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The term structure of economic management rate under the parameter analysis of the estimation model based on common differential equation

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Abstract

To reflect the country's economic growth, inflation and the implementation of monetary policies. Based on the monthly data of national debt yield from January 2015 to December 2019, these data are divided into 1 year to 30 years according to the maturity period, and the principal component analysis of the term structure of interest rate from 2012 to 2017 shows that the factors affecting the change of term structure of interest rate include level factor, skew factor and curve factor. The variance contribution rates of these factors to the variation of interest rate term structure curve are 82.2002%, 16.9948% and 0.6283% respectively. The horizontal factor represents the position of the term structure of interest rate, the skew factor represents the degree of skew of the term structure of interest rate, and the curve factor determines the interest rate.

Keywords: ordinary differential equation, CIR model, Nelson-Siegel model.**AMS 2010 codes:** 34K60

1 Introduction

In financial markets, interest rates reflect the proxy variable of asset prices, reflecting the nation on macroeconomic growth, inflation and monetary policy implementation, and the micro factor is associated with fixed income products pricing, portfolio and risk management of asset returns, which has an important reference, is one of the most core variables of the financial research field [1]. The term structure of interest rate reflects the

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functional relationship between the price of financial products at a certain point and its maturity based on the interest rate, and measures the risk contained in the recovery period of assets. At present, China is vigorously promoting the marketisation of interest rates, so it is of great practical significance and theoretical value to study the relationship between the morphological changes of the term structure curve of interest rate and macroeconomic variables. This paper takes the data of interest rate term structure implied by bond prices of the Shanghai Stock Exchange as the analysis object. Firstly, the CIR model was used to fit the selected state variables, and the optimal solution of model parameters was estimated. Secondly, the parameters are substituted into the derived ordinary differential equations to calculate the parameters into the return rate fitting equation. Finally, according to the return fitting equation to judge the fitting effect of the model on the fluctuation of government bond yield, the government bond yield is predicted and the estimated results are analysed and discussed dynamically.

As is known to all, the term structure of interest rate is of great value to the pricing of financial assets, hedging and the formulation of monetary authorities' policies, and it is of great significance to select a model to estimate the term structure of interest rate that is suitable for the characteristics of China's national debt market. The contrastive analysis of this paper is innovative. In terms of empirical comparative analysis, this paper systematically analysed and studied the Nelson-Siegel model and Svensson model from four aspects of cross-section data, in-sample, out-of-sample, and stability of model parameters, and introduced the mean price absolute error MEAP and root mean square error RMSE to make an intuitive comparison. The fitting effects of the two models were compared.

2 Research methods

This paper takes the data of interest rate term structure implied by bond prices of the Shanghai Stock Exchange as the analysis object. Firstly, the CIR model is used to fit the selected state variables, and the optimal solution of model parameters was estimated. Secondly, the parameters are substituted into the derived ordinary differential equations to calculate the parameters into the return rate fitting equation. Finally, according to the return fitting equation to judge the fitting effect of the model on the fluctuation of government bond yield, the government bond yield is predicted and the estimated results are analysed and discussed dynamically.

According to the research content, there are mainly two parts. Firstly, the data of Treasury bond yield spot interest rate are obtained, and the data from January 2015 to December 2019 are intercepted as samples, in which they are fitted and are forecasted. To reflect the ability of the model, in-sample fitting and out-of-sample prediction are done [2]. Secondly, the establishment of the term structure model of affine interest rate is described in detail. Based on the CIR model and ordinary differential equations, the establishment of the term structure model of affine interest rate and the estimation methods used are described. Based on the research of domestic and foreign scholars on the term structure of interest rates, this paper used the Nelson-Siegel model and the Svensson model to fit the monthly data from 2012 to 2016, and compared and analysed the fitting effect of the two models, to select an estimated model of the term structure of interest rates that is more suitable for the characteristics of China's national debt.

2.1 Development path of affine model

Because the price of a zero-coupon bond at maturity is 1, $P_T^T = 1$. The relationship between the yield to maturity and the price of a zero-coupon bond at any given time is as follows:

$$P_t^T e^{(T-t)R_t^T} = 1 \quad (1)$$

Where, P_t^T is the price of the bond with a maturity of t at time T , and can be the yield to maturity of the zero-coupon bond with a maturity of T at time T calculated with continuous compound interest. From the expression (1), we can get:

$$R_t^T = -\frac{\ln P_t^T}{T-t} \quad (2)$$

So R_t^T determines the yield of the zero-coupon bond with a maturity of t at time t . In addition:

$$f_t^T = \lim_{t \rightarrow T} -\frac{\ln P_t^T}{T} = -\frac{1}{P_t^T} \frac{\partial \ln P_t^T}{\partial T} \quad (3)$$

f_t^T is the instantaneous forward rate. Expression (3) describes the relationship between the instantaneous forward rate and the price of the zero-coupon bond with a maturity of t at time t .

For r_t , which is risk-free, λ , λ , λ

$$r_t = \lim_{t \rightarrow T} R_t^T = \lim_{t \rightarrow T} -\frac{1}{P_t^T} \frac{\partial P_t^T}{\partial T} = -\frac{\partial P_t^T}{\partial t} \quad (4)$$

Generally speaking, the affine model deduces the dynamic change process of the term structure of interest rate by assuming the random change process of factors or state variables. Each node of the change of these state variables may contain factors that produce random perturbations to the LIV term structure. These factors are sometimes regarded as variables that cannot be observed, such as instantaneous forward interest rate, and these variables are sometimes related to macro factors such as inflation rate. In general, it is assumed that the instantaneous interest rate follows the following process:

$$dx_t = \mu(x_t, \theta)dt + \sigma(x_t, \theta)dW_1 \quad (5)$$

Equation (5) assumes that the instantaneous interest rate follows the change process of Brownian motion, and it can be seen that the minimal change of the state variable can be expressed by the function of interest rate. Where, $\mu(X_1, R)$ is the drift term, which is not random. $\sigma(x_t, \theta)$ is the variance term, which is a random term involving standard Brownian motion. Combined with Ito's lemma and the constraint condition of no arbitrage, the functional relationship between the term structure of interest rate and the state variable can be obtained. However, due to the diversity of function forms and the complexity of solving overruns, sometimes the equation may not have a definite solution, or the solution may not be unique [3]. The initial conditions are constrained so that the equation has a unique solution, and this property does not change as more state variables are added to the model.

The following assumption is implied in the Vasicek model. Under the assumption of risk neutral, the change process of the short-term interest rate R is $dr = a(b - r)dt + \sigma dz$, where σ , A and B are constant. The feature of mean recovery is included in the model, and B is the long-term mean. Is the recovery rate, and σdz is the random disturbance term. The model concludes that

$$R_t^T = -\frac{1}{T-t} \ln A_t^T + \frac{1}{T-t} B_t^T r_t \quad (6)$$

Type $B_t^T = \frac{1 - \exp(-a(T-t))}{a}$, $\beta = (\beta_0, \beta_1, \beta_2)$

It can be seen from the above equation that when A , B , σ is determined, the term structure of interest rates can be expressed as a function of short-term interest rates.

CIR model. Cox, Ingersoll and Ross proposed the CIR model, which implicitly assumes that interest rates are always integers (which is obviously not realistic since there have been negative interest rates in the course of inflation), the fluctuation of short-term interest rate according to the model is subject to $dr = a(b - r)dt + \sigma\sqrt{r}dz$, and it can be seen from the above formula that the change process of interest rate: is also set as mean regression. Different from Vasicek's model, the volatility is no longer constant 2, but is proportional to $\sqrt{r}$, which means that the fluctuation degree of short-term interest rate increases with the increase of interest rate. In the CIR model, the bond price is marked as $P_t^T = A_t^T \exp(B_t^T r_t)$, where

$$B_t^T = \frac{2e^{r(T-t)} - 1}{(r+a)(b^{r(T-t)} - 1 + 2r)}, A_t^T = \left[\frac{2re^{(r+a)(T-t)/2}}{(r+a)(b-1) + 2r} \right]^{2ab/a^2}$$

2.2 Nelson-Siegel and its extended model

Nelson-Siegel term structure model of interest rates

Andrew Siegel and Charles Nelson first proposed the Nelson-Siegel model [M], which established the term structure model of the interest rate for the estimation of instantaneous forward interest rate parameters, there are relatively few parameters that can be directly fitted, so it is often used to estimate the forward rate function of the Nelson-Siegel term structure of interest rates with limited samples. The general expression of the forward rate function of the Nelson-Siegel term structure model is:

$$f(t, \tau, \beta) = \beta_0 + \beta_1 \exp\left(-\frac{\tau}{\lambda}\right) + \beta_2 \left(\frac{\tau}{\lambda} \exp\left(-\frac{\tau}{\lambda}\right)\right)$$

Its function expression can be obtained from the spot interest rate function as $r(\tau) = \frac{1}{\tau} \int_0^\tau f(s) ds$:

$$r(\tau) = \beta_0 + \beta_1 \left(\frac{1 - \exp\left(-\frac{\tau}{\lambda}\right)}{\frac{\tau}{\lambda}} \right) + \beta_2 \left(\frac{1 - \exp\left(-\frac{\tau}{\lambda}\right)}{\frac{\tau}{\lambda}} - \exp\left(-\frac{\tau}{\lambda}\right) \right)$$

Where: τ represents the maturity of the bond, where $\beta = (\beta_0, \beta_1, \beta_2)$ is the parameter to be estimated [4]. $\beta_0, \beta_1, \beta_2$ represents the level, slope and curve factors respectively, while the term structures of forward interest rate and long-term interest rate show different yield curve shapes under the influence of these three factors. β_1 represents the level of long-term interest rates, and β_0 is a constant with a factor load number of 1, so its effect on short, medium and long-term interest rates is synchronous, that is, with the increase of maturity τ , the spot interest rate $y(\tau)$ gradually approaches β_0 , if the maturity period is close to infinity, then the spot rate is equal to β_1 which is a monotone decreasing function with a factor load of $(1 - \exp(-\frac{\tau}{\lambda}))/\frac{\tau}{\lambda}$ which has a minimum value of zero, β_1 has only a short-term effect on interest rates, and affect the change speed is more important; The factor load number of β_2 is $1 - \exp(-\frac{\tau}{\lambda}))/\frac{\tau}{\lambda} - \exp(-\frac{\tau}{\lambda})$, as the solstice period increases, it increases first and then gradually decreases to 0, its influence on the interest rate is between β_0 and β_1 , which is a medium term factor, λ is called the decay rate, which affects the decay rate of the short and medium term terms, namely, the smaller the λ value is, the faster the short term and medium term terms in the term structure of the interest rate decay, if the value of λ is larger, the short term and medium term of the term structure of interest rate will decay more slowly, in general, in the fitting process of the term structure of long-term interest rate, the short-term term and the medium-term term decay rapidly, while in the fitting process of short-term interest rate, both the short-term term and the medium-term term decay slowly, so as to improve the fitting effect of the short-term interest rate curve. When the value of λ is fixed, the corresponding state changes as the maturity of the interest rate changes as shown in the figure below:

In the Treasury bond market, short-term interest rates are low, and long-term interest rates are high because the increase in maturities makes investors demand higher risk compensation. The rate of return increases with the increase of maturity, and the increase rate is getting smaller and smaller. As can be seen from the figure above, when the maturity is between 0 years and 5 years, the estimated spot interest rate of the two models rises relatively fast, while the rising trend of long-term bonds slows down, and bonds over 30 years are basically flat [5]. In the national debt market, the number of mid- and long-term bonds is not large, and the infrequent trading results in the slow rise of the spot interest rate curve of mid- and long-term bonds. The yield rate of China's national debt market is generally lower than that of foreign national debt markets. There is little difference between long-term and short-term national debt, so it cannot accurately reflect the risk situation.

In fitting the term structure of interest rates, we need to fit the spot interest rate at continuous time points. The Nelson-Siegel model can fit a shape of the spot interest rate at different time points, and can fit different shapes of the term structure of interest rates according to different maturities of interest rates, as shown in Figure 2.

The term structure of interest rate fitted by the Nelson-Siegel model accurately reflects the current and future changes of China's interest rate in the medium and long term. The Nelson-Siegel model has the advantages

Fig. 1 Parameter changes of Nelson-Siegel model.

Fig. 2 The yield curve that the Nelson-Siegel model can depict.

of less parameter estimation, clear economic significance of parameters, high fitting accuracy and relatively stable model, and it can well reflect the term structure of the interest rate of China's national debt. According to the results of the principal component analysis in Chapter III, the three factors that affect the yield curve are horizontal factor, slope factor and curve factor, which are also consistent with the economic meaning of the parameters of the Nelson-Siegel model and can well explain the term structure of the interest rate of China. In terms of the characteristics of China's national debt market, it is the best estimation method for both traders in the exchange market and monetary policy makers. Nelson-Siegel model can provide scientific price estimation and judgement for investors' investment decisions, and also provide an effective basis for investors' short-term, medium-term and long-term investment portfolios [6].

3 Empirical test based on affine model

The fitting structure of the state variable to the short-term interest rate is shown in Table 1. The short-term interest rate adopts the national debt yield with a maturity of half a year.

Table 1 Fitting results of state variables to short-term interest rates

The variable name	δ_0	δ_{cc}	δ_{m2}	δ_{ppi}	δ_{um}
Regression results	10.1_5 (p = 0.00) (t = _5.2_5)	-0.04 (p = 0.00) (t = 2.13)	-0.87 (p = 0.00) (t = 1.98)	0.21 (p = 0.00) (t = 3.03)	-1. 50 (p = 0.00) (t = 6.08)

As can be seen from the estimation results in Table 1, consumer confidence index, M2 growth rate and unemployment rate have a negative relationship with the short-term interest rate, while the producer price index has a positive relationship with the short-term interest rate. The reason for the inverse relationship between B1 and interest rate is that the rise of interest rate will lead to consumption, consumer loans and secondly, if the rise of interest rate is caused by the increase of government expenditure, it will lead to crowding out effect, thus further reducing consumer demand. This is also related to the consumption habits of the public. Consumers' consumption is easy to increase with the increase of income but not easy to decrease with the decrease of income. This is the ratchet effect of consumption.

3.1 Based on the data integrity standard

This paper selects the interest rate data of the Treasury bonds of the Shanghai Stock Exchange. The sample data includes the spot interest rate of the Treasury bonds with maturities of 3 months, 6 months, 9 months, 1 year, 2 years, 3 years, 4 years, _5 years, 7 years and 10 years. The time interval is from January 2010 to the closing price at the end of January 2015 (a total of 61 periods). Take the logarithm of the closing price to get the monthly yield of Shanghai bonds: $Yr = W Pr) -w Pr- \}$). The closing price data of Treasury bonds is from the Wind database. The basic nature of the data obtained by descriptive statistics on the national debt yield data is shown in Table 2:

Table 2 Descriptive statistics of government bond yields

Remaining life	The mean	The maximum	The minimum value	The median	Partial degrees	Kurtosis	The standard deviation
3 months	0.0287	0.0415	0.0138	0.0357	-0.1422	-0.0224	0.0069
6 months	0.0290	0.0416	0.0149	0.0357	-0.3901	-0.0949	0.0067
9 months	0.0296	0.0422	0.0285	0.0952	-0.4056	-0.4153	0.0066
1 year	0.0313	0.0443	0.0198	0.0251	-0.2928	0.0628	0.0059
2 years	0.338	0.0453	0.0233	0.0451	-0.0411	0.7961	0.0059
3 years	0.345	0.0446	0.0245	0.0341	0.0708	0.0.7562	0.0052
4 years	0.0365	0.0460	0.0253	0.0357	0.0357	0.0357	0.0041
5 years	0.0380	0.456	0.0293	0.0368	0.0.2449	-0.06845	0.0034
7 years	0.0413	0.0487	0.0333	0.0402	0.0357	0.0251	0.0030

3.2 The Svensson model

The Svensson model was developed based on the Nelson-Siegel model. To improve the fitting effect and get a more complex graph of the term structure of interest rates, Svensson added another term to the formula of the Nelson-Siegel model, namely, the fourth term. Then the instantaneous forward rate function is as follows:

$$r(m) = \beta_0 + \beta_1 \exp(-m/\tau_1) + \beta_2 [(m/\tau_1) \exp(-m/\tau_1) + \beta_3 [(m/\tau_2) \exp(-m/\tau_2)]$$

The addition of the fourth term can well fit the term structure of interest rates, making its graph appear a complex shape. In this equation, β_0 is the long-term interest rate, which is the asymptote of the yield curve, as M gets bigger and bigger, the yield curve should be closer to β_0 , β_1 is the short-term interest rate, and this parameter explains why the yield curve gets closer to the asymptote, when β_1 is greater than zero, the yield curve shows an upward trend, when β_0 is less than zero, the yield curve shows a downward trend, β_1 and β_3 represent different intermediate parts, τ_1 and τ_2 are the time numbers, indicating the position risk of the extreme signal point [7]. In general, the Nelson-Siegel model can fit the term structure of interest rates well, but when the term structure is very complex, the Nelson-Siegel model cannot fit the estimate well. At this time, the Svensson model can fit the estimate of the complex term structure of interest rates well. The parameters of the Nelson-Siegel model have obvious economic implications and have been widely adopted.

In conclusion, the Nelson-Siegel model is more suitable than the Svensson model to construct the term structure of the interest rate of China's national debt due to the immature development of China's national debt market, relatively few types of national debt and infrequent transactions. The Nelson-Siegel model has fewer estimated parameters and obvious economic implications of parameters. The model has the advantages of a good fitting effect and relatively stable model, so it is more suitable to construct the term structure of national debt interest rate in China. From the point of view of capital pricing theory, the conclusion is consistent with the theory. From the perspective of B_i/N in Figure 2, the reduction of the unemployment rate will lead to a faster reduction of the short-term interest rate than the long-term interest rate. Just as mentioned above, the short-term interest rate is more sensitive to state variables, which can be explained by the policy advice of the central bank and modern capital asset pricing theory proposed by Aylor's rule (1993).

From the perspective of Aylor's rule, the rise of the unemployment rate will lead to the decline of output and inflation, which will lead to the reduction of policy interest rate. From the perspective of capital asset pricing theory, the rise of the unemployment rate will cause people to increase the expected value of marginal utility of future assets, which will lead to the reduction of a risk-free interest rate. Similarly, the effect of consumer confidence on bond yields could be explained. Since the decrease of the consumer confidence index represents the increase of the expected value of the non-marginal utility of assets and the decrease of the risk-free interest rate, the reduction of the inflation rate is explained from the perspective of Ding Aylor's rule.

Similarly, when the consumer price index rises, the risk-free interest rate will rise. According to Dean Taylor's rule, we can regard this situation as the central bank's strategy to offset the rise in inflation. Finally, from the IS/LM model, when the money supply increases, it will lead to the reduction of short-term interest rates. At present, China's bond market is still undeveloped, and there are many distortions in bond pricing. The credit ratings of national bonds, corporate bonds, financial bonds and urban investment bonds are seriously differentiated, and bond pricing is highly dependent on credit rating [8].

Based on this, from the perspective of policy analysis: on the one hand, China should vigorously promote the construction of the exchange market and the OTC market interest rate system, increase reform efforts, break the rigid payment, and gradually promote the process of interest rate marketisation. On the other hand, during the implementation of monetary policy and the implementation of interest rate marketisation, it is necessary to construct an appropriate prediction model of interest rate term structure, monitor the dynamic development of the market, feedback the implementation plan of monetary policy, and gradually designate the term structure of interest rate as an economic leading indicator, to provide a basis for the implementation and monitoring of China's monetary policy [9, 10].

4 Conclusion

The variance contribution rates of the first three major factors to the variation of the interest rate term structure curve are 82.2002%, 16.9948% and 0.6283% respectively. The horizontal factor represents the position of the interest rate term structure, and the inclined factor represents the degree of the interest rate term structure. The curvature factor determines the degree of curvature of the term structure of interest rates. On the whole, the Nelson-Siegel model is smaller than the Svensson model in terms of both the mean price absolute error MEAP and the root mean square error RMSE, indicating that the overall fitting effect of the Nelson-Siegel model is better than that of the Svensson model. The level factor is negatively correlated with economic increment and money supply, and positively correlated with the consumer price index. Short - and long-term interest rate spreads are positively correlated with economic growth and money supply, and negatively correlated with consumer price index. The curvancy factor is positively correlated with economic increment and money supply, and negatively correlated with the consumer price index. In general, based on the characteristics of China's government bond market, the fitting effect of the Nelson-Siegel model for different maturities of different bonds is relatively stable, which is suitable for China's government bond market.

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