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Constructing Artistic Surface Modeling Design Based on Nonlinear Over-limit Interpolation Equation

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Abstract

The digital and physical methods of establishing minimal curved surfaces are the basis for realizing the design of the minimal curved surface modeling structure. Based on this research background, the paper showed an artistic surface modeling method based on nonlinear over-limit difference equations. The article combines parameter optimization and 3D modeling methods to model the constructed surface modeling. The research found that the nonlinear out-of-limit difference equation proposed in the paper is more accurate than the standard fractional differential equation algorithm. For this reason, the method can be extended and applied to the design of artistic surface modeling.

Keywords: Nonlinear over-limit interpolation equation; artistic surface modeling; hash data interpolation; parameter optimization; modeling

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1 Introduction

So far, the main breakthroughs of Computer Aided Geometric Design (CAGD) are the Bessel surface patch, Cons surface patch theory, and B-spline method. These methods have become the core algorithms of much well-known computer-aided design (CAD) systems. These algorithms are used to construct surfaces with various shapes ranging from airplanes, cars, and hulls to toys, jewelry, etc., and have achieved significant results.

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However, the above methods are far from meeting the needs of various applications, from conceptual design to the final, the finished product process is still too long, expensive, and complicated [1]. Ignoring specific algorithms and considering abstractly, both the Non-Uniform Rational B-Splines (NURBS) method and the Bessel method extended Schoenberg's technology to two dimensions 50 years ago. These methods require that the type value points of the constructed surface are relatively evenly distributed and the topology is quadrilateral. However, in actual production, there will be various situations different from this.

(1) The type value points are obtained by calculation or measurement, and the data is not correct, which makes it challenging to construct a smooth surface. Therefore, we must perform data preprocessing or provide a method to control the shape of the character.

(2) In reverse engineering, the model value points come from physical measurement. It isn't easy to obtain a grid of model value points with uniform distribution and quadrilateral topology [2]. At this time, the hash point interpolation method can be used to construct the surface.

(3) The construction of a non-four-sided surface. To construct an irregular n -sided surface, an appropriate point can be selected on it. Then we divide it into n four-sided surface patches and then use the traditional method to construct. Of course, it can also use the technique proposed by Gregory.

(4) The structure of closed surface entities. At present, to construct a surface entity, we first need to build a single surface and then stitch it together according to the continuous condition. This process is complicated [3]. If we use free-form deformation technology to construct and modify surface entities easily, it will be helpful to make sculpture models.

(5) The construction and control of the entire surface. If we use traditional methods to construct the character, users cannot control the surface flexibly. However, the way of solving partial di

This article intends to explain the basic principles, potential advantages, and problems of hash point interpolation surface, Gregory surface patch, FFD (Free Form Deformation), and PDE (Partial Differential Equation) methods.

2 Nonlinear overrun hash point interpolation surface

Construct a smooth binary function S for the given hash data $(x_i, y_i, z_i), i = 1, 2, \dots, n$ and make $S(x_i, y_i) = z_i, i = 1, 2, \dots, n$. There are two methods for hash data interpolation: distance-weighted interpolation and triangular interpolation. The former is suitable for multivariate interpolation. Triangular interpolation is mainly used in geometric modeling. The construction process of hash data triangular interpolation surface patch is as follows

(1) Triangulation and optimization of hash data. It includes data preprocessing, primary triangulation, main triangulation, boundary clipping, and triangulation refinement.

(2) The structure of cubic curve network and triangular surface. First, estimate the average vector at each vertex, and then construct the tangent plane of each vertex according to the intermediate vector. Next, calculate the internal control points of each side close to the vertex on the tangent plane. Thus, a control polygon can be constructed relative to each side [4]. Therefore, a total of 9 control vertices with three boundaries can be obtained for each triangle, and the control vertices inside the triangle can be calculated according to the existing formula.

(3) Splice two adjacent triangular curved surfaces according to the given persistent condition. The triangular surface patch obtained by (2) can only achieve C^0 continuity when splicing. To achieve C^1 or G^1 continuity between two adjacent surface patches, it is necessary to ensure that the three triangle pairs shown in Figure 1 are all coplanar. The key is to calculate the internal control points (the circle in Figure 1) based on the cross-boundary guide vector.

(4) The division of triangles. When the three boundaries of the triangular surface patch need to be continuously spliced with the other three surface patches simultaneously, the calculation of the internal control points will be contradictory. To increase the degree of freedom of splicing, we need to subdivide the triangular surface patch into three sub-surface patches. We make each triangular surface patch contact only one sub-triangular

Fig. 1 Joining two adjacent triangular surface patches.

surface patch.

(5) Adjustment of the control vertices inside the sub-triangular surface patch [5]. The triangular surface patch is divided into three sub-surface patches. The control vertices in each sub-surface patch can be calculated according to the requirements of cross-border continuity. To ensure the original triangular surface patch is a continuous surface, we should modify the corresponding control vertices of each sub-triangular surface patch.

(6) Generate a straight hash point interpolation surface.

If the triangular surface is required to achieve C^2 or G^2 continuity, it should be upgraded and divided according to the differential method first, and then all the control vertices that need to be adjusted are calculated according to the continuity requirements. This method is more complex and computationally expensive.

The basic theory of hash point interpolation surface is relatively mature, but the research work is mainly carried out by mathematicians [6]. Therefore, if this method is practical in CAD, all aspects need to work together.

3 Gregory surface patch

Gregory uses the following expression to define a triangular surface patch (shown in Figure 2).

$$r(V) = \sum_{i=1}^3 \gamma_i(V) \{e_i(s_i) + \lambda_i t_i(s_i)\} \quad (1)$$

In the formula: $r(V)$ represents the coordinate of the center of gravity. λ_i represents the parameter of the curve.

$$\begin{aligned} s_1 &= \lambda_3 / (\lambda_2 + \lambda_3) \\ s_2 &= \lambda_1 / (\lambda_3 + \lambda_1) \\ s_3 &= \lambda_2 / (\lambda_1 + \lambda_2) \end{aligned} \quad (2)$$

$$\gamma_i(V) = \begin{cases} (1/\lambda_i)^2 / \sum_j (1/\lambda_j)^2 & \lambda_i \neq 0 \\ 1 & \lambda_j = 0 \end{cases} \quad (3)$$

$e_i(s_i)$ represents the boundary curve i , sorted in a counterclockwise direction. $t_i(s_i)$ represents the cross-boundary derivative of the boundary curve i , sorted in counterclockwise order. According to the same principle, the n -sided surface patch can be expressed as

Fig. 2 Triangular surface patch.

$$r(V) = \sum_{i=1}^3 \gamma_i(V) \{e_i(s_i) + \lambda_i d_i(s_i)\}, s_i \in [0, 1] \quad (4)$$

In the formula: $V = \{\lambda_i, i = 1, 2, \dots, n\}$, $s_i = \lambda_{i-1} / (\lambda_{i-1} + \lambda_{i+1})$. $\gamma_i(V) = (\prod_{j \neq i}^m \lambda_j)^2 / \{\sum_{k=1}^m (\prod_{j \neq k}^m \lambda_j)^2\}$ Represents the convex combination factor. v_i represents the expansion factor. $e_i(s_i)$ means the boundary curve i , which is sorted counterclockwise. $d_i(s_i)$ represents the cross-border slope of the boundary curve i , which is sorted in a counterclockwise direction [7]. The specific information is shown in Figure 3.

Fig. 3 Gregory patch method.

For irregular surfaces, the Gregory method is more intuitive to describe the character, and countless surface patches can be spliced into a single character. The advantages are apparent, but more practice is needed

4 Free-form deformation (FFD) of three-dimensional objects

Mathematically, FFD can be defined by the Bernstein polynomial of three variables of tensor product [8]. First, define a local coordinate system $X_0 - STU$ on the cuboid, as shown in Figure 4. Then the coordinate $X(s, t, u)$ of any point in the coordinate system is

Fig. 4 Free-form deformation of a three-dimensional object.

$$X = X_0 + sS + tT + uU \quad (5)$$

The parameter in the formula: s, t, u can be expressed as a vector

$$\begin{aligned} s &= \frac{T \times U (X - X_0)}{T \times US} \\ t &= \frac{U \times S (X - X_0)}{U \times ST} \\ u &= S \times T (X - X_0) S \times TU \end{aligned} \quad (6)$$

For any point in the cuboid, $0 < s < 1, 0 < t < 1, 0 < u < 1$ Set $(l+1)(m+1)(n+1)$ points in the cuboid body and on the surface, and generate $l+1, m+1$ and $n+1$ planes in the S, T , and U directions through these points. $l+1, m+1, n+1 = 3$. The intersections constitute the control vertex grid, and the coordinates of the grid points are

$$P_{i,j,k} = X_0 + \frac{i}{l}S + \frac{j}{m}T + \frac{k}{n}U, (i = 1, 2, \dots, l; j = 1, 2, \dots, m; k = 1, 2, \dots, n) \quad (7)$$

Move $P_{i,j,k}$ to deform the object. This indicator is to calculate the position X_{ffd} of any point after deformation. (s, t, u) should be calculated first, and then the vector value three-variable Bernstein polynomial should be calculated.

$$X_{ffd} = \sum_{i=0}^l \binom{l}{i} (1-s)^{l-i} S^i \left[\sum_{j=0}^m \binom{m}{j} (1-t)^{m-j} T^j \left[\sum_{k=0}^n \binom{n}{k} (1-u)^{n-k} U^k P_{i,j,k} \right] \right] \quad (8)$$

X_{ffd} is the Cartesian coordinate vector of the moving point. P_{ijk} is the coordinate vector of the control point. If required to splice two deformed bodies, it should meet a particular continuity requirement [9]. When the

control points at the splicing place of the two objects coincide, the C^0 continuity requirement can be met. The condition that two things meet the continuous first-order derivative at the joint is

$$\begin{aligned}\frac{X_1(1,t,u)}{\partial s} &= \frac{\partial X_2(0,t,u)}{\partial s} \\ \frac{X_1(1,t,u)}{\partial t} &= \frac{\partial X_2(0,t,u)}{\partial t} \\ \frac{X_1(1,t,u)}{\partial u} &= \frac{\partial X_2(0,t,u)}{\partial u}\end{aligned}\quad (9)$$

These conditions are a direct extension of the continuous requirements of curves and surfaces. The FFD method can also be used for local deformation. To achieve the constant local deformation G^k , we only need to ensure that the control points on the K sections adjacent to the joint surface remain unchanged. As shown in Figure 5, the thick line in the figure is the joint surface [10]. Thus, the deformation process is similar to making a sculpture model. In addition, under certain conditions, the object's volume can be guaranteed to remain unchanged before and after deformation.

Fig. 5 Local deformation of FFD method.

From an engineering point of view, the characteristic of FFD is that it can complete global or local deformation under continuous conditions. This method can construct surfaces that require aesthetic appearance, certain transition surfaces, and specific functional surfaces. This method is easy to use. In expression (9), a character is generated when a particular parameter is zero. The curve is generated when the two parameters are zero. Therefore, it can be used to express curves, surfaces and curved bodies uniformly. However, FFD also has its limitations. For example, it is challenging to generate a generally rounded transition surface; simultaneously, it requires a lot of calculation.

5 Using partial differential equations (PDE) to construct surfaces

From 1992 to 1993, the CAD laboratory of MIT completed the research on the construction of free-form surfaces by solving partial differential equations. The resulting surface, like an elastic film boundary, can be fixed according to any space curve. Various points on the curved surface can also be improved. The curved surface deforms like soap bubbles when subjected to positive or negative pressure. Similar to the physical film

surface equation, the weighted sum of area and curvature is minimized. Therefore, the surface generated by it appears smooth and attractive [11]. People can control the shape of the film in different ways: (1) Fix points on the surface or curve. For example, when constructing an engine intake pipe, one can force a curved surface to pass through various sections given along the ridgeline. (2) Apply positive or negative pressure to the curved surface to deform the curved surface. The pressure can be distributed uniformly or non-uniformly. (3) Change the weight of the surface area and curvature. Changing the weight function causes the membrane's stiffness to change, and the curved surface is deformed under a given pressure. To change the shape of the surface, the operator does not need to enter a specific pressure value or weighting factor. They can use the cursor on the graphical interface to adjust the pressure on the surface until a satisfactory surface is obtained.

Physics, mechanics, engineering technology, and other natural sciences have proposed many partial differential equations. An equation containing the partial derivative of an unknown function is called a partial differential equation. If there is more than one equation, these equations are called a partial differential equation system [12].

We suppose that the function u has the continuous partial derivatives of each order appearing in the partial differential equation in the region D . If u is substituted into this partial differential equation to make it an identity in the region D , then u is called the partial differential equation in Solution in area D . $u = u(x_1, x_2, \dots, x_n)$ is a curved surface in the $n + 1$ dimensional space $(u, x_1, x_2, \dots, x_n)$. We call it the integral surface of the partial differential equation [13]. In geometric modeling, the boundary value problem is concerned. The partial differential equation is a definite solution problem with only boundary value conditions but no initial conditions at the same time. It is also required to generate a smooth surface. Therefore, we should use elliptic partial differential equations. We need that a transition surface $z = f(x, y)$, z is constructed between two semicircles and one semicircle in the region $\{\Omega : 0 \leq x \leq 2; 0 \leq y \leq 2\}$ to meet the following boundary conditions:

$$\begin{cases} z(0, 0 \leq y \leq 2) = + (1 - (y - 1)^2)^{1/2} \\ z(2, 0 \leq y \leq 1) = + (1/4 - (y - 1/2)^2)^{1/2} \\ z(2, 1 \leq y \leq 2) = - (1/4 - (y - 3/2)^2)^{1/2} \\ z(0 \leq x \leq 2, 0) = 0 \\ z(0 \leq y \leq 2, 2) = 0 \end{cases} \quad (10)$$

The surface that meets this condition can be generated by the following formula:

$$\nabla_{x,y}^2 f = \left(\frac{\partial z}{\partial x^2} + \frac{\partial z}{\partial y^2} \right) f(x, y) = 0 \quad (11)$$

Due to the smoothness of the operator ∇^2 , the generated surface is also smooth. It has no local maximum and minimum [14]. To ensure the tangential continuity between the generated surface and the adjacent surface, we need to use the fourth-order elliptic partial differential equation

$$D^4 F = \left(\frac{\partial z}{\partial u^2} + \frac{\partial z}{\partial v^2} \right) \left(\frac{\partial z}{\partial u^2} + a^2 \frac{\partial z}{\partial v^2} \right) \quad (12)$$

Where a is the shape control parameter, when the curvature is required to be continuous, a sixth-order elliptic partial differential equation is required. The advantage of using partial differential equations to construct the surface is that the entire surface can be built, the control is flexible, and the operation is convenient. The problem is that the amount of calculation is large, and the current computer can not generate the required surface at a reasonable cost. On the other hand, the modeling method is relatively new, and the application experience is insufficient.

6 Conclusion

This article briefly describes four surface modeling methods. The PDE method is the latest, and the original literature has been about six years. On the other hand, the hash point interpolation surface has the most extended history, which has reached more than 20 years. On the other hand, these methods have not been widely promoted in CAD, caused by various reasons. On the one hand, the mature application of the new process itself takes time. On the other hand, because it is difficult to modify the mathematical model of the existing CAD art surface modeling, the developers are not interested in the improvement suggestions. This is one reason why the internationally renowned software still uses the traditional method to construct the surface. In this regard, the situation in China is different. Chinese software industry is still immature, so it is possible to apply new modeling methods to CAD art surface modeling based on research and practice. The user interface is essential for the existing CAD art surface modeling's potential functions and should be paid close attention.

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