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Law of interest rate changes in financial markets based on the differential equation model of liquidity

Pengfei Wan^{1*}, Ahmed Mohamed Hamad Arbad²

¹Department of Key Construction and Development Planning, Qiqihar University, Hei Longjiang, 161005, China

²Department of Business Administration, Applied Science University, Kingdom of Bahrain

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Abstract

The paper establishes a related differential equation model about changes in financial interest rates. It uses information related to liquidity to feedback the law and stability of differential equations in interest rate changes. The article applies stochastic processes and partial differential equations to complex financial networks to confirm node yields in financial market networks. It confirms the existence of interest rate stickiness in Chinese financial markets. The advantage of this interest rate model is that when the external economic environment changes, the state of interest rates will also change accordingly.

Keywords: rate of return, liquidity, differential equation, the principle of equilibrium, financial market, interest rate changes.

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1 Introduction

In modern economics, the rate of return often refers to the rate of return to maturity of funds. Therefore, it is an important research content of finance. Investing funds in vital areas to obtain the most significant yield is the most concerning issue for all investors.

Some scholars have established basic models for calculating movements and fluctuations in the stock market, including the interest rate and liquidity equations, stock buying and selling equations, stock price change rate and so on. In addition, they put forward the basic principles of calculating the trends on the stock market from the perspective of market operation laws and the combination of financial instrument trading principles and practices [1]. As examples, we may mention the principle of potentiality and the principle of supply and demand difference.

*Corresponding author.

Email address: wpf668836@163.com

Assume that the entire financial market of the national economy is regarded as a network. It has m nodes, which are represented by Arabic numerals 1, 2,..., respectively. Each node represents various economic sectors or industries, such as the stock and securities industry, banking and finance industry, real estate industry etc. Since the rate of return between nodes changes over time, and since investors are motivated towards maximising their profit, there will be capital flows between nodes. The flow of funds between different nodes leads to changes in the instantaneous rate of return of each node, thereby obtaining the set of equations that the instantaneous rate of return of each node should satisfy [2]. Hence, it is called the stock market rate of return-liquidity equation.

In the above financial network, node i represents each financial industry with different functions. The internal capital flow of the financial industry with the same function is not considered. The financial industry with the same functions contains several different entities. They also form a network by themselves. Taking the stock and securities industry as an example, we observe that investors have multiple choices when choosing to buy stocks, and thus the flow of funds between different stocks will occur frequently. Take the banking industry as an example to analyse the financial network. There will naturally be capital flows between them. Investors will also choose according to the characteristics of each bank. Therefore, each node in the above equation is a sub-network composed of different smaller nodes. Such a network is called a complex network [3]. In a complex financial network, there are capital flows between nodes in two different sub-networks. There will also be capital flows between two different nodes within the same sub-network. Figure 1 shows the capital flow process of the bank's financial network. The literature mentioned above only considers the case of simple networks and does not study complex financial networks.

Fig. 1 Fund flow process of bank financial network.

To summarise, the primary purpose of this paper is to regard the entire financial network as a complex network composed of several sub-networks with different functions. Each sub-network represents a financial industry with particular functions. It is composed of several nodes, and each node represents a specific financial entity [4]. This article will establish and study a mathematical model similar to the stock market return-liquidity equation. We reveal the changing laws of each node's instantaneous rate of return in the complex financial network.

2 Complex financial network rate of return-liquidity equation

For the sake of simplicity, we only consider a complex financial network consisting of two sub-networks. Suppose a complex network N contains two sub-networks X and Y – i.e. $N = \{X, Y\}$. X and Y are respectively composed of m nodes $X_i, i = 1, 2, \dots, m$ and n nodes $Y_j, j = 1, 2, \dots, n$, $X = \{X_1, X_2, \dots, X_m\}, Y = \{Y_1, Y_2, \dots, Y_n\}$.

To consider the change law of the node interest rate and the number of funds in circulation with time t , we denote R as the totality of real numbers $R^+ = \{t \in R, t \geq 0\}$. For any $i = 1, 2, \dots, m, j = 1, 2, \dots, n$, let $x_i(t), y_j(t)$ be the instant return rate of nodes X_i and Y_j in sub-networks X and Y at time t , respectively. $A_i(t), B_j(t)$ is the total amount of funds of nodes X_i and Y_j at time t . A_i^*, B_j^* is the initial funds of nodes X_i and Y_j .

There is a flow of funds between the nodes of the subnet X . This capital flow has existence for any $i, j = 1, 2, \dots, m, i \neq j$. Let $A_{ij}(t)$ be the amount of funds in circulation between node X_i and node X_j at time t per unit time. $A_{ij}(t) > 0$ time represents the circulation of funds per unit time from node X_i to node X_j at time t . Hour $A_{ij}(t) < 0$ represents the circulation of funds per unit time from node X_j to node X_i . In the same way, there is also a flow of funds between the nodes of the sub-network Y . For any $i, j = 1, 2, \dots, n, i \neq j$, we denote $B_{ij}(t)$ as the amount of funds in circulation between node Y_i and node Y_j at time t per unit time.

Time $B_{ij}(t) > 0$ represents the circulation of funds per unit time from node Y_i to node Y_j at time t . $B_{ij}(t) < 0$ represents the circulation of funds per unit time from node Y_j to node Y_i . In addition, there are capital flows between the nodes of sub-network X and sub-network Y . For the existence of any $i, j = 1, 2, \dots, m, j = 1, 2, \dots, n$, let $C_{ij}(t)$ be the amount of funds in circulation between node X_i and node Y_j at time t per unit time [6]. Time $C_{ij}(t) > 0$ represents the circulation of funds per unit time from node X_i to node Y_j at time t . When $C_{ij}(t) < 0$, this represents the circulation of funds per unit time from node Y_j to node X_i .

The following relationship exists for any $t \in R^+, i, j = 1, 2, \dots, m, j = 1, 2, \dots, n$.

$$A_i(t) = A_i^* + \int_0^t \sum_{\substack{k=1 \\ k \neq i}}^m A_{ki}(\tau) d\tau - \int_0^t \sum_{k=1}^m C_{ik}(\tau) d\tau \quad (1)$$

$$B_j(t) = B_j^* + \int_0^t \sum_{\substack{k=1 \\ k \neq j}}^m B_{kj}(\tau) d\tau - \int_0^t \sum_{k=1}^m C_{kj}(\tau) d\tau \quad (2)$$

$$A_{ik}(t) = -A_{ki}(t), B_{jk}(t) = -B_{kj}(t) \quad (3)$$

We mark x_i^* (and y_j^*) as the initial rate of return of node X_i (and Y_j). It represents the rate of return of node X_i (and Y_j) when there is no capital flow between nodes. Generally speaking, different financial functional departments have different introductory rates of return. For example, the basic rate of return of banks is relatively lower than that of the stock market. This is because deposits into the bank do not require investors to worry about the operation, and the risk is relatively small. But within the same financial field, the introductory rate of return of each node of the bank's sub-network should be regarded as roughly the same [7]. Therefore, we might assume that the introductory rate of return of each node in the sub-networks X and Y are the same, i.e. $x_i^* = x^*, i = 1, 2, \dots, m, y_j^* = y^*, j = 1, 2, \dots, n$, where x^*, y^* is an average number.

According to economic market experience, we find that the cause of capital flow is the difference in the rate of return. However, the difference between the introductory rate of return is acceptable. Therefore, the difference in the introductory rate of return between the nodes will not cause the flow of funds. Similarly, if the difference between the immediate rate of return is equal to the difference between the introductory rate of return, there will be no capital flow between nodes [8]. Thus, the real reason for the flow of funds is the difference in the amount of change.

Let $(\bar{x}_i(t))\bar{y}_j(t)$ denote the difference between the instantaneous rate of return and the introductory rate of return of node $X_i(Y_j)$ at time t . Based on the above analysis, we might assume that the size of $\bar{x}_i(t) - \bar{y}_j(t)$ is

proportional to the amount of capital flow $A_{ij}(t)$ from node Y_j to node X_i per unit time [9]. The scale factor is denoted as $\gamma_{ij} > 0$. Then we obtain the following equation:

$$\bar{x}_i(t) - \bar{y}_i(t) = -\gamma_{ij}C_{ij}(t) \quad (4)$$

$$(x_i(t) - y_j(t)) - (x^* - y^*) = -\gamma_{ij}C_{ij}(t), i = 1, 2, \dots, m, j = 1, 2, \dots, n \quad (5)$$

The same can be expressed as:

$$x_i(t) - x_l(t) = a_{li}A_{li}(t) \quad (6)$$

$$y_j(t) - y_l(t) = \beta_{lj}B_{lj}(t) \quad (7)$$

On the other hand, for a certain X_i or Y_j , the faster the total amount of funds increases, the faster the immediate rate of return changes. Let us assume that they are directly proportional.

We assume that node X_i (Y_j) is an economic entity in the financial network [10]. The scale, personnel, technology etc., of any economic entity, are limited, so its total profit is limited. The more funds other nodes invest in the node, the lower the instant rate of return will be. So we get the equation:

$$x_i(t) = -c_i A_i(t) \quad (8)$$

$$\dot{y}_j(t) = -d_j \dot{B}_j(t) \quad (9)$$

Where $c_i > 0, d_j > 0$ is the proportionality factor. From Eqs (1) and (2),

$$\dot{A}_i(t) = \sum_{k=1, k \neq i}^m A_{ki}(t) - \sum_{k=1}^n C_{ik}(t) \quad (10)$$

$$\dot{B}_j(t) = \sum_{k=1, k \neq j}^n B_{kj}(t) + \sum_{k=1}^m C_{kj}(t) \quad (11)$$

Substituting Eqs (5)–(7) into Eqs (10)–(11), and then substituting Eqs (8) and (9), we obtain the differential equations of the complex financial network's instant return rate change, which is called complex financial network. The rate of return equation is the following:

$$\begin{cases} \dot{x}_i(t) = c_i \left\{ \sum_{k=1, k \neq i}^m \frac{1}{a_{ki}} [x_k(t) - x_i(t)] + \sum_{k=1}^n \frac{1}{\gamma_{ik}} [y_k(t) - x_i(t)] - (y^* - x^*) \sum_{k=1}^n \frac{1}{\gamma_{ki}} \right\}, i = 1, 2, \dots, m; \\ \dot{y}_j(t) = d_j \left\{ \sum_{k=1, k \neq j}^n \frac{1}{\beta_{kj}} [y_k(t) - y_j(t)] + \sum_{k=1}^m \frac{1}{\gamma_{kj}} [x_k(t) - y_j(t)] - (x^* - y^*) \sum_{k=1}^m \frac{1}{\gamma_{kj}} \right\}, j = 1, 2, \dots, n. \end{cases} \quad (12)$$

3 The invariance of the weighted sum of returns

We use differential equation theory to discuss Eq. (12). Eq. (12) is a system of equations composed of $m + n$ equations. First, it is easy to know $a_{ik} = a_{ki}, \beta_{jk} = \beta_{kj}$ from Eqs (3), (6) and (7). For convenience, we will express the system of Eq. (12) in the form of vectors. Then Eq. (12) can be rewritten as follows:

$$\begin{cases} \dot{x}(t) = C(Ax + Py + \xi) \\ \dot{y}(t) = D(P^T x + By + \eta) \end{cases} \quad (13)$$

Among them $C = \text{diag}(c_1, c_2, \dots, c_m)$, $D = \text{diag}(d_1, d_2, \dots, d_n)$, $A = (a_{ik})_{m \times m}$, $B = (b_{jk})_{n \times n}$.

$$a_{ik} = \begin{cases} -(\sum_{l=1, l \neq i}^m \frac{1}{a_{li}} + \sum_{l=1}^m \frac{1}{\gamma_{li}}) & i = k \\ \frac{1}{a_{ki}}, & i \neq k \end{cases} \quad (14)$$

$$b_{jk} = \begin{cases} -(\sum_{l=1, l \neq j}^n \frac{1}{\beta_{lj}} + \sum_{l=1}^m \frac{1}{\gamma_{lj}}) & j = k \\ \frac{1}{\beta_{kj}}, & j \neq k \end{cases} \quad (15)$$

C and D are both invertible matrices. So the equation system in Eq. (13) can be rewritten as

$$\begin{cases} C^{-1} \dot{x}(t) = Ax + Py + \xi \\ D^{-1} \dot{y}(t) = P^T x + By + \eta \end{cases} \quad (16)$$

We assume $e^{(m)} = (1, 1, \dots, 1) \in R^m$, $e^{(n)} = (1, 1, \dots, 1) \in R^n$. From the definition of matrices A and P and considering $a_{ik} = a_{ki}$ it can be obtained by direct calculation. For any $k = 1, 2, \dots, m$,

$$\begin{aligned} \sum_{i=1}^m a_{ik} &= \sum_{i=1, i \neq k}^m a_{ik} + a_{kk} = \sum_{i=1, i \neq k}^m \frac{1}{a_{ki}} + [-\sum_{l=1, l \neq k}^m \frac{1}{a_{lk}} - \sum_{l=1}^n \frac{1}{\gamma_{lk}}] \\ &= -\sum_{l=1}^n \frac{1}{\gamma_{kl}} e^{(m)} A + e^{(n)} P^T = 0 \end{aligned}$$

The same can be proved as:

$$\begin{aligned} e^{(m)} P + e^{(n)} B &= 0, \\ e^{(m)} \xi + e^{(n)} \eta &= 0, \end{aligned}$$

Combining the above three equations and substituting them into Eq. (16), we have

$$e^{(m)} C^{-1} \dot{x}(t) + e^{(n)} D^{-1} \dot{y}(t) = 0$$

Therefore, there are the following conclusions:

Theorem 1. We assume that $(x(t), y(t))$ is the solution of Eq. (12), where $x(t) = (x_1(t), x_2(t), \dots, x_m(t))$, $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$ is. Then there is a constant r such that for any $t \in R^+$,

$$\sum_{i=1}^m \frac{1}{c_i} x_i(t) + \sum_{j=1}^n \frac{1}{d_j} y_j(t) = r \quad (17)$$

Proof. We assume that if $x(t), y(t)$ is the solution of Eq. (12), then it satisfies Eq. (16). The above analysis shows

$$e^{(m)} C^{-1} \dot{x}(t) + e^{(n)} D^{-1} \dot{y}(t) = 0. \quad (18)$$

$$\frac{d}{dt} [e^{(m)} C^{-1} x(t) + e^{(n)} D^{-1} y(t)] = 0$$

Thus there is a constant r such that for any $t \geq 0$, there is

$$e^{(m)} C^{-1} x(t) + e^{(n)} D^{-1} y(t) = r$$

$$\sum_{i=1}^m \frac{1}{c_i} x_i(t) + \sum_{j=1}^n \frac{1}{d_j} y_j(t) = r$$

The certificate is complete.

To find the constant r , we take the instant interest rate vector at a particular initial moment

$$t_0 \geq 0$$

$$(x^{(0)}, y^{(0)}) = (x(t_0), y(t_0)) = (x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y_1^{(0)}, y_2^{(0)}, \dots, y_n^{(0)})^T$$

And substituting into Eq. (17), we can get

$$r = \sum_{i=1}^m \frac{1}{c_i} x_i^{(0)} + \sum_{j=1}^n \frac{1}{d_j} y_j^{(0)}.$$

From Eq. (17), we have

$$\sum_{i=1}^m \frac{1}{c_i} (x(t) - x_i^{(0)}) + \sum_{j=1}^n \frac{1}{d_j} (y(t) - y_j^{(0)}) = 0 \quad (19)$$

Eq. (17) shows that the weighted sum of the returns of each node in the complex financial network remains constant, and it does not change with time t . Eq. (19) shows that the weighted sum of the changes in the network is always zero. This conclusion is called the principle of interest rate equilibrium of the financial network in the literature [11]. This shows that an increased node in the network will inevitably lead to a decrease in the rate of return of other nodes. Thus, in Eqs (17) or (19), the weighting coefficients $\frac{1}{c_i}$ and $\frac{1}{d_j}$ reflect the role of nodes X_i and Y_j in maintaining the balance of network profitability.

4 Economic interpretation

Theorem 1 shows that the principle of interest rate equilibrium is still valid in a complex financial network.

Integrating both sides of Eqs. (8) and (9) from t_0 to t , we can get

$$x_i(t) = x_i(t_0) = -c_i(A_i(t) - A_i(t_0)),$$

$$y_j(t) = y_j(t_0) = -d_j(B_j(t) - B_j(t_0)),$$

It can be seen that $c_i(d_j)$ represents the sensitivity of the change in the rate of return of node in sub-network to the increased amount of capital investment.

We assume that the equation set in Eq. (12) describes the nodes of the return rate change rule of a complex financial network, including two sub-networks of Bank (X) and Stock Market (Y) $X_i, i = 1, 2, \dots, m$ represents a specific bank. The node $y_j, j = 1, 2, \dots, n$ represents a particular stock in the stock market. Since the bank's yield rate change is very small or even negligible compared to the increased amount of capital investment (deposits), c_i is very small – i.e. $\frac{1}{c_i}$ will be sufficiently large. Therefore, the bank sub-network plays a dominant role in maintaining the stability of the entire complex financial network. The stability of bank interest rates is also the guarantee for the stable development of the national economy. For the stock market sub-network Y , the change in the return rate of its stocks is more sensitive to the increased amount of capital investment. Therefore, the corresponding d_j of this type of node is relatively large and therefore $\frac{1}{d_j}$ is relatively small [12]. This shows that the change in the rate of return of such individual stocks has little effect on the change in the rate of return of the entire complex network.

$$p_i = \frac{\frac{1}{c_i}}{\sum_{k=1}^m \frac{1}{c_k} + \sum_{l=1}^n \frac{1}{d_l}}, q_i = \frac{\frac{1}{d_j}}{\sum_{k=1}^m \frac{1}{c_k} + \sum_{l=1}^n \frac{1}{d_l}} \quad (20)$$

Then p_i, q_j are, respectively, expressed as the influence factor of node X_i and Y_j on the entire network. It is easy to infer that $\sum_{i=1}^m p_i + \sum_{j=1}^n q_j = 1$. For any $t \in R^+$, let us recall that

$$z(t) = \sum_{i=1}^m p_i x_i(t) + \sum_{j=1}^n q_j y_j(t) \quad (21)$$

$z(t)$ is defined as the average rate of return of the entire complex network at time t . The research results of Theorem 1 show that the average rate of return $z(t)$ of the entire complex network is a constant, which can be regarded as an indicator of the development of the financial market.

5 Conclusion

The thesis first established the differential equation model of the first-rate flux of interest rate in the closed system. It proved that the weighted sum of interest rates at each node is a constant – i.e. the principle of interest rate equilibrium in the financial market. The interest rate limit of each node is the average interest rate of the entire network. Second, when the introductory interest rate of each node is not the same, the article establishes the inhomogeneous interest rate first-rate flux differential equation model. It proves that the weighted sum of interest rates at each node of the financial network is still a constant. The real-time interest rate difference between each node will eventually steadily tend to its primary interest rate difference.

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