Applied Mathematics and Nonlinear Sciences

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About one method of calculation in the arbitrary curvilinear basis of the Laplace operator and curl from the vector function

S. O. Gladkov[†]

Moscow Aviation Institute, National Research University (MAI), Volokolamskoye sh., 4. Moscow 125993, Russia

Submission Info

Communicated by Juan Luis García Guirao

Received December 4th 2020

Accepted April 11th 2021

Available online May 14th 2021

Abstract

A simple algorithm for calculating Christoffel symbols, a covariant projection of the result of the Laplace operator's action on the vector, vector curl and other similar operations in an arbitrary oblique base are proposed. For an arbitrary base with ortho e_i is found the expressions of vector projections $(\Delta \mathbf{A})_i$ and $(\text{rot} \mathbf{A})_i$, where A is a counter variant vector. Examples of orthonormal bases are considered and general expressions for $(\Delta \mathbf{A})_i$ and $(\text{rot} \mathbf{A})_i$ for the bases are also given. As a demonstration of the working capacity of the common formulas obtained, detailed calculations of $(\Delta \mathbf{A})_i$ and $(\text{rot} \mathbf{A})_i$ as an example are made in cases of spherical and cylindrical coordinate systems.

Keywords: curvilinear basis; orthonormal basis; covariant derivative; Christoffel symbol

1 Introduction

Our great interest in calculating the result of the action of the Laplace operator and the curl on a contravariant vector function arose mainly after reading the monographs [1, 2]. There, the results of covariant differentiation of a contravariant vector in spherical and cylindrical coordinate systems are presented in a purely formal way, with no references to the primary sources where they could be considered in detail.

This raises a quite natural question, the answer to which we have not found in the monographs mentioned above. The question is how to calculate the results of the action of the Laplace operator and curl from a vector function in arbitrary (but not only orthogonal) coordinates; for example, in parabolic or elliptic ones.

Perhaps, such a question, according to the authors of classical works [1, 2], is so simple that they did not even consider it necessary to indicate a link to the source where the answer could be found. We found no answer

[†]Corresponding author.

Email address: sgrad51@mail.ru

to this question, neither in monograph [3], nor in monograph [4].

As this problem is not given in detail in the literature, we will consider it here in this paper.

It is worth noting that the effect of the second covariant derivative on the vector function is extremely important from the point of view of its application to the problems of the theory of elasticity and hydrodynamics.

This circumstance prompted us to write this paper, where in great detail, we describe the basic concepts of calculating the projections of vectors $(\Delta \mathbf{A})_i$ and $(\text{rot} \mathbf{A})_i$, where A is a contravariant differentiable vector, on an arbitrary unit basis e_i .

To show examples that prove the correctness of the common approach described below, we demonstrate the corresponding calculations in the cases of spherical and cylindrical coordinate systems with results that automatically coincide with the commonly accepted ones.

2 Basic concepts of the method of calculation in orthonormal basis, as an alternative approach

2.1 Laplace operator

Anticipating a possible question concerning the word ‘alternative’, which we have introduced, let us immediately note that the approach that will be discussed later does not include such concepts as Lamé coefficients anywhere, provided that we are considering an orthogonal coordinate system. Our approach is based on the use of particular differentials introduced in a certain way, with the help of which all mathematical calculations are carried out due to the common calculation method proposed below.

The spatial metric in its standard form is [3]

$$dl^2 = dx_i dx^i = g_{ik} dx^i dx^k \quad (1)$$

where g_{ik} – is the covariant metric tensor, defined via transformations $x_i = x_i(x^k)$ in the usual way [3, 4] such as

$$g_{ik} = \frac{\partial x_n}{\partial x^i} \frac{\partial x_n}{\partial x^k} \quad (2)$$

where (here and below) summation is implied by repeated indexes.

The relation of the contravariant metric tensor to the covariant (2) is determined by the inverse ratio

$$g_{is} g^{sk} = \delta_i^k \quad (3)$$

where δ_i^k – is the Kronecker symbol representing the unit matrix.

We present Eq. (1) as follows:

$$dl^2 = \sum_i g_{ii} (dx^i)^2 + \sum_{i \neq k} g_{ik} dx^i dx^k. \quad (4)$$

By introducing intermediate differentials

$$\partial \tilde{x}^i = \sqrt{g_{ii}} dx^i \quad (5)$$

where i – is fixed, we get from Eq. (4) the following:

$$dl^2 = \sum_i (\partial \tilde{x}^i)^2 + \sum_{i \neq k} \frac{g_{ik}}{\sqrt{g_{ii}} \sqrt{g_{kk}}} \partial \tilde{x}^i \partial \tilde{x}^k \quad (6)$$

Clearly, the square of an element of length Eq. (6) is written in the orthogonal basis. By omitting the summation sign, we have

$$dl^2 = (\partial \tilde{x}^i) \quad (7)$$

As a rule, in all practically solved problems, only the orthogonal coordinate system is used. Therefore, expression (7) will be the main representation of the metric that we will use.

As we can see from Eq. (7), in this representation, the metric tensor, both covariant and contravariant, is simply a Kronecker symbol, that is

$$g_{ik} = g^{ik} = \delta_{ik} = \delta_i^k = \delta^{ik} \quad (8)$$

This means that in such a representation, there is a determination of Christoffel symbols of the first and second kinds, which are obviously also equal to each other (see Ref. [4] and also papers [5, 6]); their determination is in the form of the standard formula (see Refs [3, 4])

$$\Gamma_{ikl} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} \right) \quad (9)$$

not applicable, since according to Eq. (8) all $\Gamma_{ikl} = \Gamma_{kl}^i = 0$.

This simply means that we cannot use Eq. (9), but we must proceed from a direct determination, which is written as (see Ref. [3])

$$\Gamma_{ikl} = \frac{\partial x_n}{\partial x^i} \frac{\partial^2 x_n}{\partial x^k \partial x^l} \quad (10)$$

Since we know development $x_i = x_i(x^k)$, then according to Eq. (5), calculation of the Christoffel symbols will not be difficult. In fact, we have for them:

$$\Gamma_{ikl} = \Gamma_{kl}^i = \frac{\partial x_n}{\partial \tilde{x}^i} \frac{\partial^2 x_n}{\partial \tilde{x}^k \partial \tilde{x}^l} \quad (11)$$

It should be noted that calculation of the Christoffel symbols by Eq. (11) automatically results in their correct dimension $\frac{1}{L}$, where L – is the unit of length. Specific examples of how Eq. (11) ‘works’ can be found, for example, in the articles [5, 6].

Let there be some arbitrary continuous contravariant vector function $\mathbf{A}$, and we are interested in the result of the action of the Laplace operator on it, that is, vector

$$\mathbf{B} = \Delta \mathbf{A}.$$

Then its projections on the orts $\mathbf{e}_i$ will be as follows:

$$B_i = g_{ik} (\Delta \mathbf{A})^k \quad (12)$$

To calculate this expression, we first find the first and second covariant derivatives of vector $\mathbf{A} = (A^i)$.

For the first covariant derivative, we have

$$A_{,k}^i = B_k^i = \frac{\partial A^i}{\partial x^k} + \Gamma_{kl}^i A^l \quad (13)$$

where B_k^i – is the index-mixed tensor of second rank.

The second covariant derivative will be as follows:

$$B_{k,l}^i = \frac{\partial B_k^i}{\partial x^l} + \Gamma_{ls}^i B_k^s - \Gamma_{kl}^s B_s^i \quad (14)$$

which is easy to find, if we represent tensor B_k^i as a product of contravariant $\mathbf{M}$ and covariant $\mathbf{N}$ vectors, that is, as $B_k^i = M^i N_k$, and if we use the known rules for their differentiation $M_{,k}^i = \frac{\partial M^i}{\partial x^k} + \Gamma_{kl}^i M^l$, $N_{i,k} = \frac{\partial N_i}{\partial x^k} - \Gamma_{ik}^s N_s$.

By applying determination Eqs (13) and (14) for tensor B_k^i , we immediately receive

$$\begin{aligned} B_{k,l}^i &= \frac{\partial B_k^i}{\partial x^l} + \Gamma_{ls}^i B_k^s - \Gamma_{kl}^s B_s^i = \frac{\partial}{\partial x^l} \left(\frac{\partial A^i}{\partial x^k} + \Gamma_{ks}^i A^s \right) + \Gamma_{ln}^i \left(\frac{\partial A^n}{\partial x^k} + \Gamma_{ks}^n A^s \right) - \Gamma_{kl}^n \left(\frac{\partial A^i}{\partial x^n} + \Gamma_{ns}^i A^s \right) = \\ &= \frac{\partial^2 A^i}{\partial x^k \partial x^l} + \Gamma_{ks}^i \frac{\partial A^s}{\partial x^l} + \Gamma_{ls}^i \frac{\partial A^s}{\partial x^k} - \Gamma_{kl}^n \frac{\partial A^i}{\partial x^n} + \left(\frac{\partial \Gamma_{ks}^i}{\partial x^l} + \Gamma_{ln}^i \Gamma_{ks}^n - \Gamma_{ns}^i \Gamma_{kl}^n \right) A^s = \\ &= \frac{\partial^2 A^i}{\partial x^k \partial x^l} + \Gamma_{ks}^i \frac{\partial A^s}{\partial x^l} + \frac{\partial (\Gamma_{ls}^i A^s)}{\partial x^k} - \Gamma_{kl}^n \frac{\partial A^i}{\partial x^n} + R_{kls}^i A^s \end{aligned} \quad (15)$$

where the Riemann curvature tensor is defined in a conventional manner [4], as

$$R_{kls}^i = \frac{\partial \Gamma_{ks}^i}{\partial x^l} - \frac{\partial \Gamma_{ls}^i}{\partial x^k} + \Gamma_{ln}^i \Gamma_{ks}^n - \Gamma_{ns}^i \Gamma_{kl}^n \quad (16)$$

By collapsing expression (15) by indexes k, l , taking into account the explicit expression (16), we come to the following result:

$$(\Delta \mathbf{A})^i = \frac{\partial^2 A^i}{(\partial x^k)^2} + 2\Gamma_{ks}^i \frac{\partial A^s}{\partial x^k} - \Gamma_{kk}^n \frac{\partial A^i}{\partial x^n} + \left(\frac{\partial \Gamma_{ks}^i}{\partial x^k} + \Gamma_{kn}^i \Gamma_{ks}^n - \Gamma_{ns}^i \Gamma_{kk}^n \right) A^s \quad (17)$$

Highlighting here the Laplace operator in an explicit form, that is, as

$$\Delta = \frac{\partial^2}{(\partial x^i)^2} - \Gamma_{kk}^n \frac{\partial}{\partial x^n} \quad (18)$$

we finally get the following expression for vector B^i as

$$B^i = (\Delta \mathbf{A})^i = \Delta A^i + 2\Gamma_{ks}^i \frac{\partial A^s}{\partial x^k} + \left(\frac{\partial \Gamma_{ks}^i}{\partial x^k} + \Gamma_{kn}^i \Gamma_{ks}^n - \Gamma_{ns}^i \Gamma_{kk}^n \right) A^s \quad (19)$$

Thus, the covariant projection according to Eq. (12) will be

$$B_i = g_{ik} (\Delta \mathbf{A})^k = g_{ik} \left[\Delta A^k + 2\Gamma_{ks}^k \frac{\partial A^s}{\partial x^l} + \left(\frac{\partial \Gamma_{ks}^k}{\partial x^l} + \Gamma_{ln}^k \Gamma_{ks}^n - \Gamma_{ns}^k \Gamma_{ll}^n \right) A^s \right] \quad (20)$$

Let us now consider another problem. Let there be a covariant vector

$$\mathbf{B} = \Delta \mathbf{A}$$

where $\mathbf{A} = (A_i)$ and we are interested in the result

$$C_i = (\Delta \mathbf{A})_i \quad (21)$$

As above, in the case of a contravariant vector, we find the first two covariant derivatives of vector $\mathbf{A} = (A_i)$. For the first derivative, we have

$$A_{i,k} = B_{ik} = \frac{\partial A_i}{\partial x^k} - \Gamma_{ik}^s A_s \quad (22)$$

where B_{ik} – is the second rank covariant tensor.

For the second covariant derivative (representing this tensor in factorised form, as $B_{ik} = M_i N_k$), we get

$$B_{ik,l} = \frac{\partial B_{ik}}{\partial x^l} - \Gamma_{il}^s B_{sk} - \Gamma_{kl}^s B_{is} \quad (23)$$

By applying determination Eqs (22) and (23) for tensor B_{ik} , we immediately find here that

$$B_{ik,l} = A_{i,k,l} = \frac{\partial^2 A_i}{\partial x^k \partial x^l} - \Gamma_{ik}^s \frac{\partial A_s}{\partial x^l} - \Gamma_{il}^s \frac{\partial A_s}{\partial x^k} - \Gamma_{kl}^s \frac{\partial A_i}{\partial x^s} + \left(\Gamma_{il}^n \Gamma_{nk}^s + \Gamma_{kl}^n \Gamma_{in}^s - \frac{\partial \Gamma_{ik}^s}{\partial x^l} \right) A_s \quad (24)$$

By collapsing this expression by the indices k, l , we get

$$\begin{aligned} C_i &= \frac{\partial^2 A_i}{(\partial x^k)^2} - 2\Gamma_{ik}^s \frac{\partial A_s}{\partial x^k} - \Gamma_{kk}^s \frac{\partial A_i}{\partial x^s} + \left(\Gamma_{ik}^n \Gamma_{nk}^s + \Gamma_{kk}^n \Gamma_{in}^s - \frac{\partial \Gamma_{ik}^s}{\partial x^k} \right) A_s = \\ &= \Delta A_i - 2\Gamma_{ik}^s \frac{\partial A_s}{\partial x^k} + \left(\Gamma_{ik}^n \Gamma_{nk}^s + \Gamma_{kk}^n \Gamma_{in}^s - \frac{\partial \Gamma_{ik}^s}{\partial x^k} \right) A_s \end{aligned} \quad (25)$$

Comparing Eq. (25) with Eq. (20), we can see that, in general, vectors B_i and C_i , as it should be, do not coincide. To bring them in compliance, let us now consider the expression

$$A_i = g_{is} A^s$$

and find a second-rank covariant derivative of it.

Due to Ricci's lemma, according to which the covariant derivative of the metric tensor is zero, that is, we have

$$\begin{aligned} A_{i,k,l} &= g_{is} \left[\frac{\partial^2 A^s}{\partial x^k \partial x^l} + \Gamma_{kn}^s \frac{\partial A^n}{\partial x^l} + \Gamma_{ln}^s \frac{\partial A^n}{\partial x^k} - \Gamma_{kl}^n \frac{\partial A^s}{\partial x^n} + A^p \left(\frac{\partial \Gamma_{kp}^s}{\partial x^l} + \Gamma_{ln}^s \Gamma_{kp}^n - \Gamma_{kl}^n \Gamma_{np}^s \right) \right] = \\ &= g_{is} \frac{\partial^2 A^s}{\partial x^k \partial x^l} + \Gamma_{iks} \frac{\partial A^s}{\partial x^l} + \Gamma_{ils} \frac{\partial A^s}{\partial x^k} - g_{is} \Gamma_{kl}^n \frac{\partial A^s}{\partial x^n} + A^s \left(\frac{\partial \Gamma_{iks}}{\partial x^l} - \Gamma_{ks}^n \frac{\partial g_{in}}{\partial x^l} + \Gamma_{iln} \Gamma_{ks}^n - \Gamma_{kl}^n \Gamma_{ins} \right) \end{aligned}$$

But since $\frac{\partial g_{in}}{\partial x^l} = \Gamma_{inl} + \Gamma_{nil}$, it follows that

$$A_{i,k,l} = g_{is} \frac{\partial^2 A^s}{\partial x^k \partial x^l} + \Gamma_{iks} \frac{\partial A^s}{\partial x^l} + \Gamma_{ils} \frac{\partial A^s}{\partial x^k} - g_{is} \Gamma_{kl}^n \frac{\partial A^s}{\partial x^n} + A^s \left(\frac{\partial \Gamma_{iks}}{\partial x^l} - \Gamma_{ks}^n (\Gamma_{inl} + \Gamma_{nil}) + \Gamma_{iln} \Gamma_{ks}^n - \Gamma_{kl}^n \Gamma_{ins} \right) \quad (26)$$

By collapsing expression (26) by the indices k, l , we get

$$A_{i,k,k} = g_{is} \Delta A^s + 2\Gamma_{iks} \frac{\partial A^s}{\partial x^k} + A^s \left(\frac{\partial \Gamma_{iks}}{\partial x^k} - \Gamma_{ks}^n \Gamma_{nik} - \Gamma_{kk}^n \Gamma_{ins} \right) \quad (27)$$

where again, an abbreviated notation is introduced for the usual Laplace operator according to Eq. (18). Therefore,

$$C_i = g_{is} \Delta A^s + 2\Gamma_{iks} \frac{\partial A^s}{\partial x^k} + A^s \left(\frac{\partial \Gamma_{iks}}{\partial x^k} - \Gamma_{ks}^n \Gamma_{nik} - \Gamma_{kk}^n \Gamma_{ins} \right) \quad (28)$$

It is clear that in order to obtain the correct formula, we need Eqs (20) and (25) to bring in compliance with each other, but not Eqs (20) and (28).

Taking the arithmetic mean of Eqs (20) and (25), we obtain the desired common determination for calculating the result of the action of the Laplace operator on a vector function in an arbitrary curved coordinate system as

$$\begin{aligned} (\Delta \mathbf{A})_i &= \frac{g_{is} \Delta A^s + \Delta A_i}{2} + \Gamma_{isk} \frac{\partial A^s}{\partial x^k} - \Gamma_{ik}^s \frac{\partial A_s}{\partial x^k} - \frac{\Gamma_{kl}^s}{2} \left(g_{ip} \frac{\partial A^p}{\partial x^s} + \frac{\partial A_i}{\partial x^s} \right) + \\ &+ \frac{A^s}{2} \left(\frac{\partial \Gamma_{isk}}{\partial x^k} - \Gamma_{kk}^n \Gamma_{isn} - \Gamma_{ik}^n \Gamma_{nsk} \right) + \frac{A_s}{2} \left(\Gamma_{ik}^n \Gamma_{nk}^s + \Gamma_{kk}^n \Gamma_{in}^s - \frac{\partial \Gamma_{ik}^s}{\partial x^k} \right) \end{aligned} \quad (29)$$

In the case of an orthonormal basis, where, as we know, the formulas are satisfied

$$\Gamma_{kl}^i = \Gamma_{ikl}, \quad A_i = A^i, \quad g_{ik} = \delta_{ik} \quad (30)$$

It immediately follows that

$$(\Delta \mathbf{A})_i = \Delta A_i + (\Gamma_{isk} - \Gamma_{sik}) \frac{\partial A_s}{\partial x^k} + \frac{A_s}{2} \left[\frac{\partial}{\partial x^k} (\Gamma_{isk} - \Gamma_{sik}) + \Gamma_{nkk} (\Gamma_{sin} - \Gamma_{in}) + \Gamma_{nkk} (\Gamma_{snk} - \Gamma_{nsk}) \right] \quad (31)$$

Eq. (31) is the final formula that allows us to calculate the projections of vector $\Delta \mathbf{A}$ in any orthonormal coordinate system.

2.2 Vector function curl

We now calculate the projections of the curl of vector $\mathbf{A}$ on the covariant orthonormal basis $\mathbf{e}_i$. As

$$\mathbf{C}_i = (\text{rot} \mathbf{A})_i$$

then its obvious generalisation to an arbitrary curved basis can be written as

$$(\text{rot} \mathbf{A})_i = \sqrt{g} e_{ikl} g^{ln} \left(\frac{\partial A^k}{\partial x^n} + \Gamma_{sn}^k A^s \right) \quad (32)$$

On the other hand, the same expression can be represented as

$$g_{ik} (\text{rot} \mathbf{A})^k = g_{ik} \frac{e^{kls}}{\sqrt{g}} \left(\frac{\partial A_s}{\partial x^l} - \Gamma_{sl}^n A_n \right) \quad (33)$$

It is clear that they should both lead to the same result. Therefore, taking their half-sum, we find

$$(\text{Rot} \mathbf{A})_i = \frac{1}{2} \left[\sqrt{g} e_{ikl} g^{ln} \left(\frac{\partial A^k}{\partial x^n} + \Gamma_{sn}^k A^s \right) + g_{ik} \frac{e^{kls}}{\sqrt{g}} \left(\frac{\partial A_s}{\partial x^l} - \Gamma_{sl}^n A_n \right) \right] \quad (34)$$

Carrying out simple transformations here, we finally come to the general invariant expression for the curl of a contravariant vector function, which is correct in any curved coordinate system as

$$(\text{Rot} \mathbf{A})_i = \frac{e_{ikl}}{2} g^{ln} \left[\left(\sqrt{g} + \frac{1}{\sqrt{g}} \right) \frac{\partial A^k}{\partial x^n} + \left(\sqrt{g} \Gamma_{sn}^k + \frac{g^{kp}}{\sqrt{g}} \frac{\partial g_{sp}}{\partial x^n} \right) A^s \right] \quad (35)$$

In the case of orthogonal coordinates, when using the already familiar orthonormal basis, where the conditions are always satisfied (which is extremely convenient for various kinds of calculations)

$$g_{ik} = g^{ik} = \delta_{ik}, \quad \Gamma_{kl}^i = \Gamma_{ikl}, \quad A_i = A^i, \quad g = 1$$

we have

$$(\text{Rot} \mathbf{A})_i = e_{ikl} \left(\frac{\partial A_k}{\partial x^l} + \frac{1}{2} \Gamma_{ksl} A_s \right) \quad (36)$$

3 Special cases of general Eqs (31) and (36)

Let us now consider the application of Eqs (31) and (36) in specific special cases. Let us start with Eq. (31).

3.1 Spherical coordinates

Since the length element in this case is

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (37)$$

and there are intermediate differentials

$$\partial x^1 = dr, \partial x^2 = r d\theta, \partial x^3 = r \sin \theta d\varphi \quad (38)$$

then, leaving only the non-zero Christoffel characters with the help of the table given, let us say, in Ref. [5], according to which

$$\begin{aligned} \Gamma_{r\theta\theta} = \Gamma_{\theta\theta}^r = -\frac{1}{r}, \Gamma_{r\varphi\varphi} = \Gamma_{\varphi\varphi}^r = -\frac{1}{r}, \Gamma_{\theta\varphi\varphi} = \Gamma_{\varphi\varphi}^\theta = -\frac{\cot \theta}{r}, \Gamma_{\theta r\theta} = \Gamma_{\theta\theta r} = \Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r}, \\ \Gamma_{\varphi r\varphi} = \Gamma_{\varphi\varphi r} = \Gamma_{r\varphi}^\varphi = \Gamma_{\varphi r}^\varphi = \frac{1}{r}, \Gamma_{\varphi\theta\varphi} = \Gamma_{\varphi\varphi\theta} = \Gamma_{\theta\varphi}^\varphi = \Gamma_{\varphi\theta}^\varphi = \frac{\cot \theta}{r} \end{aligned} \quad (39)$$

in accordance with Eq. (31), for the projection on the orthonormal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$, we have

$$\begin{aligned}
(\Delta \mathbf{A})_r &= \Delta A_r + (\Gamma_{rsk} - \Gamma_{srk}) \frac{\partial A_s}{\partial x^k} + \frac{A_s}{2} \left[\frac{\partial}{\partial x^k} (\Gamma_{rsk} - \Gamma_{srk}) + \Gamma_{nkk} (\Gamma_{sm} - \Gamma_{rsn}) + \Gamma_{nrk} (\Gamma_{snk} - \Gamma_{nsk}) \right] = \\
&= \Delta A_r + (\Gamma_{r\theta\theta} - \Gamma_{\theta r\theta}) \frac{\partial A_\theta}{\partial x^\theta} + (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi}) \frac{\partial A_\varphi}{\partial x^\varphi} + \\
&+ \frac{A_r}{2} \left[\Gamma_{\theta r\theta} (\Gamma_{r\theta\theta} - \Gamma_{\theta r\theta}) + \Gamma_{\varphi r\varphi} (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi}) \right] + \\
&+ \frac{A_\theta}{2} \left[\frac{\partial}{\partial x^\theta} (\Gamma_{r\theta\theta} - \Gamma_{\theta r\theta}) + \Gamma_{\theta\varphi\varphi} (\Gamma_{\theta r\theta} - \Gamma_{r\theta\theta}) + \Gamma_{\varphi r\varphi} (\Gamma_{\theta\varphi\varphi} - \Gamma_{\varphi\theta\varphi}) \right] + \\
&+ \frac{A_\varphi}{2} \left[\frac{\partial}{\partial x^\varphi} (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi}) + \Gamma_{\varphi\theta\theta} (\Gamma_{\varphi r\varphi} - \Gamma_{r\varphi\varphi}) + \Gamma_{\theta r\theta} (\Gamma_{\varphi\theta\theta} - \Gamma_{\theta\varphi\theta}) \right]
\end{aligned}$$

By applying here the Christoffel symbols from Eq. (39), with the intermediate differentials of the considered orthonormal basis Eq. (38) applied to the denominators, we finally find

$$\begin{aligned}
(\Delta \mathbf{A})_r &= \Delta A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial A_\varphi}{\partial \varphi} - \frac{2A_r}{r^2} - \frac{2 \operatorname{ctg} \theta A_\theta}{r^2} = \\
&= \Delta A_r - \frac{2}{r^2} \left(A_r + \frac{1}{\sin \theta} \frac{\partial (\sin \theta A_\theta)}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial A_\varphi}{\partial \varphi} \right)
\end{aligned}$$

For the projection on the ort $\mathbf{e}_\theta$, we get

$$\begin{aligned}
(\Delta \mathbf{A})_\theta &= \Delta A_\theta + (\Gamma_{\theta sk} - \Gamma_{s\theta k}) \frac{\partial A_s}{\partial x^k} + \frac{A_s}{2} \left[\frac{\partial}{\partial x^k} (\Gamma_{\theta sk} - \Gamma_{s\theta k}) + \Gamma_{nkk} (\Gamma_{s\theta n} - \Gamma_{\theta sn}) + \Gamma_{n\theta k} (\Gamma_{snk} - \Gamma_{nsk}) \right] = \\
&= \Delta A_\theta + (\Gamma_{\theta r\theta} - \Gamma_{r\theta\theta}) \frac{\partial A_r}{\partial x^r} + (\Gamma_{\theta\varphi\varphi} - \Gamma_{\varphi\theta\varphi}) \frac{\partial A_\varphi}{\partial x^\varphi} + \\
&+ \frac{A_r}{2} \left[\frac{\partial}{\partial x^\theta} (\Gamma_{\theta r\theta} - \Gamma_{r\theta\theta}) + \Gamma_{\theta\varphi\varphi} (\Gamma_{r\theta\theta} - \Gamma_{\theta r\theta}) + \Gamma_{\varphi\theta\varphi} (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi}) \right] + \\
&+ \frac{A_\theta}{2} \left[\Gamma_{r\theta\theta} (\Gamma_{\theta r\theta} - \Gamma_{r\theta\theta}) + \Gamma_{\varphi\theta\theta} (\Gamma_{\theta\varphi\theta} - \Gamma_{\varphi\theta\theta}) + \Gamma_{\varphi\theta\varphi} (\Gamma_{\theta\varphi\varphi} - \Gamma_{\varphi\theta\varphi}) \right] + \\
&+ \frac{A_\varphi}{2} \left[\frac{\partial}{\partial x^\theta} (\Gamma_{\theta\varphi\varphi} - \Gamma_{\varphi\theta\varphi}) + \frac{\partial}{\partial x^\varphi} (\Gamma_{\theta\varphi\varphi} - \Gamma_{\varphi\theta\varphi}) + \Gamma_{\theta\varphi\varphi} (\Gamma_{\varphi\theta\theta} - \Gamma_{\theta\varphi\theta}) + \Gamma_{\varphi\theta\theta} (\Gamma_{\varphi\theta\varphi} - \Gamma_{\theta\varphi\varphi}) + \Gamma_{\theta\theta\varphi} (\Gamma_{\varphi\theta\varphi} - \Gamma_{\theta\varphi\varphi}) \right]
\end{aligned}$$

By applying here the non-zero values of the Christoffel symbols according to Eq. (39) and the intermediate differentials from Eq. (38), we get

$$(\Delta \mathbf{A})_\theta = \Delta A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2 \operatorname{ctg} \theta}{r^2 \sin \theta} \frac{\partial A_\varphi}{\partial \varphi} - \frac{A_\theta}{r^2 \sin^2 \theta} \quad (40)$$

Finally, the projection on the ort $\mathbf{e}_\varphi$ gives

Using Eqs (38) and (39), we have from here

$$(\Delta \mathbf{A})_\varphi = \Delta A_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \varphi} + \frac{2 \operatorname{ctg} \theta}{r^2 \sin \theta} \frac{\partial A_\theta}{\partial \varphi} - \frac{A_\varphi}{r^2 \sin^2 \theta} \quad (41)$$

where the usual Laplace operator in a spherical coordinate system is

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \quad (42)$$

Putting Eqs (40)-(42) together, we find

$$\begin{cases} (\Delta \mathbf{A})_r = \Delta A_r - \frac{2}{r^2} \left(A_r + \frac{1}{\sin \theta} \frac{\partial (\sin \theta A_\theta)}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial A_\varphi}{\partial \varphi} \right), \\ (\Delta \mathbf{A})_\theta = \Delta A_\theta + \frac{2}{r^2} \left(\frac{\partial A_r}{\partial \theta} - \frac{\operatorname{ctg} \theta}{\sin \theta} \frac{\partial A_\varphi}{\partial \varphi} - \frac{A_\theta}{2 \sin^2 \theta} \right) \\ (\Delta \mathbf{A})_\varphi = \Delta A_\varphi + \frac{2}{r^2 \sin \theta} \left(\frac{\partial A_r}{\partial \varphi} + \operatorname{ctg} \theta \frac{\partial A_\theta}{\partial \varphi} - \frac{A_\varphi}{2 \sin \theta} \right) \end{cases} \quad (43)$$

Comparing projections (44) with the formulas given, for example, in monograph [1] on page 615 (see also Ref. [2]), we see their complete coincidence. At the same time, we emphasise that the authors of both monographs do not specify the source of information in which the corresponding calculations are given.

3.2 Cylindrical coordinate system

this, there are non-zero Christoffel symbols

$$\Gamma_{r\varphi\varphi} = -\frac{1}{r}, \Gamma_{\varphi r\varphi} = \Gamma_{\varphi\varphi r} = \frac{1}{r} \quad (44)$$

and intermediate differentials

$$\partial x^r = dr, \partial x^\varphi = r d\varphi, \partial x^z = dz \quad (45)$$

Using the general Eq. (31) for projections of vector $\mathbf{A}$ on an orthonormal unit basis $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z$, we find for projection on $\mathbf{e}_r$:

$$\begin{aligned} (\Delta \mathbf{A})_r &= \Delta A_r + (\Gamma_{rsk} - \Gamma_{srk}) \frac{\partial A_s}{\partial x^k} + \frac{A_s}{2} \left[\frac{\partial}{\partial x^k} (\Gamma_{rsk} - \Gamma_{srk}) + \Gamma_{nkk} (\Gamma_{sm} - \Gamma_{rsn}) + \Gamma_{nrk} (\Gamma_{snk} - \Gamma_{nsk}) \right] = \\ &= \Delta A_r + (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi}) \frac{\partial A_\varphi}{\partial x^\varphi} + \frac{A_r}{2} [\Gamma_{\varphi r\varphi} (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi})] + \frac{A_\varphi}{2} \frac{\partial}{\partial x^\varphi} (\Gamma_{r\varphi\varphi} - \Gamma_{\varphi r\varphi}) \end{aligned}$$

By applying Eqs (45) and (46) here, we get

$$(\Delta \mathbf{A})_r = \Delta A_r - \frac{2}{r^2} \frac{\partial A_\varphi}{\partial \varphi} - \frac{A_r}{r^2} \quad (46)$$

For the projection on the ort $\mathbf{e}_\varphi$:

$$\begin{aligned} (\Delta \mathbf{A})_\varphi &= \Delta A_\varphi + (\Gamma_{\varphi sk} - \Gamma_{s\varphi k}) \frac{\partial A_s}{\partial x^k} + \frac{A_s}{2} \left[\frac{\partial}{\partial x^k} (\Gamma_{\varphi sk} - \Gamma_{s\varphi k}) + \Gamma_{nkk} (\Gamma_{s\varphi n} - \Gamma_{\varphi sn}) + \Gamma_{n\varphi k} (\Gamma_{snk} - \Gamma_{nsk}) \right] = \\ &= \Delta A_\varphi + (\Gamma_{\varphi r\varphi} - \Gamma_{r\varphi\varphi}) \frac{\partial A_r}{\partial x^r} + \frac{A_r}{2} \frac{\partial}{\partial x^r} (\Gamma_{\varphi r\varphi} - \Gamma_{r\varphi\varphi}) + \frac{A_\varphi}{2} \Gamma_{r\varphi\varphi} (\Gamma_{\varphi r\varphi} - \Gamma_{r\varphi\varphi}) \end{aligned}$$

By applying Eqs (45) and (46) here, we will finally have

$$(\Delta \mathbf{A})_\varphi = \Delta A_\varphi + \frac{2}{r^2} \frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r^2} \quad (47)$$

For the projection on the ort $\mathbf{e}_z$:

$$(\Delta \mathbf{A})_z = \Delta A_z \quad (48)$$

Putting Eqs (47)-(49) together, we have

$$\begin{cases} (\Delta \mathbf{A})_r = \Delta A_r - \frac{2}{r^2} \frac{\partial A_\varphi}{\partial \varphi} - \frac{A_r}{r^2} \\ (\Delta \mathbf{A})_\varphi = \Delta A_\varphi + \frac{2}{r^2} \frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r^2} \\ (\Delta \mathbf{A})_z = \Delta A_z. \end{cases} \quad (49)$$

Thus, we made certain that the general Eq. (31) is correct, and it can always be used in calculations in any orthogonal coordinate system.

Let us now proceed for verification of Eq. (36).

Let us consider spherical coordinates, where, according to Eq. (38), the intermediate differentials are $\partial x^1 = dr, \partial x^2 = r d\theta, \partial x^3 = r \sin \theta d\varphi$.

Then, for the projections of the curl on the unit orts $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$, from Eq. (36) we find as follows:

$$\begin{cases} (rot \mathbf{A})_r = \frac{\partial A_\theta}{\partial x^3} - \frac{\partial A_\varphi}{\partial x^2} + \frac{1}{2} (\Gamma_{\theta\varphi s} - \Gamma_{\varphi\theta s}) A_s, \\ (rot \mathbf{A})_\theta = \frac{\partial A_r}{\partial x^3} - \frac{\partial A_\varphi}{\partial x^1} + \frac{1}{2} (\Gamma_{r\varphi s} - \Gamma_{\varphi r s}) A_s \\ (rot \mathbf{A})_\varphi = \frac{\partial A_r}{\partial x^2} - \frac{\partial A_\theta}{\partial x^1} + \frac{1}{2} (\Gamma_{r\theta s} - \Gamma_{\theta r s}) A_s. \end{cases} \quad (50)$$

By applying the differentials Eq. (38) to the denominators, according to Eq. (51), and using the values of the Christoffel symbols Eq. (39), we get

$$\begin{cases} (rot \mathbf{A})_r = \frac{1}{r \sin \theta} \left[\frac{\partial(\sin \theta A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} \right], \\ (rot \mathbf{A})_\theta = \frac{1}{r} \frac{\partial}{\partial r} (r A_\varphi) - \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \varphi} \\ (rot \mathbf{A})_\varphi = \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta}. \end{cases} \quad (51)$$

The projections of the curl of vector Eq. (52) are the well-known expressions that are given in every source that is somehow associated with curved coordinates (see, for example, the already mentioned monograph [1]).

In the case of a cylindrical SC, the result will also be correct.

Thus, using the general Eq. (36), we can calculate the expressions for vector curl in any orthogonal coordinate system (and not just in spherical and cylindrical SC, as is usually done) thanks to the method of transition to an orthonormal basis demonstrated above.

4 Conclusion

In conclusion, we emphasise several important factors.

An algorithm for calculating the projections of vectors $(\Delta \mathbf{A})_i$ and $(rot \mathbf{A})_i$ in an arbitrary curvilinear basis is proposed;

General expressions for $(\Delta \mathbf{A})_i$ and $(rot \mathbf{A})_i$ in an arbitrary orthonormal basis are obtained;

Special cases of calculation $(\Delta \mathbf{A})_i$ and $(rot \mathbf{A})_i$ in spherical and cylindrical coordinate systems have been considered.

References

- [1] Landau L.D., Lifshitz E.M. Electrodynamics of Continuous Media. Vol. 8. Moscow. Science. 1982.
- [2] L. I. Sedov. Continuum Mechanics. In 2 volumes. Moscow. Science. 1973.
- [3] A. J. Mc Connell. Application of Tensor Analysis. Dover Publications Inc. New York. 1957.
- [4] P. K. Rashevskiy. Riemannian Geometry and Tensor Analysis. Moscow. Science. 1967.
- [5] Gladkov, S.O. On the application of the second covariant derivative of the vector function to the problems of hydrodynamics and the theory of elasticity. Bulletin of the Moscow State Regional University. 2019. No. 3. P. 42-67. DOI: 10.18384-2310-7251-2019-3-42-67.
- [6] Gladkov, S.O. About alternative calculation of covariant derivatives with an application to the problems of mechanics, physics and geometry. Bulletin of the Moscow State Regional University. 2019. No. 1. P. 16 - 45. DOI: 10.18384-2310-7251-2019-1-16-45.

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