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Travelling wave solutions to the proximate equations for LWSW

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Abstract

By feat of Maple 17 and a subsidiary ordinary differential equation, a new extension algebraic method is chosen to construct the travelling wave solutions to the proximate equation set involving an arbitrary parameter for long waves over shallow-water. Multiple triangle periodic solutions and new Jacobi elliptic function solutions are obtained. This procedure is applicable to other nonlinear partial differential equations as well.

Keywords: auxiliary equation; algebraic method; travelling wave solutions

1 Introduction

Investigating exact solutions of nonlinear partial differential equations (PDEs) has always been a key subject in mathematical physics. Some direct ways are presented, for example, the tanh-function means [1–3], Jacobi elliptic function expansion way [4–6], the F-expansion [7], (G'/G) -expansion way [8–13] and the homogeneous balance means [14, 15]. In this paper, a kind of new extension algebraic means is presented to construct the explicit solutions of the following proximate long water wave equations in shallow water [12]

$$u_t - uu_x - v_x + \alpha u_{xx} = 0 \quad (1a)$$

$$v_t - (uv)_x - \alpha v_{xx} = 0 \quad (1b)$$

where $\alpha \neq 0$ is an arbitrary constant.

Eqs (1a) and (1b) express the interaction of long water waves in time and one spatial dimension. The special case of Eqs (1a) and (1b)

$$u_t - uu_x - v_x + \frac{1}{2}u_{xx} = 0 \quad (2a)$$

$$v_t - (uv)_x - \frac{1}{2}v_{xx} = 0 \quad (2b)$$

was discussed by Broer [16] and Whitham [17, 18]; Broer first discovered that Eqs (2a) and (2b) are equivalent to Boussinesq equations for two-way dispersive long waves in shallow water by an appropriate variational

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principles. Kupershmidt [19] studied the symmetry and conservation law of Eq. (2). Furthermore, Wang and Zhou [20], Zhang [21], Yan [22] and Manafiana et al. [23] obtained abundant periodic solutions, soliton solutions and soliton-like solutions of Eq. (2) by using the homogeneous balance and extended homogeneous balance. Yan et al. [20, 21] derived some hyperbolic function solutions and periodic solutions of Eq. (2) by deleting tail auxiliary function method. Eqs (1a) and (1b) are the generalised case of Eq. (2), which involves an arbitrary constant. Wang et al. [12] obtained many hyperbolic solutions and trigonometric period solutions of Eqs (1a) and (1b) by using (G'/G) -expansion means. This paper tries to seek some important and undiscovered travelling wave solutions, especially Jacobi elliptic function solutions of Eqs (1a) and (1b) by using an auxiliary ODE (ordinary differential equation).

2 The new extension algebraic means

We'd like to provide the main steps of a new extension algebraic means in this section based on (G'/G) -expansion way in literatures [8-13] and the auxiliary equation methods in Refs [25–32]. Provided that a general nonlinear PDE with respect to two independent variables x and t expressed as

$$F(u_t, u_x, u_{tt}, u_{xt}, \dots) = 0 \quad (3)$$

where $u = u(x, t)$ is an unknown smooth function, $F(u_t, u_x, u_{tt}, u_{xt}, \dots)$ is a polynomial with respect to $u(x, t)$ and its various partial derivatives.

where is an unknown smooth function, is a polynomial with respect to and its various partial derivatives.

We assume the travelling wave transformations

$$u(x, t) = u(\xi), \quad \xi = \mu x - ct \quad (4)$$

reduces Eq. (3) to an ODE for $u(\xi)$ as follows

$$H(u, u', u'', u''', \dots) = 0 \quad (5)$$

where and in Eq. (4) are arbitrary constants to be determined later and the superscripts indicate the ordinary derivatives in regard to variable where μ and c in Eq. (4) are arbitrary constants to be determined later and the superscripts indicate the ordinary derivatives in regard to variable ξ

If Eq. (5) is integrable, then integrating term by term one or more times yields constant(s) of integration.

Assume that the solution of Eq. (5) can be shown by the following form

$$u = c_0 + \sum_{i=1}^n \left(\frac{a_{i0} + a_{i1} \phi'}{b_{i0} + b_{i1} \phi} \right)^i, \quad \alpha_{n1} b_{n1} \neq 0, \quad (6)$$

where $c_0, a_{i0}, a_{i1}, b_{i0}, b_{i1} (i = 1, 2, \dots, n)$ are arbitrary constants, n is a positive integer and $\phi = \phi(\xi)$ satisfies the following assistant ODE

$$\phi' = \varepsilon \sqrt{\sum_{j=0}^m e_j \phi^j}, \quad \varepsilon = \pm 1 \quad (7)$$

where $e_j, j = 0, 1, 2, 3, \dots, m$ are arbitrary constants and m is a positive integer. The solutions of Eq. (7) are given by Table 1 in the appendix.

The integer m and n can be obtained through using the homogeneous balance between the highest-order nonlinear terms and the highest-order derivatives in Eq. (5). Substituting Eq. (6) along with Eq. (7) into Eq. (5) with the aim of Maple17 and equating the coefficients of $\phi^i \sqrt{\sum_{j=0}^m e_j \phi^j} (i = 0, 1, 2, 3, 4, \dots)$ to zero forms an algebraic equation set for $\mu, c, c_0, a_{i0}, b_{i0} (i = 0, 1, 2, \dots, n)$. Suppose the numerical values of the foregoing constants can be gotten by solving the algebraic equations set, now that the solutions of Eq. (7) have been known, substituting the numerical values of $\mu, c, c_0, a_{i0}, b_{i0} (i = 0, 1, 2, \dots, n), e_j (j = 0, 1, 2, \dots, m)$ and the solutions of Eq. (7) into Eq. (6), so the travelling wave solutions of Eq. (3) will be found.

In Section 3, we obtain new and multiple travelling wave solutions of the proximate long water Eq. (1) by using the proposed means.

3 The proximate long water equations

The travelling wave variable is assumed that

$$u(x, t) = u(\xi), v(x, t) = v(\xi) \quad \xi = \mu x - ct \quad (8)$$

The travelling wave transformations Eq. (8) can convert Eq. (1) into the following ODEs for $u = u(\xi)$ and $v = v(\xi)$

$$-cu' - \mu uu' - \mu v' + \alpha \mu^2 u'' = 0 \quad (9a)$$

$$-cv' - \mu(uv)' - \alpha \mu^2 v'' = 0 \quad (9b)$$

Integrating the ODEs above in regard to ξ yields

$$F_1 - cu - \frac{1}{2}\mu u^2 - \mu v + \alpha \mu^2 u' = 0 \quad (10a)$$

$$F_2 - cv - \mu uv - \alpha \mu^2 v' = 0 \quad (10b)$$

where F_1 and F_2 are integration constants to be determined later.

Eqs (9a) and (10a) can be described respectively in the following forms

$$v' = -\frac{c}{\mu}u' - uu' + \alpha \mu u'' \quad (11a)$$

$$v = \frac{F_1}{\mu} - \frac{c}{\mu}u - \frac{1}{2}u^2 + \alpha \mu u' \quad (11b)$$

Substituting Eq. (11) into Eq. (10b) and simplifying it, Eqs (1a) and (1b) are reduced to an ODE as follows

$$(F_2 - \frac{c}{\mu}F_1) + (\frac{c^2}{\mu} - F_1)u + \frac{3c}{2}u^2 + \frac{1}{2}\mu u^3 - \alpha^2 \mu^3 u'' = 0 \quad (12)$$

Making full use of the homogeneous balance between the highest derivative u'' and the highest nonlinear term u^3 in Eq. (12) yields that m is an arbitrary positive integer and $n = 1$. We only analyse one case in the following discussion.

Let $n = 1, m = 4$,

In this situation, Eqs (6) and (7) are transformed into the following equations, respectively.

$$u(\xi) = c_0 + \frac{a_0 + a_1 \phi^4}{b_0 + b_1 \phi^4}, \quad (13)$$

$$\phi' = \varepsilon \sqrt{e_0 + e_1 \phi + e_2 \phi^2 + e_3 \phi^3 + e_4 \phi^4} \quad (14)$$

By substituting Eq. (13) into Eq. (12) with Eq. (14), the left-hand side of Eq. (12) is converted into another polynomial of ϕ^i ($i = 0, 1, 2, 3, 4, \dots, 8$) and $\phi^i \sqrt{\sum_{j=0}^4 e_j \phi^j}$ ($i = 0, 1, 2, 3, 4, \dots, 7$). And setting the coefficients of ϕ^i ($i = 0, 1, 2, \dots$) and $\phi^i \sqrt{\sum_{j=0}^4 e_j \phi^j}$ ($i = 0, 1, 2, 3, 4, \dots, 6$) to zero gives rise to an algebraic equation set. Solving the algebraic equation set with Maple 17 yields the followings:

$$\text{Case 1 } a_1 = \pm 2\alpha \mu b_1, c_0 = -\frac{c}{\mu}, a_0 = 0, F_2 = 0, b_0 = \frac{e_3}{4e_4} b_1, e_1 = \frac{e_3(4e_2e_4 - e_3^2)}{8e_4^2}, F_1 = -\frac{3\alpha^2 \mu^4 e_3^2 - 8\alpha^2 \mu^4 e_2e_4 + 2c^2 e_4}{4\mu e_4}.$$

Where e_0, e_2, e_3 are arbitrary constants, and μ, c, e_4 and b_1 are arbitrary nonzero constants.

Case 2

$$a_1 = 0, b_0 = 0, b_1 = \frac{a_0}{2\alpha \mu \sqrt{e_0}}, F_2 = \frac{\alpha^3 \mu^4 \sqrt{e_0}(2e_3e_0 - e_1e_2)}{2e_0}, F_1 = \frac{\mu(\pm 2c_0\alpha \mu e_1 \sqrt{e_0} - 2c_0^2e_0 - 4\alpha^2 \mu^2 e_0e_2 + \alpha^2 \mu^2 e_1^2)}{4e_0},$$

$$c = \frac{\mu(\pm \alpha \mu e_1 - 2c_0 \sqrt{e_0})}{2\sqrt{e_0}}, \text{ where } c_0, e_1, e_2, e_3, a_0 \text{ are arbitrary constants, and } \mu \neq 0, e_0 > 0, e_4 \neq 0.$$

Choosing different values of e_j ($j = 0, 1, 2, 3, 4$), Eq. (11) has many different fundamental solutions, substituting all solutions of Eq. (15) into Eq. (13), and considering Eq. (11b) yields multiple Jacobi elliptic and solution-like solutions of Eq. (1) and are listed as follows.

For case1, Jacobi elliptic function solutions have

$$1.1) e_0 = 1, e_2 = -(1 - k^2), e_4 = k^2, e_1 = e_3 = 0,$$

$$u_{1.1.1} = -\frac{c}{\mu} + 2\mu \alpha (1 - k^2) \text{cs}(\xi) \text{dn}(\xi), v_{1.1.1} = -4\alpha^2 \mu^2 \text{ns}(\xi)^2; u_{1.1.2} = -\frac{c}{\mu} - 2\mu \alpha (1 - k^2) \text{cs}(\xi) \text{dn}(\xi), v_{1.1.2} = -4\alpha^2 \mu^2 k^2 \text{sn}(\xi)^2;$$

$$u_{1.1.3} = -\frac{c}{\mu} - 2\mu \alpha (1 - k^2) \text{sc}(\xi) \text{nd}(\xi), v_{1.1.3} = -4\alpha^2 \mu^2 \text{dc}(\xi)^2; u_{1.1.4} = -\frac{c}{\mu} + 2\mu \alpha (1 - k^2) \text{sc}(\xi) \text{nd}(\xi), v_{1.1.4} = -4\alpha^2 \mu^2 k^2 \text{cd}(\xi)^2,$$

$$1.2) e_0 = 1 - k^2, e_2 = 2k^2 - 1, e_4 = -k^2, e_1 = e_3 = 0,$$

$$u_{1.2.1} = -\frac{c}{\mu} - 2\mu \alpha \text{sc}(\xi) \text{dn}(\xi), v_{1.2.1} = -4\alpha^2 \mu^2 (1 - k^2) \text{nc}(\xi)^2; u_{1.2.2} = -\frac{c}{\mu} + 2\mu \alpha \text{sc}(\xi) \text{dn}(\xi), v_{1.2.2} = 4\alpha^2 \mu^2 k^2 \text{cn}(\xi)^2;$$

$$\begin{aligned}
&1.3) e_0 = k^2 - 1, e_2 = 2 - k^2, e_4 = -1, e_1 = e_3 = 0 \\
&u_{1.3.1} = -\frac{c}{\mu} - 2\mu\alpha k^2 \operatorname{sd}(\xi) \operatorname{cn}(\xi), v_{1.3.1} = -4\alpha^2 \mu^2 (k^2 - 1) \operatorname{nd}(\xi)^2, \\
&1.4) e_0 = k^2, e_2 = -(1 + k^2), e_4 = 1, e_1 = e_3 = 0, \\
&u_{1.4.1} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{dn}(\xi) \operatorname{cs}(\xi), v_{1.3.1} = -4\alpha^2 \mu^2 k^2 \operatorname{sn}(\xi)^2; u_{1.4.2} = -\frac{c}{\mu} + 2\mu\alpha \operatorname{dn}(\xi) \operatorname{cs}(\xi), v_{1.4.2} = -4\alpha^2 \mu^2 \operatorname{ns}(\xi)^2, \\
&u_{1.4.4} = -\frac{c}{\mu} - 2\mu\alpha (1 - k^2) \operatorname{nd}(\xi) \operatorname{sc}(\xi), v_{1.4.4} = -4\alpha^2 \mu^2 \operatorname{dc}(\xi)^2, \\
&1.5) e_0 = -k^2, e_2 = 2k^2 - 1, e_4 = 1 - k^2, e_1 = e_3 = 0, \\
&u_{1.5.1} = -\frac{c}{\mu} + 2\mu\alpha \operatorname{sn}(\xi) \operatorname{dc}(\xi), v_{1.5.1} = 4\alpha^2 \mu^2 k^2 \operatorname{cn}(\xi)^2; u_{1.5.2} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{sn}(\xi) \operatorname{dc}(\xi), v_{1.5.2} = -4\alpha^2 \mu^2 (1 - k^2) \operatorname{nc}(\xi)^2, \\
&1.6) e_0 = -1, e_2 = 2 - k^2, e_4 = k^2 - 1, e_1 = e_3 = 0, \\
&u_{1.6.1} = -\frac{c}{\mu} + 2\mu\alpha k^2 \operatorname{sn}(\xi) \operatorname{cd}(\xi), v_{1.6.1} = 4\alpha^2 \mu^2 \operatorname{dn}(\xi)^2; u_{1.6.2} = -\frac{c}{\mu} - 2\mu\alpha k^2 \operatorname{sn}(\xi) \operatorname{cd}(\xi), v_{1.6.2} = -4\alpha^2 \mu^2 (k^2 - 1) \operatorname{nd}(\xi)^2; \\
&1.7) e_0 = 1 - k^2, e_2 = 2 - k^2, e_4 = 1, e_1 = e_3 = 0, \\
&u_{1.7.1} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{nc}(\xi) \operatorname{ds}(\xi), v_{1.7.1} = -4\alpha^2 \mu^2 (1 - k^2) \operatorname{sc}(\xi)^2; u_{1.7.2} = -\frac{c}{\mu} + 2\mu\alpha \operatorname{nc}(\xi) \operatorname{ds}(\xi), v_{1.7.2} = -4\alpha^2 \mu^2 \operatorname{cs}(\xi)^2, \\
&1.8) e_0 = 1, e_2 = 2 - k^2, e_4 = 1 - k^2, e_1 = e_3 = 0, \\
&u_{1.8.1} = -\frac{c}{\mu} + 2\mu\alpha \operatorname{ns}(\xi) \operatorname{dc}(\xi), v_{1.8.1} = -4\alpha^2 \mu^2 \operatorname{cs}(\xi)^2; u_{1.8.2} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{ns}(\xi) \operatorname{dc}(\xi), v_{1.8.2} = -4\alpha^2 \mu^2 (1 - k^2) \operatorname{sc}(\xi)^2, \\
&1.9) e_0 = 1, e_2 = 2k^2 - 1, e_4 = k^2(k^2 - 1), e_1 = e_3 = 0, \\
&u_{1.9.1} = -\frac{c}{\mu} + 2\mu\alpha \operatorname{ns}(\xi) \operatorname{cd}(\xi), v_{1.9.1} = -4\alpha^2 \mu^2 \operatorname{ds}(\xi)^2; u_{1.9.2} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{ns}(\xi) \operatorname{cd}(\xi), v_{1.9.2} = -4\alpha^2 \mu^2 k^2 (k^2 - 1) \operatorname{sd}(\xi)^2, \\
&1.10) e_0 = k^2(k^2 - 1), e_2 = 2k^2 - 1, e_4 = 1, e_1 = e_3 = 0, \\
&u_{1.10.1} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{cd}(\xi) \operatorname{ns}(\xi), v_{1.10.1} = -4\alpha^2 \mu^2 k^2 (k^2 - 1) \operatorname{sd}(\xi)^2; u_{1.10.2} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{cd}(\xi) \operatorname{ns}(\xi), v_{1.10.2} = -4\alpha^2 \mu^2 \operatorname{ds}(\xi)^2, \\
&1.11) e_0 = 0, e_2 = -(1 + k^2), e_4 = k^2, e_1 = e_3 = 0, \\
&u_{1.11.1} = -\frac{c}{\mu} - 2\mu\alpha \operatorname{cs}(\xi) \operatorname{dn}(\xi), v_{1.11.1} = -4\alpha^2 \mu^2 k^2 \operatorname{sn}(\xi)^2; \\
&1.12) e_0 = \frac{k^4}{4}, e_2 = \frac{1}{2}(k^2 - 2), e_4 = \frac{1}{4}, e_1 = e_3 = 0, \\
&v_{1.14.1} = -\frac{\alpha^2 \mu^2 k^4}{(\operatorname{ns}(\xi) \pm \operatorname{ds}(\xi))^2}; v_{1.14.2} = -\alpha^2 \mu^2 (\operatorname{ns}(\xi) \pm \operatorname{ds}(\xi))^2, \\
&1.13) e_0 = \frac{k^2}{4}, e_2 = \frac{(k^2 - 2)}{2}, e_4 = \frac{k^2}{4}, e_1 = e_3 = 0, \\
&u_{1.13.1} = -\frac{c}{\mu} \mp 2i\alpha\mu \sqrt{1 - k^2} \operatorname{nd}(\xi), v_{1.13.1} = -\frac{\alpha^2 \mu^2 k^2}{(i\sqrt{1 - k^2} \operatorname{sn}(\xi) \pm \operatorname{cn}(\xi))^2}; u_{1.13.2} = -\frac{c}{\mu} \pm 2i\alpha\mu \sqrt{1 - k^2} \operatorname{nd}(\xi), v_{1.13.2} = -\alpha^2 \mu^2 k^2 (i\sqrt{1 - k^2} \operatorname{sn}(\xi) \pm \operatorname{cn}(\xi))^2; \\
&u_{1.13.3} = -\frac{c}{\mu} \mp 2i\alpha\mu \operatorname{dn}(\xi), v_{1.13.3} = -\frac{\alpha^2 \mu^2 k^2}{(\operatorname{sn}(\xi) \pm i \operatorname{cn}(\xi))^2}; u_{1.13.4} = -\frac{c}{\mu} \pm 2i\alpha\mu \operatorname{dn}(\xi), v_{1.13.4} = -\alpha^2 \mu^2 k^2 (\operatorname{sn}(\xi) \pm i \operatorname{cn}(\xi))^2, \\
&1.14) e_0 = 1, e_2 = 2 - 4k^2, e_4 = 1, e_1 = e_3 = 0, \\
&u_{1.14.1} = -\frac{c}{\mu} + 2\alpha\mu (\operatorname{nc}(\xi) \operatorname{ds}(\xi) - k^2 \operatorname{sn}(\xi)) \operatorname{cd}(\xi), v_{1.14.1} = -4\alpha^2 \mu^2 \operatorname{sc}(\xi)^{-2} \operatorname{dn}(\xi)^{-2}; \\
&u_{1.14.2} = -\frac{c}{\mu} - 2\alpha\mu (\operatorname{nc}(\xi) \operatorname{ds}(\xi) - k^2 \operatorname{sn}(\xi)) \operatorname{cd}(\xi), v_{1.14.2} = -4\alpha^2 \mu^2 \operatorname{sc}(\xi)^2 \operatorname{dn}(\xi)^2, \\
&1.15) e_0 = \frac{(k-1)^2}{4D_1}, e_2 = \frac{1}{2}(1 + 6k + k^2), e_4 = \frac{D_1^2(k-1)^2}{4}, e_1 = e_3 = 0, \\
&u_{1.15.1} = -\frac{c}{\mu} - \frac{2\alpha\mu (\operatorname{sn}(\xi) (k^2 \operatorname{cn}^2(\xi) + \operatorname{dn}^2(\xi)) (1 + (k+1) \operatorname{sn}(\xi) + k \operatorname{sn}^2(\xi)) + \operatorname{cn}^2(\xi) \operatorname{dn}^2(\xi) (k+1 + 2k \operatorname{sn}(\xi)))}{(1 + (k+1) \operatorname{sn}(\xi) + k \operatorname{sn}^2(\xi)) \operatorname{dn}(\xi) \operatorname{cn}(\xi)}, \\
&v_{1.15.1} = -\frac{\alpha^2 \mu^2 (k-1)^2 D_1 (1 + \operatorname{sn}(\xi))^2 (1 + k \operatorname{sn}(\xi))^2}{\operatorname{dn}(\xi)^2 \operatorname{cn}(\xi)^2}, \\
&u_{1.15.2} = -\frac{c}{\mu} + \frac{2\alpha\mu (\operatorname{sn}(\xi) (k^2 \operatorname{cn}^2(\xi) + \operatorname{dn}^2(\xi)) (1 + (k+1) \operatorname{sn}(\xi) + k \operatorname{sn}^2(\xi)) + \operatorname{cn}^2(\xi) \operatorname{dn}^2(\xi) (k+1 + 2k \operatorname{sn}(\xi)))}{(1 + (k+1) \operatorname{sn}(\xi) + k \operatorname{sn}^2(\xi)) \operatorname{dn}(\xi) \operatorname{cn}(\xi)},
\end{aligned}$$

$$v_{1.15.2} = -\frac{\alpha^2 \mu^2 (k-1)^2 \operatorname{dn}(\xi)^2 \operatorname{cn}(\xi)^2}{(1+\operatorname{sn}(\xi))^2 (1+k \operatorname{sn}(\xi))^2},$$

$$1.16) e_0 = \frac{(k+1)^2}{4D_1}, e_2 = \frac{1}{2}(1-6k+k^2), e_4 = \frac{D_1^2(k+1)^2}{4}, e_1 = e_3 = 0,$$

$$u_{1.16.1} = -\frac{c}{\mu} - \frac{2\alpha\mu(\operatorname{sn}(\xi)(k^2 \operatorname{cn}^2(\xi) + \operatorname{dn}^2(\xi))(1+(-k+1)\operatorname{sn}(\xi)-k \operatorname{sn}^2(\xi)) + \operatorname{cn}^2(\xi) \operatorname{dn}^2(\xi)(-k+1-2k \operatorname{sn}(\xi)))}{(1+(-k+1)\operatorname{sn}(\xi)-k \operatorname{sn}^2(\xi)) \operatorname{dn}(\xi) \operatorname{cn}(\xi)},$$

$$v_{1.16.1} = -\frac{\alpha^2 \mu^2 (k+1)^2 D_1 (1+\operatorname{sn}(\xi))^2 (1-k \operatorname{sn}(\xi))^2}{\operatorname{dn}(\xi)^2 \operatorname{cn}(\xi)^2}; v_{1.16.2} = -\frac{\alpha^2 \mu^2 (k+1)^2 \operatorname{dn}(\xi)^2 \operatorname{cn}(\xi)^2}{(1+\operatorname{sn}(\xi))^2 (1-k \operatorname{sn}(\xi))^2},$$

$$1.17) e_0 = k^4 - 2k^3 + k^2, e_2 = 6k - k^2 - 1, e_4 = -\frac{4}{k}, e_1 = e_3 = 0,$$

$$u_{1.17.1} = -\frac{c}{\mu} - \frac{2\mu\alpha((k^2 \operatorname{cn}(\xi)^2 + \operatorname{dn}(\xi)^2)(1+k \operatorname{sn}^2(\xi)) + 2k \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2)}{(1+k \operatorname{sn}(\xi)^2) \operatorname{dn}(\xi) \operatorname{cs}(\xi)},$$

$$v_{1.17.1} = -\frac{4\alpha^2 \mu^2 (k^4 - 2k^3 + k^2)(1+k \operatorname{sn}(\xi)^2)}{k^2 \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2};$$

$$u_{1.17.2} = -\frac{c}{\mu} + \frac{2\mu\alpha((k^2 \operatorname{cn}(\xi)^2 + \operatorname{dn}(\xi)^2)(1+k \operatorname{sn}^2(\xi)) + 2k \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2)}{(1+k \operatorname{sn}(\xi)^2) \operatorname{dn}(\xi) \operatorname{cs}(\xi)}, v_{1.17.2} = \frac{16\alpha^2 \mu^2 k \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2}{(1+k \operatorname{sn}(\xi)^2)^2},$$

$$1.18) e_0 = k^4 + 2k^3 + k^2, e_2 = -6k - k^2 - 1, e_4 = \frac{4}{k}, e_1 = e_3 = 0,$$

$$v_{1.18.1} = -\frac{4\alpha^2 \mu^2 (k^4 + 2k^3 + k^2)(-1+k \operatorname{sn}(\xi)^2)^2}{k^2 \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2};$$

$$v_{1.18.2} = -\frac{16\alpha^2 \mu^2 k \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2}{(-1+k \operatorname{sn}(\xi)^2)^2},$$

$$1.19) e_0 = 2 + 2k_1 - k^2, e_2 = 6k_1 - k^2 + 2, e_4 = 4k_1, e_1 = e_3 = 0,$$

$$u_{1.19.1} = -\frac{c}{\mu} + \frac{2\mu\alpha((\operatorname{cn}(\xi)^2 - \operatorname{sn}(\xi)^2)(k_1 - \operatorname{dn}(\xi)^2) - 2k^2 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2)}{(k_1 - \operatorname{dn}(\xi)^2) \operatorname{sn}(\xi) \operatorname{cd}(\xi)}, v_{1.19.1} = -\frac{4\alpha^2 \mu^2 (2+2k_1-k^2)(k_1 - \operatorname{dn}(\xi)^2)^2}{k^4 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2}; v_{1.19.2} = -\frac{16\alpha^2 \mu^2 k_1 k^4 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2}{(k_1 + \operatorname{dn}(\xi)^2)^2},$$

$$1.20) e_0 = 2 - 2k_1 - k^2, e_2 = -6k_1 - k^2 + 2, e_4 = -4k_1, e_1 = e_3 = 0,$$

$$u_{1.20.1} = -\frac{c}{\mu} - \frac{2\mu\alpha((\operatorname{cn}(\xi)^2 - \operatorname{sn}(\xi)^2)(k_1 + \operatorname{dn}(\xi)^2) + 2k^2 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2)}{(k_1 + \operatorname{dn}(\xi)^2) \operatorname{sn}(\xi) \operatorname{cd}(\xi)}, v_{1.20.1} = -\frac{4\alpha^2 \mu^2 (2-2k_1-k^2)(k_1 + \operatorname{dn}(\xi)^2)^2}{k^4 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2}; u_{1.20.2} = -\frac{c}{\mu} + \frac{2\mu\alpha((\operatorname{cn}(\xi)^2 - \operatorname{sn}(\xi)^2)(k_1 + \operatorname{dn}(\xi)^2) + 2k^2 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2)}{(k_1 + \operatorname{dn}(\xi)^2) \operatorname{sn}(\xi) \operatorname{cd}(\xi)}, v_{1.20.2} = \frac{16\alpha^2 \mu^2 k_1 k^4 \operatorname{sn}(\xi)^2 \operatorname{cn}(\xi)^2}{(k_1 - \operatorname{dn}(\xi)^2)^2},$$

$$1.21) e_0 = \frac{k^2-1}{4(D_3^2+D_2^2)}, e_2 = \frac{k^2+1}{2}, e_4 = \frac{(D_3^2 k^2 - D_2^2)(k^2-1)}{4}, e_1 = e_3 = 0,$$

$$u_{1.21.1} = -\frac{c}{\mu} + \frac{2\mu\alpha\left(D_2 \operatorname{dn}(\xi) + D_3 \operatorname{cn}(\xi) + D_2 \sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} \operatorname{sn}(\xi) \operatorname{dn}(\xi) + D_3 k^2 \sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} \operatorname{cn}(\xi) \operatorname{sn}(\xi)\right)}{(D_2 \operatorname{cn}(\xi) + D_3 \operatorname{dn}(\xi)) \left(\sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} + \operatorname{sn}(\xi)\right)},$$

$$v_{1.21.1} = -\frac{\alpha^2 \mu^2 (k^2 - 1) (D_2 \operatorname{cn}(\xi) + D_3 \operatorname{dn}(\xi))^2}{(D_3^2 + D_2^2) \left(\sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} + \operatorname{sn}(\xi)\right)^2};$$

$$u_{1.21.2} = -\frac{c}{\mu} - \frac{2\mu\alpha\left(D_2 \operatorname{dn}(\xi) + D_3 \operatorname{cn}(\xi) + D_2 \sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} \operatorname{sn}(\xi) \operatorname{dn}(\xi) + D_3 k^2 \sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} \operatorname{cn}(\xi) \operatorname{sn}(\xi)\right)}{(D_2 \operatorname{cn}(\xi) + D_3 \operatorname{dn}(\xi)) \left(\sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} + \operatorname{sn}(\xi)\right)},$$

$$v_{1.21.2} = -\frac{\alpha^2 \mu^2 (D_3^2 k^2 - D_2^2) (k^2 - 1) \left(\sqrt{\frac{D_2^2 - D_3^2}{D_2^2 - D_3^2 k^2}} + \operatorname{sn}(\xi) \right)^2}{(D_2 \operatorname{cn}(\xi) + D_3 \operatorname{dn}(\xi))^2},$$

$$1.22) e_0 = \frac{k^4}{4(D_3^2 + D_2^2)}, e_2 = \frac{k^2}{2} - 1, e_4 = \frac{D_3^2 + D_2^2}{4}, e_1 = e_3 = 0,$$

$$u_{1.22.1} = -\frac{c}{\mu} - \frac{2\mu\alpha \left(k^2 \operatorname{sn}(\xi) \operatorname{cn}(\xi) (D_2 \operatorname{sn}(\xi) + D_3 \operatorname{cn}(\xi)) + \operatorname{dn}(\xi) \left(\sqrt{\frac{D_2^2 - (k^2 - 1)D_3^2}{D_2^2 + D_3^2}} + \operatorname{dn}(\xi) \right) (D_2 \operatorname{cn}(\xi) - D_3 \operatorname{sn}(\xi)) \right)}{(D_2 \operatorname{sn}(\xi) + D_3 \operatorname{cn}(\xi)) \left(\sqrt{\frac{D_2^2 - (k^2 - 1)D_3^2}{D_2^2 + D_3^2}} + \operatorname{dn}(\xi) \right)},$$

$$v_{1.22.1} = -\frac{\alpha^2 \mu^2 k^4 (D_2 \operatorname{sn}(\xi) + D_3 \operatorname{cn}(\xi))^2}{(D_3^2 + D_2^2) \left(\sqrt{\frac{D_2^2 - (k^2 - 1)D_3^2}{D_2^2 + D_3^2}} + \operatorname{dn}(\xi) \right)^2};$$

$$u_{1.22.2} = -\frac{c}{\mu} + \frac{2\mu\alpha \left(k^2 \operatorname{sn}(\xi) \operatorname{cn}(\xi) (D_2 \operatorname{sn}(\xi) + D_3 \operatorname{cn}(\xi)) + \operatorname{dn}(\xi) \left(\sqrt{\frac{D_2^2 - (k^2 - 1)D_3^2}{D_2^2 + D_3^2}} + \operatorname{dn}(\xi) \right) (D_2 \operatorname{cn}(\xi) - D_3 \operatorname{sn}(\xi)) \right)}{(D_2 \operatorname{sn}(\xi) + D_3 \operatorname{cn}(\xi)) \left(\sqrt{\frac{D_2^2 - (k^2 - 1)D_3^2}{D_2^2 + D_3^2}} + \operatorname{dn}(\xi) \right)},$$

$$v_{1.22.2} = -\frac{\alpha^2 \mu^2 (D_3^2 + D_2^2) \left(\sqrt{\frac{D_2^2 - (k^2 - 1)D_3^2}{D_2^2 + D_3^2}} + \operatorname{dn}(\xi) \right)^2}{(D_2 \operatorname{sn}(\xi) + D_3 \operatorname{cn}(\xi))^2},$$

$$1.23) e_0 = \frac{2k - k^2 - 1}{D_2^2}, e_2 = 2k^2 + 2, e_4 = -D_2^2(k^2 + 2k + 1), e_1 = e_3 = 0,$$

$$u_{1.23.1} = -\frac{c}{\mu} + \frac{8\mu\alpha \operatorname{ksn}(\xi) \operatorname{cn}(\xi) \operatorname{dn}(\xi)}{k^2 \operatorname{sn}(\xi)^4 - 1}, v_{1.23.1} = -\frac{4\alpha^2 \mu^2 (2k - k^2 - 1) (k \operatorname{sn}(\xi)^2 + 1)^2}{(k \operatorname{sn}(\xi)^2 - 1)^2};$$

$$u_{1.23.2} = -\frac{c}{\mu} - \frac{8\mu\alpha \operatorname{ksn}(\xi) \operatorname{cn}(\xi) \operatorname{dn}(\xi)}{k^2 \operatorname{sn}(\xi)^4 - 1}, v_{1.23.2} = \frac{4\alpha^2 \mu^2 (k^2 + 2k + 1) (k \operatorname{sn}(\xi)^2 - 1)^2}{(k \operatorname{sn}(\xi)^2 + 1)^2},$$

$$1.24) e_0 = \frac{-2k - k^2 - 1}{D_2^2}, e_2 = 2k^2 + 2, e_4 = -D_2^2(k^2 + 2k + 1), e_1 = e_3 = 0,$$

$$u_{1.24.2} = -\frac{c}{\mu} + \frac{8\mu\alpha \operatorname{ksn}(\xi) \operatorname{cn}(\xi) \operatorname{dn}(\xi)}{k^2 \operatorname{sn}(\xi)^4 - 1}, v_{1.24.2} = \frac{4\alpha^2 \mu^2 (k^2 + 2k + 1) (k \operatorname{sn}(\xi)^2 + 1)^2}{(k \operatorname{sn}(\xi)^2 - 1)^2},$$

$$1.25) e_0 = e_4 = \frac{1}{4}, e_2 = \frac{1 - 2k^2}{2}, e_1 = e_3 = 0,$$

$$u_{1.25.3} = -\frac{c}{\mu} + \frac{2\mu\alpha \operatorname{ksn}(\xi) (\operatorname{dn}(\xi) \mp i \operatorname{ksn}(\xi))}{(k \operatorname{sn}(\xi) \pm i \operatorname{dn}(\xi))}, v_{1.25.3} = -\frac{\alpha^2 \mu^2}{(k \operatorname{sn}(\xi) \pm i \operatorname{dn}(\xi))^2}; u_{1.25.4} = -\frac{c}{\mu} - \frac{2k \operatorname{cn}(\xi) (\operatorname{dn}(\xi) \mp i \operatorname{ksn}(\xi))}{(k \operatorname{sn}(\xi) \pm i \operatorname{dn}(\xi))}, v_{1.25.4} = -\alpha^2 \mu^2 (k \operatorname{sn}(\xi) \pm i \operatorname{dn}(\xi))^2; u_{1.25.7} = -\frac{c}{\mu} \pm 2\mu\alpha \operatorname{ds}(\xi), v_{1.25.7} = -\alpha^2 \mu^2 \operatorname{ns}(\xi)^2 (1 \pm \operatorname{cn}(\xi))^2; u_{1.25.8} = -\frac{c}{\mu} \mp 2\mu\alpha \operatorname{ds}(\xi), v_{1.25.8} = -\alpha^2 \mu^2 \operatorname{sn}(\xi)^2 (1 \pm \operatorname{cn}(\xi))^{-2},$$

$$1.26) e_0 = e_4 = \frac{k^2 - 1}{4}, e_2 = \frac{k^2 + 1}{2}, e_1 = e_3 = 0,$$

$$u_{1.26.1} = -\frac{c}{\mu} \pm 2\mu\alpha \operatorname{kcd}(\xi), v_{1.26.1} = -\frac{\alpha^2 \mu^2 (k^2 - 1)}{(k \operatorname{sd}(\xi) \pm \operatorname{nd}(\xi))^2};$$

$$u_{1.26.2} = -\frac{c}{\mu} \mp 2\mu\alpha \operatorname{kcd}(\xi), v_{1.26.2} = -\alpha^2 \mu^2 (k^2 - 1) (k \operatorname{sd}(\xi) \pm \operatorname{nd}(\xi))^2;$$

$$u_{1.26.3} = -\frac{c}{\mu} \mp 2\mu\alpha \operatorname{kdc}(\xi), v_{1.26.3} = -\frac{\alpha^2 \mu^2 (k^2 - 1) (1 \pm k \operatorname{sn}(\xi))^2}{\operatorname{dn}(\xi)^2};$$

$$u_{1.26.4} = -\frac{c}{\mu} \pm 2\mu\alpha \operatorname{kdc}(\xi), v_{1.26.4} = -\frac{\alpha^2 \mu^2 (k^2 - 1) \operatorname{dn}(\xi)^2}{(1 \pm k \operatorname{sn}(\xi))^2},$$

$$1.27) e_0 = -\frac{(1 - k^2)^2}{4}, e_2 = \frac{k^2 + 1}{2}, e_4 = -\frac{1}{4}, e_1 = e_3 = 0,$$

$$\begin{aligned}
u_{1.27.1} &= -\frac{c}{\mu} \mp \mu \alpha \operatorname{sn}(\xi), v_{1.27.1} = \alpha^2 \mu^2 (1-k^2)^2 (k \operatorname{cn}(\xi) \pm \operatorname{dn}(\xi))^{-2}; u_{1.27.2} = -\frac{c}{\mu} \pm \mu \alpha \operatorname{sn}(\xi), \\
v_{1.27.2} &= \alpha^2 \mu^2 (k \operatorname{cn}(\xi) \pm \operatorname{dn}(\xi))^2, \\
1.28) e_0 &= \frac{1}{4}, e_2 = \frac{k^2+1}{2}, e_4 = \frac{(1-k^2)^2}{4}, e_1 = e_3 = 0, \\
u_{1.28.1} &= -\frac{c}{\mu} \pm 2\mu \alpha \operatorname{ns}(\xi), v_{1.28.1} = -\alpha^2 \mu^2 (\operatorname{dn}(\xi) \pm \operatorname{cn}(\xi))^2 \operatorname{sn}(\xi)^{-2}; u_{1.28.2} = -\frac{c}{\mu} \mp 2\mu \alpha \operatorname{ns}(\xi), v_{1.28.2} = \\
&-\alpha^2 \mu^2 (1-k^2)^2 \operatorname{sn}(\xi)^2 (\operatorname{dn}(\xi) \pm \operatorname{cn}(\xi))^{-2}, 1.29) e_0 = \frac{1}{4}, e_2 = \frac{(k^2-2)^2}{2}, e_4 = \frac{k^4}{4}, e_1 = e_3 = 0, \\
u_{1.29.1} &= -\frac{c}{\mu} \mp 2\mu \alpha \sqrt{1-k^2} \operatorname{sc}(\xi), v_{1.29.1} = -\frac{\alpha^2 \mu^2 (\sqrt{1-k^2} \pm \operatorname{dn}(\xi))^2}{\operatorname{cn}(\xi)^2}; \\
u_{1.29.2} &= -\frac{c}{\mu} \pm 2\mu \alpha \sqrt{1-k^2} \operatorname{sc}(\xi), v_{1.29.2} = -\frac{\alpha^2 \mu^2 k^4 \operatorname{cn}(\xi)^2}{(\sqrt{1-k^2} \pm \operatorname{dn}(\xi))^2}.
\end{aligned}$$

The solutions of hyperbolic function have

$$1.30) e_0 = 0, e_2 > 0, e_4 < 0, e_1 = e_3 = 0,$$

$$u_{1.30.1} = -\frac{c}{\mu} + 2\alpha \mu \sqrt{e_2} \tanh(\sqrt{e_2} \xi), v_{1.30.1} = 4\alpha^2 \mu^2 e_2 \operatorname{sech}^2(\sqrt{e_2} \xi),$$

$$1.31) e_0 = 0, e_2 > 0, e_4 > 0, e_1 = e_3 = 0,$$

$$u_{1.31.1} = -\frac{c}{\mu} + 2\alpha \mu \sqrt{e_2} \coth(\sqrt{e_2} \xi), v_{1.31.1} = -4\alpha^2 \mu^2 e_2 \operatorname{csc} h^2(\sqrt{e_2} \xi),$$

$$1.32) e_0 = \frac{e_2^2}{4e_4}, e_2 < 0, e_4 > 0, e_1 = e_3 = 0,$$

$$\begin{aligned}
u_{1.32.1} &= -\frac{c}{\mu} + 2\alpha \mu \sqrt{\frac{-e_2}{2}} \operatorname{sech}(\sqrt{\frac{-e_2}{2}} \xi) \operatorname{csch}(\sqrt{\frac{-e_2}{2}} \xi), v_{1.32.1} = 2\alpha^2 \mu^2 e_2 \tanh^{-2}(\sqrt{\frac{-e_2}{2}} \xi); v_{1.32.2} = \\
&2\alpha^2 \mu^2 e_2 \tanh^2(\sqrt{\frac{-e_2}{2}} \xi),
\end{aligned}$$

$$1.33) e_0 = 0, e_2 = 1, e_4 = \frac{1}{2}, e_1 = e_3 = 0,$$

$$u_{1.33.1} = -\frac{c}{\mu} - 2\sqrt{2} \mu \alpha \coth(D - \xi), v_{1.33.1} = -2\alpha^2 \mu^2 \tanh^2(D - \xi) (2 - 2 \tanh^2(D - \xi))^{-1},$$

Triangular periodic solutions have

$$1.34) e_0 = 0, e_2 < 0, e_4 > 0, e_1 = e_3 = 0,$$

$$u_{1.34.1} = -\frac{c}{\mu} - 2\alpha \mu \sqrt{-e_2} \tan(\sqrt{-e_2} \xi), v_{1.34.1} = 4\alpha^2 \mu^2 e_2 \sec^2(\sqrt{-e_2} \xi);$$

$$u_{1.34.2} = -\frac{c}{\mu} + 2\alpha \mu \sqrt{-e_2} \cot(\sqrt{-e_2} \xi), v_{1.34.2} = 4\alpha^2 \mu^2 e_2 \csc^2(\sqrt{-e_2} \xi); 1.35) e_0 = \frac{e_2^2}{4e_4}, e_2 < 0, e_4 > 0, e_1 = e_3 = 0,$$

$$u_{1.36.1} = -\frac{c}{\mu} + 2\sqrt{2e_2} \mu \alpha \csc(\sqrt{2e_2} \xi), v_{1.36.1} = -2\alpha^2 \mu^2 e_2 \cot^2(\sqrt{\frac{e_2}{2}} \xi);$$

$$u_{1.36.2} = -\frac{c}{\mu} - 2\sqrt{2e_2} \mu \alpha \csc(\sqrt{2e_2} \xi), v_{1.36.2} = -2\alpha^2 \mu^2 e_2 \tan^2(\sqrt{\frac{e_2}{2}} \xi),$$

Rational solutions have

$$1.36) e_0 = 0, e_2 = 0, e_4 > 0, e_1 = e_3 = 0,$$

$$u_{1.36.1} = -\frac{c}{\mu} - \frac{2\alpha \mu \sqrt{e_4}}{\sqrt{e_4} \xi + \xi_0}, v_{1.36.1} = -\frac{4\alpha^2 \mu^2 e_4}{(\sqrt{e_4} \xi + \xi_0)^2},$$

$$1.37) e_0 = k, e_1 = -4k, e_2 = -1 + 6k - k^2, e_3 = 2(k-1)^2, e_4 = -(k-1)^2,$$

$$u_{1.37.1} = -\frac{c}{\mu} - \frac{4\alpha \mu \sqrt{k} \operatorname{cn}(\xi) \operatorname{dn}(\xi)}{1 - k \operatorname{sn}(\xi)^2},$$

$$v_{1.37.1} = 2\alpha^2 \mu^2 (k^2 + 6k + 1) - \frac{16\alpha^2 \mu^2 k \operatorname{cn}(\xi)^2 \operatorname{dn}(\xi)^2}{(1 - k \operatorname{sn}(\xi)^2)^2} + \frac{4\alpha^2 \mu^2 ((-1 - k^2 + 6k)(\sqrt{k} \operatorname{sn}(\xi) + 1)^2 - 2(k-1)^2)}{(\sqrt{k} \operatorname{sn}(\xi) + 1)^2 (1 - \sqrt{k} \operatorname{sn}(\xi))}; u_{1.37.2} = -\frac{c}{\mu} +$$

$$\frac{4\alpha \mu \sqrt{k} \operatorname{cn}(\xi) \operatorname{dn}(\xi)}{1 - k \operatorname{sn}(\xi)^2}, v_{1.37.2} = 2\alpha^2 \mu^2 (k^2 + 6k + 1) - \frac{4\alpha^2 \mu^2 ((-1 - k^2 + 6k)(\sqrt{k} \operatorname{sn}(\xi) + 1)^2 - 2(k-1)^2)}{(\sqrt{k} \operatorname{sn}(\xi) + 1)^2 (1 - \sqrt{k} \operatorname{sn}(\xi))};$$

$$u_{1.37.3} = -\frac{c}{\mu} - \frac{4\alpha \mu k^3 \sqrt{k} \operatorname{cn}(\xi) \operatorname{dn}(\xi) \operatorname{sn}^2(\xi)}{k^3 \operatorname{sn}(\xi)^2 - (1 - \operatorname{dn}(\xi)^2)^2},$$

$$v_{1.37.3} = 2\alpha^2\mu^2(k^2 + 6k + 1) - \frac{16\alpha^2\mu^2k^7\text{cn}^2(\xi)\text{dn}^2(\xi)\text{sn}^4(\xi)}{(k^3\text{sn}(\xi)^2 - (1 - \text{dn}(\xi)^2)^2)} + \frac{4\alpha^2\mu^2((-1 - k^2 + 6k)(k\sqrt{k}\text{sn}(\xi) - \text{dn}(\xi)^2 + 1)^2 - 2(k-1)^2k^7\text{sn}(\xi)^2)k\sqrt{k}\text{sn}(\xi)}{(k\sqrt{k}\text{sn}(\xi) - \text{dn}(\xi)^2 + 1)(k^3\text{sn}(\xi)^2 - (1 - \text{dn}(\xi)^2)^2)};$$

$$u_{1.37.4} = -\frac{c}{\mu} + \frac{4\alpha\mu k^3\sqrt{k}\text{cn}(\xi)\text{dn}(\xi)\text{sn}^2(\xi)}{k^3\text{sn}(\xi)^2 - (1 - \text{dn}(\xi)^2)^2},$$

$$v_{1.37.4} = 2\alpha^2\mu^2(k^2 + 6k + 1) - \frac{4\alpha^2\mu^2((-1 - k^2 + 6k)(k\sqrt{k}\text{sn}(\xi) - \text{dn}(\xi)^2 + 1)^2 - 2(k-1)^2k^7\text{sn}(\xi)^2)k\sqrt{k}\text{sn}(\xi)}{(k\sqrt{k}\text{sn}(\xi) - \text{dn}(\xi)^2 + 1)(k^3\text{sn}(\xi)^2 - (1 - \text{dn}(\xi)^2)^2)};$$

$$1.38) e_0 = -\frac{e_1}{4}, e_1 < 0, e_2 = -e_1(2 - k^2), e_3 = 2e_1(1 - k^2), e_4 = -e_1(1 - k^2),$$

$$u_{1.38.1} = -\frac{c}{\mu} - \frac{4\mu\alpha\sqrt{-e_1}\text{dn}(\sqrt{-e_1}\xi)(1 + \text{cn}(\sqrt{-e_1}\xi))}{(1 + \text{cn}(\sqrt{-e_1}\xi))^2 - \text{sn}(\sqrt{-e_1}\xi)^2},$$

$$v_{1.38.1} = -\alpha^2\mu^2e_1(k^2 + 1) + \frac{16\alpha^2\mu^2e_1\text{dn}(\sqrt{-e_1}\xi)^2(\text{cn}(\sqrt{-e_1}\xi) + 1)^2}{((\text{cn}(\sqrt{-e_1}\xi) + 1)^2 - \text{sn}(\sqrt{-e_1}\xi)^2)^2} + \frac{4\alpha^2\mu^2e_1((k^2 - 2)(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)^2 - 2(1 - k^2)\text{sn}(\sqrt{-e_1}\xi)^2)\text{sn}(\sqrt{-e_1}\xi)^2}{(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)((\text{cn}(\sqrt{-e_1}\xi) + 1)^2 - \text{sn}(\sqrt{-e_1}\xi)^2)}; u_{1.38.2} = -\frac{c}{\mu} + \frac{4\mu\alpha\sqrt{-e_1}\text{dn}(\sqrt{-e_1}\xi)(1 + \text{cn}(\sqrt{-e_1}\xi))}{(1 + \text{cn}(\sqrt{-e_1}\xi))^2 - \text{sn}(\sqrt{-e_1}\xi)^2},$$

$$v_{1.38.2} = -\alpha^2\mu^2e_1(k^2 + 1) - \frac{4\alpha^2\mu^2e_1((k^2 - 2)(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)^2 - 2(1 - k^2)\text{sn}(\sqrt{-e_1}\xi)^2)\text{sn}(\sqrt{-e_1}\xi)^2}{(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)((\text{cn}(\sqrt{-e_1}\xi) + 1)^2 - \text{sn}(\sqrt{-e_1}\xi)^2)};$$

$$u_{1.38.3} = -\frac{c}{\mu} + \frac{4\mu\alpha\sqrt{-e_1}\text{dn}(\sqrt{-e_1}\xi)(\text{cn}(\sqrt{-e_1}\xi) + 1)}{\text{sn}(\sqrt{-e_1}\xi)^2 - (\text{cn}(\sqrt{-e_1}\xi) + 1)^2},$$

$$v_{1.38.3} = -\alpha^2\mu^2e_1(k^2 + 1) + \frac{16\alpha^2\mu^2e_1\text{dn}(\sqrt{-e_1}\xi)^2(\text{cn}(\sqrt{-e_1}\xi) + 1)^2}{(\text{sn}(\sqrt{-e_1}\xi)^2 - (\text{cn}(\sqrt{-e_1}\xi) + 1)^2)^2} + \frac{4\alpha^2\mu^2e_1((k^2 - 2)(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)^2 - 2(1 - k^2)(\text{cn}(\sqrt{-e_1}\xi) + 1)^2)(\text{cn}(\sqrt{-e_1}\xi) + 1)}{(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)((\text{cn}(\sqrt{-e_1}\xi) + 1)^2 - \text{sn}(\sqrt{-e_1}\xi)^2)};$$

$$u_{1.38.4} = -\frac{c}{\mu} - \frac{4\mu\alpha\sqrt{-e_1}\text{dn}(\sqrt{-e_1}\xi)(\text{cn}(\sqrt{-e_1}\xi) + 1)}{\text{sn}(\sqrt{-e_1}\xi)^2 - (\text{cn}(\sqrt{-e_1}\xi) + 1)^2},$$

$$v_{1.38.4} = -\alpha^2\mu^2e_1(k^2 + 1) - \frac{4\alpha^2\mu^2e_1((k^2 - 2)(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)^2 - 2(1 - k^2)(\text{cn}(\sqrt{-e_1}\xi) + 1)^2)(\text{cn}(\sqrt{-e_1}\xi) + 1)}{(\text{sn}(\sqrt{-e_1}\xi) + \text{cn}(\sqrt{-e_1}\xi) + 1)((\text{cn}(\sqrt{-e_1}\xi) + 1)^2 - \text{sn}(\sqrt{-e_1}\xi)^2)};$$

$$1.39) e_0 > 0, e_1 = -4e_0, e_2 = 4e_0(2 - k^2), e_3 = -8e_0(1 - k^2), e_4 = 4e_0(1 - k^2),$$

$$u_{1.39.1} = -\frac{c}{\mu} - \frac{8\mu\alpha\sqrt{e_0(1 - k^2)}(\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))}{\text{cn}(2\sqrt{e_0}\xi)^2 - (\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2},$$

$$v_{1.39.1} = -4\alpha^2\mu^2e_0(k^2 + 1) - \frac{2^9\alpha^2\mu^2e_0(1 - k^2)(\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2}{(\text{cn}(2\sqrt{e_0}\xi)^2 - (\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2)^2} + \frac{16\alpha^2\mu^2e_0(2 - k^2)((\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{cn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2 + \text{cn}(2\sqrt{e_0}\xi)^2)\text{cn}(2\sqrt{e_0}\xi)}{(\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{cn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2(-\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{cn}(2\sqrt{e_0}\xi) - \text{dn}(2\sqrt{e_0}\xi))}.$$

$$u_{1.39.2} = -\frac{c}{\mu} + \frac{8\mu\alpha\sqrt{e_0(1 - k^2)}(\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))}{\text{cn}(2\sqrt{e_0}\xi)^2 - (\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2},$$

$$v_{1.39.2} = -4\alpha^2\mu^2e_0(k^2 + 1) - \frac{16\alpha^2\mu^2e_0(2 - k^2)((\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{cn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2 + \text{cn}(2\sqrt{e_0}\xi)^2)\text{cn}(2\sqrt{e_0}\xi)}{(\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{cn}(2\sqrt{e_0}\xi) + \text{dn}(2\sqrt{e_0}\xi))^2(-\sqrt{1 - k^2}\text{sn}(2\sqrt{e_0}\xi) + \text{cn}(2\sqrt{e_0}\xi) - \text{dn}(2\sqrt{e_0}\xi))}.$$

$$1.40) e_0 = 0, e_1 = 0, e_2 > 0, e_3^2 = 4e_4e_2 \neq 0,$$

$$u_{1.40.1} = -\frac{c}{\mu} \pm \frac{8\alpha\mu e_4 e_2 \sqrt{e_2} \sec h^2\left(\frac{\sqrt{e_2}}{2}\xi\right) \left(e_3^2 \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right) + e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2\right)}{\left(e_3^2 - e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2\right) \left(e_3^2 - e_4 e_2 \left(3 \tanh^2\left(\frac{\sqrt{e_2}}{2}\xi\right) + 2 \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right) - 5\right)\right)},$$

$$v_{1.40.1} = \frac{\alpha^2 \mu^2 (3e_3^2 - 8e_4 e_2)}{4e_4} - \frac{32\alpha^2 \mu^2 e_4^3 e_3 \operatorname{sech}^4\left(\frac{\sqrt{e_2}}{2}\xi\right) \left(e_3^2 \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right) + e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2\right)^2}{\left(e_3^2 + e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2\right)^2 \left(e_3^2 - e_4 e_2 \left(3 \tanh^2\left(\frac{\sqrt{e_2}}{2}\xi\right) + 2 \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right) - 5\right)\right)^2}$$

$$- \frac{8\alpha^2 \mu^2 e_4 e_2^2 e_3 \left(e_3^2 - e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2 - 2e_4 e_2 e_3^2 \operatorname{sech}^4\left(\frac{\sqrt{e_2}}{2}\xi\right)\right) \operatorname{sech}^2\left(\frac{\sqrt{e_2}}{2}\xi\right)}{\left(e_3^2 - e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2\right)^2 \left(e_3^2 - e_4 e_2 \left(1 - \tanh\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)^2 - 2e_4 e_2 \operatorname{sech}^2\left(\frac{\sqrt{e_2}}{2}\xi\right)\right)}$$

where $k(0 < k < 1)$ denotes the modulus of the Jacobi elliptic function, $k_1 = \sqrt{1 - k^2}$, $D_1, D_2, D_3 (D_1 D_2 D_3 \neq 0, \mu$ and $c(\mu c \neq 0)$ are arbitrary constants, and $\xi = \mu x - ct$.

For case 2, we can obtain a lot of travelling wave solutions using the same process as that in case 1. For the sake of simplicity, only interesting ones are listed,

2.1) $e_1 = e_3 = 0, e_0 = k^2, e_2 = -(1 + k^2), e_4 = 1, \xi = \mu x + c_0 \mu t$
 $u_{2.1.1} = c_0 + \frac{2\alpha\mu}{k} \operatorname{sn}(\xi), v_{2.1.1} = \alpha^2 \mu^2 (1 + k^2) - \frac{2\alpha^2 \mu^2 (1 - k \operatorname{ds}(\xi) \operatorname{cs}(\xi))}{k^2 \operatorname{ns}(\xi)^2}; u_{2.1.2} = c_0 + \frac{2\alpha\mu}{k} \operatorname{cd}(\xi), v_{2.1.2} = \alpha^2 \mu^2 (1 + k^2) - \frac{2\alpha^2 \mu^2 (1 + k(1 - k^2) \operatorname{nc}(\xi) \operatorname{sc}(\xi))}{k^2 \operatorname{dc}(\xi)^2},$

2.2) $e_1 = e_3 = 0, e_0 = \frac{k^4}{4}, e_2 = \frac{1}{2}(k^2 - 2), e_4 = \frac{1}{4}, \xi = \mu x + c_0 \mu t$

$$u_{2.2.1} = c_0 + \frac{4\alpha\mu}{k^2 (\operatorname{ns}(\xi) \pm \operatorname{ds}(\xi))}, v_{2.2.1} = -\alpha^2 \mu^2 (k^2 - 2) - \frac{4\alpha^2 \mu^2 (2 - k^2 \operatorname{cs}(\xi) (\operatorname{ds}(\xi) \pm \operatorname{ns}(\xi)))}{k^4 (\operatorname{ns}(\xi) \pm \operatorname{ds}(\xi))^2},$$

2.3) $e_1 = e_3 = 0, e_0 = \frac{k^2}{4}, e_2 = \frac{(k^2 - 2)}{2}, e_4 = \frac{k^2}{4}, \xi = \mu x + c_0 \mu t$

$$u_{2.3.1} = c_0 + \frac{4\alpha\mu}{k (\operatorname{sn}(\xi) \pm i \operatorname{cn}(\xi))}, v_{2.3.1} = -\frac{1}{2} \alpha^2 \mu^2 (k^2 - 2) - \frac{4\alpha^2 \mu^2 (2 - i k \operatorname{dn}(\xi) (i \operatorname{cn}(\xi) \pm \operatorname{sn}(\xi)))}{k^2 (\operatorname{sn}(\xi) \pm i \operatorname{cn}(\xi))^2},$$

2.4) $e_0 = k, e_1 = -4k, e_2 = -1 + 6k - k^2, e_3 = 2(k - 1)^2, e_4 = -(k - 1)^2, \xi = \mu x + \mu(c_0 \pm 2\alpha\mu\sqrt{k})t$

$$u_{2.4.1} = c_0 + \frac{2\alpha\mu(\sqrt{k} \operatorname{sn}(\xi) + 1)}{\sqrt{k}}, v_{2.4.1} = -2\alpha^2 \mu^2 (k - 1)^2 + 4\alpha^2 \mu^2 k \pm 4\alpha^2 \mu^2 k \left(\sqrt{k} \operatorname{sn}(\xi) + 1\right) - \frac{2\alpha^2 \mu^2 ((\sqrt{k} \operatorname{sn}(\xi) + 1)^2 - k \operatorname{dn}(\xi) \operatorname{cd}(\xi))}{k},$$

2.5) $e_0 = -\frac{e_1}{4}, e_1 < 0, e_2 = -e_1(2 - k^2), e_3 = 2e_1(1 - k^2), e_4 = -e_1(1 - k^2), \xi = \mu x + \mu(c_0 \pm \alpha\mu\sqrt{-e_1})t$

$$u_{2.5.1} = c_0 + \frac{4\alpha\mu}{\sqrt{-e_1}} (\operatorname{ns}(\sqrt{-e_1}\xi) + \operatorname{cs}(\sqrt{-e_1}\xi) + 1),$$

$$v_{2.5.1} = -\alpha^2 \mu^2 e_1 k^2 - 14\alpha^2 \mu^2 e_1 \pm 4\alpha^2 \mu^2 (\operatorname{ns}(\sqrt{-e_1}\xi) + \operatorname{cs}(\sqrt{-e_1}\xi) + 1) + 4\alpha^2 \mu^2 e_1^{-2} \left(2 (\operatorname{ns}(\sqrt{-e_1}\xi) + \operatorname{cs}(\sqrt{-e_1}\xi) + 1)^2 - e_1 \operatorname{ds}(\sqrt{-e_1}\xi)^2 (1 + \operatorname{cn}(\sqrt{-e_1}\xi))^2\right);$$

$$u_{2.5.2} = c_0 + \frac{4\alpha\mu (\operatorname{sn}(\sqrt{-e_1}\xi) + \operatorname{cn}(\sqrt{-e_1}\xi) + 1)}{\sqrt{-e_1} (\operatorname{cn}(\sqrt{-e_1}\xi) + 1)},$$

$$v_{2.5.2} = -\alpha^2 \mu^2 e_1 k^2 - 14\alpha^2 \mu^2 e_1 \pm \frac{4\alpha^2 \mu^2 (\operatorname{sn}(\sqrt{-e_1}\xi) + \operatorname{cn}(\sqrt{-e_1}\xi) + 1)}{e_1 (\operatorname{cn}(\sqrt{-e_1}\xi) + 1)} + \frac{4\alpha^2 \mu^2 (2 (\operatorname{sn}(\sqrt{-e_1}\xi) + \operatorname{cn}(\sqrt{-e_1}\xi) + 1)^2 + e_1 \operatorname{dn}(\sqrt{-e_1}\xi)^2 (1 + \operatorname{cn}(\sqrt{-e_1}\xi))^2)}{e_1^2 \operatorname{dn}(\sqrt{-e_1}\xi)^2 (1 + \operatorname{cn}(\sqrt{-e_1}\xi))^2},$$

2.6) $e_0 = -\frac{e_1}{4} \neq 0, e_1 = -(\beta\xi)^2, e_2 = -(2e_1 + \beta^2), e_3 = 2(e_1 + \beta^2), e_4 = -(e_1 + \beta^2), \xi = \mu x + \mu(c_0 \pm \alpha\mu k\beta)t$

$$u_{2.6.1} = c_0 + \frac{4\alpha\mu}{\sqrt{-e_1}k} (k + \operatorname{ds}(\beta\xi) - \operatorname{ns}(\beta\xi)),$$

$$v_{2.6.1} = \frac{\alpha^2\mu^2(2e_1 + \beta^2) + \alpha^2\mu^2k^2\beta^2 \pm \frac{4\alpha^2\mu^2}{k}(k - \operatorname{ns}(\beta\xi) + \operatorname{ds}(\beta\xi)) - 4\alpha^2\mu^2(2(k\operatorname{sn}(\beta\xi) + \operatorname{dn}(\beta\xi) - 1)^2 + \sqrt{-e_1}k\beta(1 - \operatorname{cn}(\beta\xi)))}{-e_1k^2\operatorname{sn}(\beta\xi)^2},$$

$$u_{2.6.2} = c_0 + \frac{4\alpha\mu(k\operatorname{sn}(\beta\xi) - \operatorname{dn}(\beta\xi) - 1)}{\sqrt{-e_1}(1 + \operatorname{dn}(\beta\xi))},$$

$$v_{2.6.2} = \frac{\alpha^2\mu^2(2e_1 + \beta^2) + \alpha^2\mu^2k^2\beta^2 \pm \frac{4\alpha^2\mu^2(k\operatorname{sn}(\beta\xi) - \operatorname{dn}(\beta\xi) - 1)}{(1 + \operatorname{dn}(\beta\xi))} + 4\alpha^2\mu^2(2(k\operatorname{sn}(\beta\xi) - \operatorname{dn}(\beta\xi) - 1)^2 + \sqrt{-e_1}k^2\beta(1 + \operatorname{dn}(\beta\xi)))}{e_1(1 + \operatorname{dn}(\beta\xi))^2},$$

2.7) $e_0 = -\frac{e_2^2}{4e_4}, e_2 < 0, e_4 > 0, e_1 = e_3 = 0, \xi = \mu x + c_0\mu t$

$$u_{2.7.2} = c_0 - \frac{4\alpha\mu e_4}{e_2\sqrt{\frac{-e_2}{2}\tanh\sqrt{\frac{-e_2}{2}}(\xi)}}, v_{2.7.2} =$$

$$-\alpha^2\mu^2 e_2 + \frac{4\alpha^2\mu^2 e_4}{e_2^3} \left(4e_4 + e_2^2 \csc h\left(\sqrt{\frac{-e_2}{2}}\xi\right) \sec h\left(\sqrt{\frac{-e_2}{2}}\xi\right) \coth^2\sqrt{\frac{-e_2}{2}}(\xi) \right),$$

2.8) $e_0 = -\frac{e_2^2}{4e_4}, e_2 > 0, e_4 > 0, e_1 = e_3 = 0, \xi = \mu x + c_0\mu t$

where $k(0 < k < 1)$ expresses the modulus of the Jacobi elliptic function, $i^2 = -1$, c_0, μ, β and $c(\beta\mu c \neq 0)$ are arbitrary constants.

Remark 1. All the solutions to Eqs (1a) and (1b) obtained in this paper are checked by Maple17.

Remark 2. We also can suppose that $u(\xi) = c_0 + \frac{a_0 + a_1\phi'}{b_0 + b_1\phi}, v(\xi) = d_0 + \frac{f_{10} + f_{11}\phi'}{g_{10} + g_{11}\phi} + \left(\frac{f_{20} + f_{21}\phi'}{g_{20} + g_{21}\phi}\right)^2$, which are substituted into Eq. (10) instead of Eq. (12). The same solutions of Eqs (1a) and (1b) can be obtained.

Remark 3. We only consider one special case to Eqs (1a) and (1b). We are able to get more solutions to Eqs (1a) and (1b) for other conditions such as other positive integers of n, m by the same process as that in the above. The solutions for Eq. (7) are listed in literature [33].

4 Conclusions

Making full use of solutions of one nonlinear ODE, many varieties of explicit and exact travelling wave solutions for Eq. (1) are found, which include Jacobi elliptic function solutions and combined Jacobi elliptic function solutions, hyperbolic function solutions, triangular periodic solutions and rational solutions. As far as our information goes, some solutions to Eqs (1a) and (1b) obtained in this paper are not proposed in any literature existing until now. The above procedures can be extended to many other nonlinear PDEs.

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