Applied Mathematics and Nonlinear Sciences

<https://www.sciendo.com>

Digital model creation and image meticulous processing based on variational partial differential equation

Pei Wang¹, Hani Jamal Sulaimani², Sae Hoon Kim^{1†}

¹Sejong University, Seoul, 05006, Korea

²Department of Computer Information Systems, Faculty of Computing and Information Technology,
King Abdulaziz University, Jeddah, Saudi Arabia

Submission Info

Communicated by Juan Luis García Guirao

Received June 6th 2021

Accepted September 24th 2021

Available online November 22nd 2021

Abstract

In the application of digital model of animation scene, image restoration technology and image denoising technology are the basic tasks of practical operation, which are closely related, but there exist also essential differences. The reason is that both of them want to obtain the original image from the degraded noise image or damaged image, but generally speaking, as there is no sufficient constraint information to accurately recover the original image, both of them are unwell-posed inverse problems. Therefore, on the basis of understanding the basic content and application research status of variational partial differential equations (PDEs), this paper discusses the application value of variational PDEs in image denoising and restoration according to the image processing requirements in the digital model of animation scenes.

Keywords: variational, partial differential equation, animation scene, digital model, images, denoising.

AMS 2010 codes: 62G08

1 Introduction

In the development of digital model construction of animation scene, the related digital image processing has a long history of development. Previous image processing theories mainly involve Fourier analysis, filtering theory and other content; the processing methods used are mostly simple and single heuristic methods and thus the practical application has a strong limit, and cannot be used to calculate part of the characteristics and performance of the algorithm for in-depth discussion. But as more and more researchers became involved in the field of digital image processing, the technology gradually shifted from alchemy to modern chemistry. Partial differential equation (PDE) method, the most critical content in mathematical analysis, is closely related to the

[†]Corresponding author.

Email address: kim0500@163.com

physical world. For example, common heat conduction equation and wave equation belong to PDE [1, 2]. In early studies, PDE was mainly used to represent physical and mechanical phenomena, while in today's research and discussion, PDE used in various fields, such as finance, biology, image processing and so on. According to the conclusion of Koenderink et al.'s study, the Gaussian convolution of the image is equal to the solution of the thermal diffusion equation. Thus, the relation between PDE dedrying and classical filter dedrying is constructed, and the result of the fusion with the concept of scale space proposed by Witkin is the basic content of PDE image processing. At the same time, Hummel also put forward the maximum and minimum value criterion of scale space according to the above content, which can be regarded as the mathematical explanation of causality. Guided by this, Perona and Malik et al. first proposed a denoising model based on nonlinear diffusion in the 1990s. However, after entering the 21st century, Tschumperle et al. analysed and verified that the diffusion tensor in the divergence operator framework could not directly present the local diffusion behaviour of the model by using the divergence mode of TV denoising model. Therefore, they finally proposed a denoising model with trace operator as the core. Finally, a highly efficient geometric adaptive denoising model based on the expression of trace operator is obtained by extending the framework of divergence group proposed by Weickert to the framework of trace operator. It should be noted that although this method can improve the drying level and edge retention level, there are still some problems that need to be improved in practical application. Therefore, on the basis of understanding the variational PDE, this paper makes a comparative analysis of the creation of the digital model of the animation scene and the image denoising and restoration technology, and finally makes clear the application advantages of variational PDE in the comparative analysis of the experimental effect [3, 4].

2 Digital model analysis of animation scene based on variational PDE

2.1 Image denoising

Definition 1. First, variational expression. Assuming it has been clear that the average value of Gaussian white noise $N \mu = 0$ meets this condition, then combined with the analysis of maximum likelihood criterion, an approximation of I can be obtained by using the computational least squares problem. The specific formula is shown as follows:

$$\min_I \int_{\Omega} ||I_{noisy} - I||^2 d\Omega$$

Theorem 1. In the above formula, $||I_{noisy} - I||^2$ represents a measure of image difference during denoising, as shown below:

$$||I_{noisy} - I||^2$$

Since n represents the Gaussian white noise with the mean $\mu = 0$ and the standard deviation of σ , all points in a radius sphere with I_{noisy} as the centre of the circle can be approximated by I , in other words, they may all be solutions of the above formula. Therefore, in order to obtain more valuable solutions, the related problems need to be normalised. If the smoothing property of the image can be measured by using the normalised term $R(I)$, and the corresponding I becomes more positive with the subsequent decline, then the optimal solution of I can be obtained according to the level set of R .

Proposition 2. Second, divergence operator expression. The divergence operator model framework proposed by Weickert regards the PDE image denoising problem as a chemical concentration diffusion process following certain physical laws. In this way, the variational PDE denoising model is sorted into a more general framework based on divergence operator. The specific formula is shown as follows:

$$\frac{\partial I}{\partial t} = \text{div}(D\nabla I)$$

Lemma 3. In the above formula, ∇I stands for the gradient field of the image, which corresponds to the concentration gradient of the chemical concentration diffusion, and D refers to the diffusion tensor field constructed

based on the diffusion tensor. At this time, the diffusion behaviour is combined with the following formula, and the corresponding D needs to be designed, so as to obtain the following formula:

$$\frac{\partial I}{\partial t} = \operatorname{div}(D \nabla I)$$

By comparison with the above formula, it can be seen that Φ functional expression is a special case of divergence expression, and the diffusion tensor corresponding to it is:

$$D = \frac{\phi'(\|\nabla I\|)}{\|\nabla I\|} Id$$

In the above formula, Id represents the identity matrix of 2×2 , and the following formula can also be regarded as a special case of the divergence expression, and the corresponding diffusion heat formula is:

Heat diffusion equation: $\frac{\partial I}{\partial t} = \Delta I = \operatorname{div}(\nabla I)$

The corresponding diffusion tensor is: $D = Id$

P-M model equation: $\frac{\partial I}{\partial t} = \operatorname{div}[g(\|\nabla I\|)\nabla I]$

The corresponding diffusion tensor is: $D = g(\|\nabla I\|)Id$

Selective smoothing model: $\frac{\partial I}{\partial t} = \operatorname{div}(g(\|G_{\sigma} * \nabla I\|)\nabla I)$

The corresponding diffusion tensor is: $D = g(\|G_{\sigma} * \nabla I\|)Id$

Corollary 4. Thirdly, trace operator expression. Combined with the above research and analysis on thermal diffusion, P-M model, Φ functional expression and directional smoothing, the expression form of trace operator can be seen as follows:

$$\frac{\partial I}{\partial t} = c_1 I_{uu} + c_2 I_{vv}$$

In the above formula, $u, v \in \mathbb{R}^2$ and $\perp v, c_1, c_2 \geq 0$, I_{uu} and I_{vv} represent the second directional derivative of image I in the direction u and v , respectively. Combined with the expression of the second order directional derivative, the expression formula of trace operator of its diffusion tensor can be clearly expressed as:

$$\begin{aligned} \frac{\partial I}{\partial t} &= c_1 I_{uu} + c_2 I_{vv} \\ &= c_1 \operatorname{trace}(Huu^T) + c_2 \operatorname{trace}(Hvv^T) \\ &= \operatorname{trace}[H(c_1 uu^T + c_2 vv^T)] \\ &= \operatorname{trace}(HT) \end{aligned}$$

Conjecture 5. In the above formula, $\operatorname{trace}()$ represents the trace operator and H represents the Hessian matrix. To accurately distinguish the diffusion tensor D contained in the discrete form, T should be used to represent the diffusion tensor expressed by the trace operator. The specific formula is as follows:

$$T = c_1 uu^T + c_2 vv^T$$

Meanwhile, since the Hessian matrix H and the diffusion tensor D belong to the symmetric matrix, the expression formula of the trace operator above can be transformed into:

$$\frac{\partial I}{\partial t} = \operatorname{trace}(TH)$$

In geometric adaptive denoising model, assuming that the η^*, ξ^* and μ_1^*, μ_2^* represent smooth structure tensor S * characteristic vector and the corresponding characteristic value, and meets the $\mu_1^* \geq \mu_2^*$ condition, so in order to make smooth behaviour can directly show the image of local structure, can be combined with

orthogonal smooth direction represent the related gradient direction and the direction of the illumination, such as line, In other words, let the eigenvectors of T , U , V and S^* be consistent:

$$\begin{cases} u = \eta^* \\ v = \xi^* \end{cases}$$

Example 6. In order to remove the noise effectively and retain the edge as much as possible, it should be processed in accordance with the direction of the relevant gradient and the smoothing intensity of the direction of the isoluminance line. In other words, the characteristic value of the diffusion tensor T , λ_1, λ_2 should be adjusted autonomically according to the characteristics of the region in which it is located. When processing different image features, Tschumperle uses $\sqrt{\mu_1^* + \mu_2^*}$ to distinguish them. In this case, λ_1, λ_2 represents the $\sqrt{\mu_1^* + \mu_2^*}$ function. In this process, the following design principles need to be observed: First, the isotropic smoothing and effective denoising is implemented in the non-edge region, and the specific formula is as follows:

$$\frac{\partial I}{\partial t} \approx \Delta I = \text{trace}(H) \quad \text{namely:} \quad \lim_{\sqrt{\mu_1^* + \mu_2^*} \rightarrow 0} T = Id$$

And should be reflected in: λ_1, λ_2

$$\lim_{\sqrt{\mu_1^* + \mu_2^*} \rightarrow 0} \lambda_2 = \lim_{\sqrt{\mu_1^* + \mu_2^*} \rightarrow 0} \lambda_2 = 1$$

Second, the anisotropic smoothing process is implemented in the edge region, and the edge region is preserved as far as possible while the noise is removed. At this time, smoothing should be processed with the direction of ξ^* of isoluminance line. The specific formula is as follows:

$$\frac{\partial I}{\partial t} \approx \lambda_2 \xi^* \xi^{*T} = \text{trace}(\lambda_2 \xi^* \xi^{*T} H)$$

Also available:

$$\lim_{\sqrt{\mu_1^* + \mu_2^*} \rightarrow \infty} \frac{\lambda_1}{\lambda_2} = 0$$

The choices proposed by Tschumperle that meet the above requirements are:

$$\begin{cases} \lambda_1 = \frac{1}{1 + \mu_1^* + \mu_2^*} \\ \lambda_2 = \frac{1}{\sqrt{1 + \mu_1^* + \mu_2^*}} \end{cases}$$

By referring the processing formula of the edge region above to the calculation formula of the trace operator's diffusion tensor, the expression formula of T can be obtained:

$$T = \frac{1}{1 + \mu_1^* + \mu_2^*} \eta^* \eta^{*T} + \frac{1}{\sqrt{1 + \mu_1^* + \mu_2^*}} \xi^* \xi^{*T}$$

2.2 Image restoration

Note 7. First, the repair model based on variational expression. The expression form of divergence operator of TV repair model can be obtained by combining with Euler-Lagrange equation. Without analysing the normalisation factor A , the actual expression formula is as follows:

$$\frac{\partial I}{\partial t} = \text{div} \left(\frac{\nabla I}{\|\nabla I\|} \right) + \lambda_e (I - I^0)$$

The calculation formula of the corresponding diffusion tensor D is:

$$D = \frac{1}{\|\nabla I\|} Id$$

Therefore, it is proved that D belongs to a diagonal matrix, and part of the behaviour of TV repair model can be regarded as nonlinear isotropic diffusion. At the same time, combining with the above analysis and research on the Φ functional, it can be clear that the trace operator expression formula of the corresponding model is:

$$\frac{\partial I}{\partial t} = 0I_{\eta\eta} + \frac{1}{\|\nabla I\|} I_{\xi\xi} = \text{trace}(TH)$$

And the expression formula of T is:

$$T = c_{\eta}\eta\eta^T + c_{\xi}\xi\xi^T = \frac{1}{\|\nabla I\|}\xi\xi^T = \frac{1}{\|\nabla I\|^3} \begin{pmatrix} -I_x I_y & I_y^2 \\ I_x^2 & I_x I_y \end{pmatrix}$$

At this point, the eigenvector of T represents gradient direction η and vertical direction ξ , and the corresponding eigenvalues are 0 and.

Considering the above repair algorithm from the perspective of comprehensive adaptive threshold and multi-scale, it can be seen that the constant A needs to be introduced into the following formula, whose essence is to prevent 0 items from appearing in the denominator. Meanwhile, the uniqueness of this repair model can be determined after studying the influence of different choices of A on the repair.

$$\frac{\partial I}{\partial t} = \text{div} \left(\frac{\nabla I}{\sqrt{a^2 + \|\nabla I\|^2}} \right) + \lambda_e(I - I^0)$$

Combined with the above formula analysis, it can be seen that under the condition of a $\|\nabla I\|$, it can be approximated as:

$$\begin{aligned} \frac{\partial I}{\partial t} &\approx \text{div} \left(\frac{\nabla I}{a} \right) + \lambda_e(I - I^0) \\ &\approx \Delta I + \lambda_e(I - I^0) \end{aligned}$$

Open Problem 8. In other words, under a $\|\nabla I\|$, the effect of the above formula is approximately linear isotropic diffusion.

Second, the repair model based on divergence operator expression. Although using the variational method to deal with and analyse the denoising model can provide a global explanation for the denoising process, part of the flexibility of model design will be lost, and an efficient repair model cannot be obtained. Therefore, it is necessary to use more local expression forms to deal with the problem, namely discrete operator and trace operator. Taking the curvature-driven (CDD) repair model as an example, D is regarded as the area waiting to be repaired, and E is regarded as the outer neighbourhood of the area waiting to be repaired. Under the condition of no noise analysis, the TV repair model can be simplified as:

$$\min_{I \in E \cup D} \|\nabla I\| dx dy$$

The actual equation of the most rapid descent is:

$$\frac{\partial I}{\partial t} = \text{div} \left(\frac{1}{\|\nabla I\|} \nabla I \right)$$

The Fast Curvature Driven (FCDD) repair model improves the actual computing speed on the basis of the original model, and the specific form is shown as follows:

$$\frac{\partial I}{\partial t} = \text{div}(g(|k|)\nabla I)$$

Thirdly, the repair model based on trace operator expression. The repair model with divergence operator as the core mainly regards the repair process as a process of information diffusion to the damaged area. The core of the application of this model is that the diffusion characteristic of each point in the damaged area is the corresponding diffusion tensor. Since the diffusion tensor contained in the circuit operator cannot fully and intuitively present the actual diffusion situation, the reference trace operator should be written to express it. Generally speaking, trace operator expression will push information into the damaged area one by one in a smooth way. It is same as the expression form of divergence operator, and its design core is also a diffusion tensor. Taking the self-aware repair model as an example, it is assumed that I_0 represents the original image, D represents the area waiting to be repaired, and D^c refers to the complement of D , that is, the clear area without damage in the image. In this case, the image repair model based on trace operator expression is:

$$\begin{cases} \frac{\partial I}{\partial t} = \text{trace}(TH), X \in D \\ I = I^0, X \in D^c \end{cases}$$

In the above formula, H represents the Hessian matrix and T represents the diffusion tensor, which can intuitively present the smooth behaviour of some regions of the model.

The key difficulty of using trace operator expression to design image repair model lies in diffusion tensor T , which is related to the corresponding feature value λ_1, λ_2 as follows:

$$T = \lambda_1 uu^T + \lambda_2 vv^T$$

In the curvature-driven directional 1DLaplacians repair model, the final solution can be calculated using the classical time-stepping method, which can be divided into the following points:

- (1) The central difference method is used to analyse the first-order spatial derivatives I_x, I_y and the second-order spatial derivatives I_{xx}, I_{yy} and I_{xy} ;
- (2) Calculate the curvature K and integrate it into the structural tensor field S and Hessian matrix field H , as follows:

$$k = \frac{I_{xx}I_y^2 - 2I_xI_{xy} + I_{yy}I_x^2}{(I_x^2 + I_y^2)^{3/2}}$$

$$S = \begin{pmatrix} I_x^2 & I_{xy} \\ I_{xy} & I_y^2 \end{pmatrix}$$

$$H = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix}$$

- (3) Clarifying the smooth connecting tensor field S^* , and clarifying the local structure geometry μ_1^*, μ_2^* and η^*, ξ^* on the basis of the decomposition of characteristic values.
- (4) Calculate the diffusion tensor T with the following formula:

$$T = \frac{1}{1 + 1/|k|} \eta^* \eta^{*T} + \xi^* \xi^{*T}$$

Calculate the iteration rate β of PDE:

$$\beta = \text{trace}(TH)$$

- (6) Update the image:

$$I^{(n+1)} = I^{(n)} + \Delta t \beta$$

In the above formula, ΔT represents the length of the time distribution, and I represents the value of the waiting repair point at the time of $N \Delta T$.

Repeat the above steps until the repair work is completed [5, 6].

3 Result analysis

From the combination of the hierarchical diagram analysis of the expression of variational, divergence and trace operators as shown in Figure 1, it can be observed that three expression models for image denoising and restoration are proposed in order to optimise the application effect of the digital model of the animation scene in this paper. Figure 2 shows the repair effect. In the variational restoration model, the results obtained by the restoration of this image are shown in Figure 3.

Fig. 1 Hierarchical relationship analysis diagram.

Fig. 2 Repair effect of scratches.

According to the analysis of Figure 3, (a) represents the original TV model algorithm, (b) represents the adaptive threshold model algorithm and (c) represents the comprehensive adaptive threshold and multi-scale repair algorithm. It can be seen from the variation of the curve value that the actual repair time of the original algorithm will get higher and higher as the number of iterations increases. The same is true for the adaptive threshold, the comprehensive adaptive threshold and the multi-scale repair algorithm, but compared with the comprehensive adaptive threshold and the multi-scale repair algorithm, the number of iterations and repair time are less [7].

In the model based on divergence operator expression, the time required by CDD and FCDD algorithm in image restoration and the final rendering effect are compared and analysed, as shown in the Table 1.

Fig. 3 Curve change diagram of image change.

Table 1 The number and time of iterations required by CDD and FCDD models

	Bar		The ring		Face		LAKE	
	CDD	FCDD	CDD	FCDD	CDD	FCDD	CDD	FCDD
Number of iterations	19980	824	47012	3972	47841	550	13822	598
Repair time	94.3 s	7.3 s	1433.3 s	125.0 s	987.6 s	11.9 s	307 s	13.3 s
Number of restored pixels	150		1659		894		992	

CDD, curvature-driven; FCDD, fast curvature driven.

In the model based on trace operator expression, the simulated image and the real image are used for comparative experimental analysis, and the final results of the overview model, the geometric adaptive repair model and the TV model are compared and studied. Through the observation test in this paper, on the edge of the algorithm to the damaged area larger vertical and inclined to repair ability, the damaged area is to initialise white before repair, and the results of the TV to repair the false edge, geometric adaptive model repair results cannot be damaged area of the inner region surrounding the spread of information, This is bound to form a false edge in the damaged edge area. From the observation of the repair results of the model outlined in this paper, we can see that the information is fully diffused to the damaged area and the sharpness of the repair edge can be ensured [8,9].

4 Conclusion

To sum up, combining the research and analysis of the variational PDE image denoising and restoration technology, it can be seen that the requirement of image detailed processing is extremely high in the application of the creation of digital model of animation scene based on variational PDEs. Therefore, although some achievements have been made in the practical development, there are still many challenges. So, in the future, researchers, with the continuous optimisation and promotion of network technology, should make use of variational PDEs to do image denoising and restoration according to the requirements of image processing on the basis of a comprehensive understanding of the creation requirements of digital models of animation scenes.

Acknowledgement: Anhui Province Humanities and Social Sciences Key Project ‘a study on the Construction of Animation Semiotics in City Brand Promotion—considered Wuhu, a national civilised city, as an example’ (SK2018A0721).

References

- [1] Ningli Luan, Cong. Jin Total Variational Image Denoising Model Based on Weighting Function [J]. *Electronic Measurement Technology*, 2018, 041(007):58-63.
- [2] Kumar B, Halim A. A linear fourth-order PDE-based gray-scale image inpainting model[J]. *Computational and Applied Mathematics*, 2019, 38(1):-.
- [3] F Tony, Chan Shen., Jianhong (Jackie)Shen, Luminivase, et al. A Variational Partial Differential Equation Model for Image Processing (I)[J]. *Mathematics Translin*, 2004.
- [4] Jing He, Xiaoqian You. Research on Parallel Image Enhancement Based on Total Variational Model. *Microcomputer Information*, 2008, 24(003):314-316.
- [5] Shen J. A Stochastic-Variational Model for Soft Mumford-Shah Segmentation[J]. *Int J Biomed Imaging*, 2006(14):92329.
- [6] Shaomei Fang, Duanshan Huang, Zhong Chen. Mathematical problems in image processing [J]. *Journal of Shaoguan University*, 2005, 026(009):1-3.
- [7] Wei W, Zhou B, D Połap, et al. A regional adaptive variational PDE model for computed tomography image reconstruction[J]. *Pattern Recognition*, 2019, 92:64-81.
- [8] Barcelos C Z. A new stochastic variational PDE model for soft Mumford–Shah segmentation[J]. *Journal of Mathematical Analysis & Applications*, 2011, 384(1):104-114.
- [9] Wu Y D, Sun Y, Zhang H Y, et al. Variational PDE based image restoration using neural network[J]. *Iet Image Processing*, 2007, 1(1):85-93.

This page is intentionally left blank