Applied Mathematics and Nonlinear Sciences

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Applications of the extended rational sine-cosine and sinh-cosh techniques to some nonlinear complex models arising in mathematical physics

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Submission Info

Communicated by Juan Luis García Guirao

Received September 23rd 2020

Accepted January 31st 2021

Available online April 6th 2021

Abstract

This study presents the applications of the extended rational sine-cosine/sinh-cosh schemes to the Klein-Gordon-Zakharov equations and the (2+1)-dimensional Maccari system. Various wave solutions such as singular periodic, periodic wave, topological, topological kink-type, dark and singular soliton solutions are successfully revealed. To display the physical features of the reported solutions, we use some appropriate choice of parameters in plotting the 3D, 2D, and contour graphs of some attained solutions.

Keywords: NLEEs; Soliton; Periodic waves

1 Introduction

Nonlinear evolution equations (NLEEs) may be utilized to explain diverse complex nonlinear aspects arising in different areas of nonlinear sciences, like quantum mechanics, mechanics, chemistry, optic fiber, engineering, photonics and so on. Exploring the wave solutions to the NLEEs play a vital role in explaining the physical features of these equations, that makes it of paramount important to secure their solutions. To find the exact solutions, particularly solitary wave solutions to NLEEs in mathematical physics plays a significant role in the field of soliton theory. The physical problems are usually mathematical modelled by NLEEs and thus it is vital to investigate the exact solutions of NLEEs. These solutions of NLEEs provide better evidence about its physical structures. These equations represent mathematical models of complex physical phenomena that are utilized in various branches of nonlinear sciences. A variety of an efficient and reliable computational techniques have been designed to explore such kind of problems [1–39].

This work is aimed at investigating two complex nonlinear models, namely; the Klein-Gordon-Zakharov equation [42,43] and the (2+1)-dimensional Maccari system [44] by using the newly extended rational sine/cosine

and sinh/cosh methods. We report various wave solutions that may be helpful in explaining the physical meaning of different nonlinear complex models arising the fields of engineering, applied sciences, mathematical physics, etc..

The Klein-Gordon-Zakharov equations is given by [42]

$$\begin{aligned}\chi_{tt} - \chi_{xx} + \chi + \lambda \psi \chi &= 0, \\ \psi_{tt} - \psi_{xx} - \sigma(|\chi|^2)_{xx} &= 0,\end{aligned}\quad (1)$$

where χ is a complex-valued function and stands for the fast time scale component of electric field raised by electrons, and ψ is a real valued-function which stands for the deviation of ion density from its equilibrium, λ and σ are two nonzero real constants. The Klein-Gordon-Zakharov system portrays the interaction between the Langmuir wave and the ion acoustic wave in a high frequency plasma [43].

The (2+1)-dimensional nonlinear complex coupled Maccari system is given by [44]

$$\begin{aligned}i\chi_t + \chi_{xx} + \chi\psi &= 0, \\ \psi_t + \psi_y + (|\chi|^2)_x &= 0.\end{aligned}\quad (2)$$

Eq. (2) describes the motion of the isolated waves, localized in a small part of space, in different fields such as hydrodynamic, plasma physics, nonlinear optics etc [44].

The remaining part of the paper is organised as follows: In section 2, the overview of the applied methods is presented. In section 3, we present the applications of the methods presented in section 2. In section 4, we present the graphical representation of the reported results in section 3. We give the conclusion of this study in section 5.

2 Overview of the Methods

This section discusses the steps involved in the extended rational sine/cosine and sinh/cosh methods.

Consider the general form of nonlinear partial differential equation (NPDE)

$$F(\chi_x \chi^2, \chi_{xx}, \chi_{xt}, \dots) = 0, \quad (3)$$

where $\chi = \chi(x, t)$ is an unknown function and F is a polynomial in χ and its partial derivatives. Suppose that

$$\chi(x, y, t) = \Theta(\eta), \quad \eta = x - kt, \quad (4)$$

Then, by using (4), Eq. (3) can be turned into the following nonlinear ordinary differential equation (NODE) w.r.t. η :

$$G(\Theta, \Theta', \Theta'', \dots) = 0. \quad (5)$$

2.1 Extended rational sine/cosine method

Suppose that the solution of Eq. (5) can be written in the following forms:

$$\Theta(\eta) = \frac{a_0 \sin(\mu \eta)}{a_2 + a_1 \cos(\mu \eta)}, \quad \cos(\mu \eta) \neq -\frac{a_2}{a_1}, \quad (6)$$

$$\Theta(\eta) = \frac{a_0 \cos(\mu \eta)}{a_2 + a_1 \sin(\mu \eta)}, \quad \sin(\mu \eta) \neq -\frac{a_2}{a_1}, \quad (7)$$

where a_0, a_1 and a_2 are parameters to be found in terms of the other parameters. The non-zero constant μ is the wave number. The derivatives of the predicted solutions are

$$\Theta'(\eta) = \frac{a_0 \mu [\cos(\mu \eta) a_2 + a_1]}{[a_2 + a_1 \sin(\mu \eta)]^2}, \quad (8)$$

$$(9)$$

$$\Theta''(\eta) = \frac{a_0 \mu^2 \sin(\mu \eta) [2a_1^2 + a_1 \cos(\mu \eta) a_2 - a_2^2]}{[a_2 + a_1 \cos(\mu \eta)]^3}, \quad (10)$$

in the first form and

$$\Theta'(\eta) = -\frac{a_0 \mu [\sin(\mu \eta) a_2 + a_1]}{[a_2 + a_1 \sin(\mu \eta)]^2}, \quad (11)$$

$$(12)$$

$$\Theta''(\eta) = \frac{a_0 \mu^2 \cos(\eta \xi) [2a_1^2 + a_1 \sin(\eta \xi) a_2 - a_2^2]}{[a_2 + a_1 \sin(\eta \xi)]^3}. \quad (13)$$

in the second form.

We substitute Eqs. (8) or (11) into Eq. (5) and get a polynomial in trigonometric functions. Collecting the coefficients of the same power of $\cos^m(\mu \eta)$ or $\sin^m(\mu \eta)$ and equating each summation to zero, we get a set of algebraic equations. We solve the system of equations to get the solutions of the equation into consideration.

2.2 Extended rational sinh/cosh method

Assume that the solutions of Eq. (5) may be written in the forms

$$\Theta(\eta) = \frac{a_0 \sinh(\mu \eta)}{a_2 + a_1 \cosh(\mu \eta)}, \quad \cosh(\mu \eta) \neq -\frac{a_2}{a_1}, \quad (14)$$

$$\Theta(\eta) = \frac{a_0 \cosh(\mu \eta)}{a_2 + a_1 \sinh(\mu \eta)}, \quad \sinh(\mu \eta) \neq -\frac{a_2}{a_1}, \quad (15)$$

where a_0, a_1 and a_2 are parameters to be found in terms of the other parameters. The non-zero constant μ is the wave number. The derivatives of the predicted solutions are

$$\Theta'(\eta) = \frac{a_0 \mu [\cosh(\mu \eta) a_2 + a_1]}{[a_2 + a_1 \sinh(\mu \eta)]^2}, \quad (16)$$

$$(17)$$

$$\Theta''(\eta) = -\frac{a_0 \mu^2 \sinh(\mu \eta) [2a_1^2 + a_1 \cosh(\mu \eta) a_2 - a_2^2]}{[a_2 + a_1 \cosh(\mu \eta)]^3}, \quad (18)$$

in the first form and

$$\Theta'(\eta) = \frac{a_0 \mu [\sinh(\mu \eta) a_2 - a_1]}{[a_2 + a_1 \sinh(\mu \eta)]^2}, \quad (19)$$

$$(20)$$

$$\Theta''(\eta) = \frac{a_0 \mu^2 \cosh(\mu \eta) [2a_1^2 - a_1 \sinh(\mu \eta) a_2 + a_2^2]}{[a_2 + a_1 \sinh(\mu \eta)]^3}. \quad (21)$$

in the second form.

We substitute Eqs. (16) or (19) into the reduced form of the governing equation obtained above in Eq. (5). Collecting the coefficients of the $\cosh^m(\mu\eta)$ and/or $\sinh^m(\mu\eta)$ of the same power, and equating each summation to zero, we get a set of algebraic equations. We solve the system of equations to get the solutions of the equation into consideration.

3 Applications

In this section, we present the applications of the extended rational sine/cosine and sinh/cosh methods to the Klein-Gordon-Zakharov equations, and the (2+1)-dimensional Maccari system.

3.1 Application to Eq. (1)

Here, we apply the methods the Klein-Gordon-Zakharov equations.

Consider the wave transformation

$$\chi = \Theta(\eta)e^{i\theta}, \quad \psi = \Phi(\eta), \quad \eta = x - kt, \quad \theta = px + rt. \quad (22)$$

Placing Eq. (22) in Eq. (1), provides:

$$(k^2 - 1)(1 + p^2 - r^2)\Theta + \lambda\sigma\Theta^3 + \mu^2(k^2 - 1)^2\Theta'' = 0 \quad (23)$$

from the real part, and $k = -\frac{p}{r}$ from the imaginary part.

Suppose that Eq. (23) have the solutions of the form

$$\Theta(\eta) = \frac{a_0 \sin(\mu\eta)}{a_2 + a_1 \cos(\mu\eta)} \quad (24)$$

Placing Eq. (24) and its derivatives into Eq. (23) gives a polynomial in power of trigonometric functions. Collecting the coefficients of the same power of $\cos(\mu\eta)$, and equating each summation to zero, provides a system of equations. Solving this system of equations, provides

Class-1: When

$$a_0 = \frac{a_1 \sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2}}{\sqrt{\lambda}r\sqrt{\sigma}}, \quad a_2 = 0, \quad \mu = -\frac{\sqrt{-p^2r^2 + r^4 - r^2}}{\sqrt{2}\sqrt{p^2 - r^2}},$$

we get

$$\chi_{1a}(x, t) = -\frac{\sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2} e^{i(px+rt)} \tan\left(\frac{\sqrt{-p^2r^2 + r^4 - r^2}(\frac{pt}{r} + x)}{\sqrt{2}\sqrt{|p^2 - r^2|}}\right)}{\sqrt{\lambda}r\sqrt{\sigma}}, \quad (25)$$

$$\psi_{1a}(x, t) = \frac{(p^4 - 2p^2r^2 + p^2 + r^4 - r^2) \tan^2\left(\frac{\sqrt{-p^2r^2 + r^4 - r^2}(\frac{pt}{r} + x)}{\sqrt{2}\sqrt{|p^2 - r^2|}}\right)}{\lambda r^2 \left(\frac{p^2}{r^2} - 1\right)}. \quad (26)$$

Class-2: When

$$a_0 = \frac{a_1 \sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2}}{\sqrt{\lambda}r\sqrt{\sigma}}, \quad a_2 = a_1, \quad \mu = -\frac{\sqrt{2}\sqrt{-p^2r^2 + r^4 - r^2}}{\sqrt{p^2 - r^2}},$$

we get

$$\chi_{2a}(x, t) = - \frac{\sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2} e^{i(px+rt)} \sin\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)}{\sqrt{\lambda}r\sqrt{\sigma}\left(\cos\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)+1\right)}, \quad (27)$$

$$\psi_{2a}(x, t) = \frac{(p^4 - 2p^2r^2 + p^2 + r^4 - r^2) \sin^2\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)}{\lambda r^2\left(\frac{p^2}{r^2}-1\right)\left(\cos\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)+1\right)^2}. \quad (28)$$

Suppose that Eq. (23) secures the solution of the form

$$\Theta(\eta) = \frac{a_0 \cos(\mu\eta)}{a_2 + a_1 \sin(\mu\eta)}. \quad (29)$$

Placing Eq. (29) and its derivatives into Eq. (23) gives a polynomial in power of trigonometric functions. Collecting the coefficients of the same power of $\sin(\mu\eta)$, and equating each summation to zero, provides a system of equations. Solving this system of equations, provides

Class-1: When

$$a_0 = \frac{a_1 \sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2}}{\sqrt{\lambda}r\sqrt{\sigma}}, \quad a_2 = 0, \quad \mu = \frac{\sqrt{-p^2r^2 + r^4 - r^2}}{\sqrt{2}\sqrt{p^2 - r^2}},$$

we get

$$\chi_{1b}(x, t) = \frac{\sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2} e^{i(px+rt)} \cot\left(\frac{\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{2}\sqrt{|p^2-r^2|}}\right)}{\sqrt{\lambda}r\sqrt{\sigma}}, \quad (30)$$

$$\psi_{1b}(x, t) = \frac{(p^4 - 2p^2r^2 + p^2 + r^4 - r^2) \cot^2\left(\frac{\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{2}\sqrt{|p^2-r^2|}}\right)}{\lambda r^2\left(\frac{p^2}{r^2}-1\right)}. \quad (31)$$

Class-2: When

$$a_0 = \frac{a_1 \sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2}}{\sqrt{\lambda}r\sqrt{\sigma}}, \quad a_2 = -a_1, \quad \mu = \frac{\sqrt{2}\sqrt{-p^2r^2 + r^4 - r^2}}{\sqrt{p^2 - r^2}},$$

we get

$$\chi_{2b}(x, t) = \frac{\sqrt{p^4 - 2p^2r^2 + p^2 + r^4 - r^2} e^{i(px+rt)} \cos\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)}{\sqrt{\lambda}r\sqrt{\sigma}\left(\sin\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)-1\right)}, \quad (32)$$

$$\psi_{2b}(x, t) = \frac{(p^4 - 2p^2r^2 + p^2 + r^4 - r^2) \cos^2\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)}{\lambda r^2\left(\frac{p^2}{r^2}-1\right)\left(\sin\left(\frac{\sqrt{2}\sqrt{|-p^2r^2+r^4-r^2|}\left(\frac{pt}{r}+x\right)}{\sqrt{|p^2-r^2|}}\right)-1\right)^2}. \quad (33)$$

Assuming that Eq. (23) secures the solution of the form

$$\Theta(\eta) = \frac{a_0 \sinh(\mu\eta)}{a_2 + a_1 \cosh(\mu\eta)}. \quad (34)$$

Placing Eq. (34) and its derivatives into Eq. (23) gives a polynomial in power of hyperbolic functions. Collecting the coefficients of the same power of $\cosh(\mu\eta)$, and equating each summation to zero, provides a system of equations. Solving this system of equations, provides

Class-1: When

$$a_0 = \frac{a_1 \sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2}}{\sqrt{\lambda} r \sqrt{\sigma}}, \quad a_2 = 0, \quad \mu = -\frac{r \sqrt{p^2 - r^2 + 1}}{\sqrt{2p^2 - 2r^2}},$$

we get

$$\chi_{1c}(x, t) = -\frac{\sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2} e^{i(px+rt)} \tanh\left(\frac{r \sqrt{|p^2 - r^2 + 1|} \left(\frac{pt}{r} + x\right)}{\sqrt{|2p^2 - 2r^2|}}\right)}{\sqrt{\lambda} r \sqrt{\sigma}}, \quad (35)$$

$$\psi_{1c}(x, t) = \frac{(-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2) \tanh^2\left(\frac{r \sqrt{|p^2 - r^2 + 1|} \left(\frac{pt}{r} + x\right)}{\sqrt{|2p^2 - 2r^2|}}\right)}{\lambda r^2 \left(\frac{p^2}{r^2} - 1\right)}. \quad (36)$$

Class-2: When

$$a_0 = \frac{a_1 \sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2}}{\sqrt{\lambda} r \sqrt{\sigma}}, \quad a_2 = -a_1, \quad \mu = \frac{\sqrt{2} \sqrt{p^2 r^2 - r^4 + r^2}}{\sqrt{p^2 - r^2}},$$

$$\chi_{2c}(x, t) = \frac{\sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2} e^{i(px+rt)} \sinh\left(\frac{\sqrt{2} \sqrt{|p^2 r^2 - r^4 + r^2|} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right)}{\sqrt{\lambda} r \sqrt{\sigma} \left(\cosh\left(\frac{\sqrt{2} \sqrt{|p^2 r^2 - r^4 + r^2|} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right) - 1\right)}, \quad (37)$$

$$\psi_{2c}(x, t) = \frac{(-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2) \sinh^2\left(\frac{\sqrt{2} \sqrt{|p^2 r^2 - r^4 + r^2|} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right)}{\lambda r^2 \left(\frac{p^2}{r^2} - 1\right) \left(\cosh\left(\frac{\sqrt{2} \sqrt{|p^2 r^2 - r^4 + r^2|} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right) - 1\right)^2}. \quad (38)$$

Assuming that Eq. (23) secures the solution of the form

$$\Theta(\eta) = \frac{a_0 \cosh(\mu\eta)}{a_2 + a_1 \sinh(\mu\eta)}. \quad (39)$$

Placing Eq. (39) and its derivatives into Eq. (23) gives a polynomial in power of hyperbolic functions. Collecting the coefficients of the same power of $\sinh(\mu\eta)$, and equating each summation to zero, provides a system of equations. Solving this system of equations, provides

Class-1: When

$$a_0 = \frac{a_1 \sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2}}{\sqrt{\lambda} r \sqrt{\sigma}}, \quad a_2 = 0, \quad \mu = -\frac{r \sqrt{p^2 - r^2 + 1}}{\sqrt{2p^2 - 2r^2}},$$

we get

$$\chi_{1d}(x, t) = -\frac{\sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2} e^{i(px+rt)} \coth\left(\frac{r \sqrt{|p^2 - r^2 + 1|} \left(\frac{pt}{r} + x\right)}{\sqrt{|2p^2 - 2r^2|}}\right)}{\sqrt{\lambda} r \sqrt{\sigma}}, \quad (40)$$

$$\psi_{1d}(x, t) = \frac{(-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2) \coth^2\left(\frac{r \sqrt{|p^2 - r^2 + 1|} \left(\frac{pt}{r} + x\right)}{\sqrt{|2p^2 - 2r^2|}}\right)}{\lambda r^2 \left(\frac{p^2}{r^2} - 1\right)}. \quad (41)$$

Class-2: When

$$a_0 = \frac{a_1 \sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2}}{\sqrt{\lambda} r \sqrt{\sigma}}, \quad a_2 = i a_1, \quad \mu = -\frac{\sqrt{2} \sqrt{p^2 r^2 - r^4 + r^2}}{\sqrt{p^2 - r^2}},$$

we get

$$\chi_{2d}(x, t) = \frac{\sqrt{-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2} e^{i(px+rt)} \cosh\left(\frac{\sqrt{2} \sqrt{p^2 r^2 - r^4 + r^2} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right)}{\sqrt{\lambda} r \sqrt{\sigma} \left(-\sinh\left(\frac{\sqrt{2} \sqrt{p^2 r^2 - r^4 + r^2} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right) + i\right)}, \quad (42)$$

$$\psi_{2d}(x, t) = \frac{(-p^4 + 2p^2 r^2 - p^2 - r^4 + r^2) \cosh^2\left(\frac{\sqrt{2} \sqrt{p^2 r^2 - r^4 + r^2} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right)}{\lambda r^2 \left(\frac{p^2}{r^2} - 1\right) \left(-\sinh\left(\frac{\sqrt{2} \sqrt{p^2 r^2 - r^4 + r^2} \left(\frac{pt}{r} + x\right)}{\sqrt{|p^2 - r^2|}}\right) + i\right)^2}. \quad (43)$$

3.2 Application to Eq. (2)

Here, we present the application of the extended rational sine/cosine and sinh/cosh methods to the coupled Maccari system.

Consider the wave transformation

$$\chi(x, y, t) = e^{i\theta} \Theta(\eta), \quad \psi(x, y, t) = \Phi(\eta), \quad \eta = x + y + kt, \quad \theta = ax + by + rt. \quad (44)$$

Placing Eq. (44) into Eq. (2), provides the following NODE:

$$\Theta^3 + (1 + c)(a^2 + r)\Theta - (1 + c)\Theta'' = 0 \quad (45)$$

from the real part, and the relation $k = -2a$ from the imaginary part.

Assuming that Eq. (45) secures (24) as its trial solution

Proceedings as before, we secure set of solutions as:

Class-1: When

$$a_0 = a_1 \left(-\sqrt{(1-2a)(a^2+r)}\right), \quad a_2 = 0, \quad \mu = \frac{\sqrt{a^2+r}}{\sqrt{2}},$$

we get

$$\chi_{1a}(x, y, t) = -\sqrt{a^2 + (1-2a)r} e^{i(ax+by+rt)} \tan\left(\frac{\sqrt{|a^2+r|}(-2at+x+y)}{\sqrt{2}}\right), \quad (46)$$

$$\psi_{1a}(x, y, t) = -(a^2 + r) \tan^2\left(\frac{\sqrt{|a^2+r|}(-2at+x+y)}{\sqrt{2}}\right). \quad (47)$$

Class-2:

$$a_0 = a_1 \sqrt{(1-2a)(a^2+r)}, \quad a_2 = -a_1, \quad \mu = -\sqrt{2} \sqrt{a^2+r},$$

we get

$$\chi_{2a}(x, y, t) = -\frac{\sqrt{(1-2a)(a^2+r)} e^{i(ax+by+rt)} \sin\left(\sqrt{2} \sqrt{|a^2+r|}(-2at+x+y)\right)}{\cos\left(\sqrt{2} \sqrt{|a^2+r|}(-2at+x+y)\right) - 1}, \quad (48)$$

$$\psi_{2a}(x, y, t) = -\frac{(a^2 + r) \sin^2 \left(\sqrt{2} \sqrt{|a^2 + r|} (-2at + x + y) \right)}{\left(\cos \left(\sqrt{2} \sqrt{|a^2 + r|} (-2at + x + y) \right) - 1 \right)^2}. \quad (49)$$

Assuming that Eq. (45) secures (29) as its trial solution

Proceedings as before, we get the following set of solutions:

Class-1: When

$$a_0 = a_1 \left(-\sqrt{(1-2a)(a^2+r)} \right), \quad a_2 = 0, \quad \mu = \frac{\sqrt{a^2+r}}{\sqrt{2}},$$

we get

$$\chi_{1b}(x, y, t) = \sqrt{(1-2a)(a^2+r)} \left(-e^{i(ax+by+rt)} \right) \cot \left(\frac{\sqrt{|a^2+r|}(-2at+x+y)}{\sqrt{2}} \right), \quad (50)$$

$$\psi_{1b}(x, y, t) = -(a^2+r) \cot^2 \left(\frac{\sqrt{|a^2+r|}(-2at+x+y)}{\sqrt{2}} \right). \quad (51)$$

Class-2: When

$$a_0 = a_1 \left(-\sqrt{(1-2a)(a^2+r)} \right), \quad a_2 = -a_1, \quad \mu = \sqrt{2} \sqrt{a^2+r},$$

we get

$$\chi_{2b}(x, y, t) = -\frac{\sqrt{(1-2a)(a^2+r)} e^{i(ax+by+rt)} \cos \left(\sqrt{2} \sqrt{|a^2+r|} (-2at+x+y) \right)}{\sin \left(\sqrt{2} \sqrt{|a^2+r|} (-2at+x+y) \right) - 1}, \quad (52)$$

$$\psi_{2b}(x, y, t) = -\frac{(a^2+r) \cos^2 \left(\sqrt{2} \sqrt{|a^2+r|} (-2at+x+y) \right)}{\left(\sin \left(\sqrt{2} \sqrt{|a^2+r|} (-2at+x+y) \right) - 1 \right)^2}. \quad (53)$$

Assuming that Eq. (45) secures (34) as its trial solution

Proceedings as before, we get the following set of solutions:

Class-1: When

$$a_0 = a_1 \sqrt{(2a-1)(a^2+r)}, \quad a_2 = 0, \quad \mu = -\frac{\sqrt{-a^2-r}}{\sqrt{2}},$$

we get

$$\chi_{1c}(x, y, t) = \sqrt{(2a-1)(a^2+r)} \left(-e^{i(ax+by+rt)} \right) \tanh \left(\frac{\sqrt{|-a^2-r|}(-2at+x+y)}{\sqrt{2}} \right), \quad (54)$$

$$\psi_{1c}(x, y, t) = -\frac{(2a-1)(a^2+r) \tanh^2 \left(\frac{\sqrt{|-a^2-r|}(-2at+x+y)}{\sqrt{2}} \right)}{1-2a}. \quad (55)$$

Class-2: When

$$a_0 = a_1 \left(-\sqrt{(2a-1)(a^2+r)} \right), \quad a_2 = -a_1, \quad \mu = \sqrt{2} \sqrt{-a^2-r},$$

we get

$$\chi_{2c}(x, y, t) = -\frac{\sqrt{(2a-1)(a^2+r)} e^{i(ax+by+rt)} \sinh \left(\sqrt{2} \sqrt{|-a^2-r|} (-2at+x+y) \right)}{\cosh \left(\sqrt{2} \sqrt{|-a^2-r|} (-2at+x+y) \right) - 1}, \quad (56)$$

$$\psi_{2c}(x, y, t) = -\frac{(2a-1)(a^2+r)\sinh^2\left(\sqrt{2}\sqrt{|-a^2-r|}(-2at+x+y)\right)}{(1-2a)\left(\cosh\left(\sqrt{2}\sqrt{|-a^2-r|}(-2at+x+y)\right)-1\right)^2}. \quad (57)$$

Assuming that Eq. (45) secures (39) as its trial solution

Proceedings as before, we secure the following set of solutions:

Class-1: When

$$a_0 = a_1\sqrt{(2a-1)(a^2+r)}, \quad a_2 = 0, \quad \mu = \frac{\sqrt{-a^2-r}}{\sqrt{2}},$$

we get

$$\chi_{1d}(x, y, t) = \sqrt{(2a-1)a^2+re^{i(ax+by+rt)}} \coth\left(\frac{\sqrt{|-a^2-r|}(-2at+x+y)}{\sqrt{2}}\right), \quad (58)$$

$$\psi_{1d}(x, y, t) = -\frac{(2a-1)(a^2+r)\coth^2\left(\frac{\sqrt{|-a^2-r|}(-2at+x+y)}{\sqrt{2}}\right)}{1-2a}. \quad (59)$$

Class-2: When

$$a_0 = a_1\left(-\sqrt{(2a-1)(a^2+r)}\right), \quad a_2 = -ia_1, \quad \mu = -\sqrt{2}\sqrt{-a^2-r},$$

we get

$$\chi_{2d}(x, y, t) = -\frac{\sqrt{(2a-1)(a^2+r)}e^{i(ax+by+rt)}\cosh\left(\sqrt{2}\sqrt{|-a^2-r|}(-2at+x+y)\right)}{-\sinh\left(\sqrt{2}\sqrt{|-a^2-r|}(-2at+x+y)\right)-i}, \quad (60)$$

$$\psi_{2d}(x, y, t) = -\frac{(2a-1)(a^2+r)\cosh^2\left(\sqrt{2}\sqrt{|-a^2-r|}(-2at+x+y)\right)}{(1-2a)\left(-\sinh\left(\sqrt{2}\sqrt{|-a^2-r|}(-2at+x+y)\right)-i\right)^2}. \quad (61)$$

4 Graphical description

In this section, we present the 3D, 2D, and their crossponding contour graphs to some of the reported solutions under a suitable choice of parameters.

Fig. 1 The (a) 2D, 3D and (b) contour surfaces of Eq. (17).

Fig. 2 The (a) 2D, 3D and (b) contour surfaces of Eq. (24).

Fig. 3 The (a) 2D, 3D and (b) contour surfaces of Eq. (25).

5 Conclusions

This study revealed the singular periodic, periodic wave, topological, topological kink-type, dark and singular soliton solutions to two important nonlinear complex mathematical models, namely; the Klein-Gordon-Zakharov equations, the (2+1)-dimensional Maccari system via the extended rational sine-cosine/ rational sinh-cosh methods. All the reported solutions satisfy the studied nonlinear models. To display the physical features of the studied models, the 3D, 2D and the contour graphs to some of the obtained solutions are presented. It is believed that the reported solutions may play an important role describing the physical features of various nonlinear complex models. The methods used are efficient and important mathematical tools that may be applied in obtaining solutions to various nonlinear models.

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