}$, $b \neq d$, and $k^2 - c^2 > 0$.

Here, α, b, c, d, k, E are non-zero real constants. Considering some values of parameters, the various figures of Eq. (47) may be seen in Figures 21 and 22.

Case 2.6. For $b \neq d$, we can consider the coefficients

$$a_0 = 0, a_1 = 0, a_3 = 0, a_4 = -\frac{\alpha(a_2)^2}{24\sqrt{2}d^{3/2}(-c^2 + k^2)^{1/4}}, b = -\frac{\alpha a_2}{24\sqrt{2}\sqrt{d}(-c^2 + k^2)^{1/4}}, \quad (48)$$

$$w = -\frac{(-c^2 + k^2)^{1/4}}{\sqrt{2}\sqrt{d}}.$$

Putting these variables into Eq. (35) by considering Eq. (10), we get the following exponential function solution for Eq. (2):

$$u(x, y, t) = \frac{576d^2\sqrt{\psi}E \left(24\sqrt{2}d^{3/2}(\psi)^{1/4}E - e^{2d\tau}\alpha a_2 \right) (1152d^3\sqrt{\psi}E^2 + e^{4d\tau}\alpha^2 a_2^2)}{\alpha(-1327104d^6\psi E^4 + e^{8d\tau}\alpha^4 a_2^4)}, \quad (49)$$

where $\tau = -ct + kx - (-c^2 + k^2)^{1/4}y 2^{-1/2}d^{-1/2}$, $\psi = -c^2 + k^2$ and $k^2 - c^2 > 0$ are the strain conditions.

Fig. 22 The 2D graph of Eq. (47) for $E = 0.1, w = 0.2, c = 0.3, b = 0.4, d = 0.5, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

Here, α, a_2, c, d, k, E are non-zero real constants. The surfaces of Eq. (49) can be observed in Figures 23 and 24 with the appropriate values of parameters.

Fig. 23 The 3D and contour surfaces of Eq. (49) when the values are $E = 0.1, w = 0.2, c = 0.3, d = 0.4, a_2 = 0.5, \alpha = 0.6, k = 0.8, y = 0.03$.

Fig. 24 The 2D graph of Eq. (49) when the values are $E = 0.1, w = 0.2, c = 0.3, d = 0.4, a_2 = 0.5, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

Case 2.7. For $b \neq d$, if we take the following values,

$$\begin{aligned} a_0 = 0, a_1 = 0, a_3 = 0, a_4 &= \frac{i\alpha(a_2)^2}{24\sqrt{2}d^{3/2}(-c^2+k^2)^{1/4}}, b = \frac{i\alpha a_2}{24\sqrt{2}\sqrt{d}(-c^2+k^2)^{1/4}}, \\ w &= -\frac{i(-c^2+k^2)^{1/4}}{\sqrt{2}\sqrt{d}}, \end{aligned} \quad (50)$$

we find another complex function solution to Eq. (2) as follows:

$$u(x, y, t) = \frac{576id^2\sqrt{\psi}E\left(24\sqrt{2}d^{3/2}(\psi)^{3/2}E + ie^{2d\tau}\alpha a_2\right)\left(1152d^3\sqrt{\psi}E^2 - e^{4d\tau}\alpha^2 a_2^2\right)}{\alpha(-1327104d^6\psi E^4 + e^{8d\tau}\alpha^4 a_2^4)}, \quad (51)$$

where $\tau = -ct + kx - i(-c^2 + k^2)^{1/4}y2^{-1/2}d^{-1/2}$, $\psi = -c^2 + k^2$ and $k^2 - c^2 > 0$ are the strain conditions. Here, α, a_2, c, d, k, E are non-zero real constants. The wave simulations may be seen by using appropriate values of variables.

Fig. 25 The 3D and contour surfaces of Eq. (51) when the values are $E = 0.1, c = 0.3, d = 0.4, a_2 = 0.5, \alpha = 0.6, k = 0.8, y = 0.03$.

Case 2.8. If

$$\begin{aligned} a_0 &= -\frac{(4+4i)d^{3/2}(-c^2+k^2)^{1/4}}{\alpha}, a_1 = 0, a_3 = 0, a_4 = -\frac{(1/48-i/48)\alpha(a_2)^2}{d^{3/2}(-c^2+k^2)^{1/4}}, \\ b &= -\frac{(1/48-i/48)\alpha a_2}{\sqrt{d}(-c^2+k^2)^{1/4}}, w = -\frac{(1/2+i/2)(-c^2+k^2)^{1/4}}{\sqrt{d}}, b \neq d, \end{aligned} \quad (52)$$

we obtain the following complex solution for Eq. (2):

$$u = \frac{(4+4i)d^{3/2}\psi^{1/4}\left(\tau + \left((72+72i)\sqrt{d}E\left(-(27648-27648i)\psi d^{9/2}E^3 - \varpi + \kappa - e^{6d\tau}\psi^{1/4}\alpha^3(a_2)^3\right)\right)\right)}{\alpha\left(1327104\psi d^6E^4 + e^{8d\tau}\alpha^4(a_2)^4\right)} \quad (53)$$

in which $\tau = ct - kx + \left(\frac{1}{2} + \frac{i}{2}\right)(-c^2 + k^2)^{1/4}yd^{-1/2}$, $\psi = -c^2 + k^2$, $\varpi = 1152id^3e^{2d\tau}\psi^{3/4}\alpha E^2 a_2$,

Fig. 26 The 2D graph of Eq. (51) when the values are $E = 0.1, c = 0.3, d = 0.4, a_2 = 0.5, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

$\kappa = (24 + 24i)d^3/2e^{4d\tau}\sqrt{\psi}\alpha^2E(a_2)^2$ and $k^2 - c^2 > 0$ are the strain conditions. Here, α, a_2, c, d, k, E and α, a_2, c, d, k, E are non-zero real constants. The following graphics show the surfaces of Eq. (53) with some values of parameters in Figures 27 and 28.

Fig. 27 The 3D and contour surfaces of Eq. (53) when the values are $E = 0.1, c = 0.3, d = 0.4, a_2 = 0.5, \alpha = 0.6, k = 0.8, y = 0.03$.

4 Conclusions

In this paper, we have successfully applied BSEFM to some powerful non-linear models, such as the integro-partial differential equation and fifth-order integrable model. We have reported some strain conditions for the validity of the obtained results. Moreover, the travelling wave solutions, such as Eqs. (16, 18, 20, 29), obtained by using BSEFM are the new exponential function solutions for Eq. (1), compared with the paper previously

Fig. 28 The 2D graph of Eq. (53) when the values are $E = 0.1, c = 0.3, d = 0.4, a_2 = 0.5, \alpha = 0.6, k = 0.8, y = 0.03, t = 0.01$.

found in literature [28]. Using powerful computational package programs, we observe that all solutions verify Eqs. (1, 2). Under specific values of parameters, we revise our results into existing solutions. Moreover, we have found many other entirely new analytical and complex travelling wave solutions for governing models. As far as we know, BSEFM has not been applied to Eq. (1) earlier. The projected method in this paper may be used to seek more travelling wave solutions of non-linear evolution equations for some applications, such as easy calculations, writing programs for obtaining variables, and so on.

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