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Nonlinear differential equations based on the B-S-M model in the pricing of derivatives in financial markets

Limin Tao[1], Liping Xu[2†], Hani Jamal Sulaimani[3]

[1] Xianda College of Economics and Humanities Shanghai International Studies University, Shanghai 200083, China

[2] Shanghai Branch, HNA Futures Company Limited, Shanghai 200122, China

[3] Department of Computer Information Systems, Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia

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Abstract

The pricing and hedging of financial derivatives have become one of the hot research issues in mathematical finance today. In the case of non-risk neutrality, this article uses the martingale method and probability measurement method to study the pricing method and hedging strategy of financial derivatives. This paper also further studies the hedging strategy of financial derivatives in the incomplete market based on the BSM model and converts the solution of this problem into solving a vector on the Hilbert space to its closure. The problem of space projection is to use projection theory to decompose financial derivatives under a given martingale measure. In the imperfect market, the vertical projection theory is used to obtain the approximate pricing method and hedging strategy of financial derivatives in which the underlying asset follows the martingale process; the projection theory is further expanded, and the pricing problem of financial derivatives under the mixed-asset portfolio is obtained. Approximate pricing of financial derivatives; in the discrete state, the hedging investment strategy of financial derivatives H in the imperfect market is found through the method of variance approximation.

Keywords: B-S-M model, nonlinear differential equation, financial market, financial derivatives
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1 Introduction

Financial derivatives are derived from basic financial assets and are a very important part of global financial innovation in the 1970s and 1980s. Financial derivatives have the functions of arbitrage, risk transfer and price

†Corresponding author.
Email address: lipingxu009@163.com

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discovery. Financial derivatives can be divided into four types of contracts: forwards, futures, options and swaps according to their own product forms [1,2]. If divided according to the relationship between the price of financial derivatives and the price of its underlying assets, financial derivatives can be divided into two categories: linear derivatives and non-linear derivatives: linear derivatives include forward contracts and futures contracts With swap contracts, there is a linear relationship between the price of such derivatives and the price of the underlying asset; nonlinear derivatives include options, structured derivative securities, and exotic derivative securities, and their prices are very close to the price of the underlying asset [3,4]. Complex nonlinear relationship. Because the structure of linear derivatives is relatively simple, this article will only briefly explain it. This article mainly studies the pricing and hedging of nonlinear derivatives, which mainly refers to the pricing and hedging of options.

2 The basic theory of financial derivatives pricing

2.1 Pricing of linear derivatives

Futures contracts and forward contracts have similar properties. The main difference between the two is that futures contracts are customised and issued by exchanges. They are standardised contracts and are generally settled centrally by settlement companies and have a unique settlement system; forward contracts are generally traded off-exchange by both parties to the transaction. Therefore, in many literature research records at home and abroad, it is believed that futures and forwards can use the same pricing model. When the forward and futures contracts of the same underlying asset have the same expiry date, their contract prices are also very similar [5,6]. Any financial product pricing model must have certain prerequisites, while the forward and futures contract pricing models include the following four basic assumptions: (1) There are no transaction costs, and the number of transactions can be subdivided indefinitely; (2) The tax rate for all trading profits is the same; (3) All traders borrow or lend funds at the same risk-free interest rate; (4) There is no risk-free arbitrage opportunity in the market.

The pricing includes the following three situations: The pricing formula of a forward contract in which the underlying asset does not pay income: $F = Se^{r(T-t)}$. The pricing formula of a forward contract for which a security pays a known cash return: $F = (S-I)e^{r(T-t)}$. The pricing formula of a forward contract for which a security pays a known dividend rate: $F = Se^{(r-q)(T-t)}$. Swap can be regarded as a combination of a series of forward contracts. Therefore, the pricing of the swap can be obtained through the combination of a series of forward contract prices, and the pricing method of the swap contract can be derived from the pricing of the forward contract.

2.2 The pricing of nonlinear derivatives in a non-risk-neutral sense

We assume that the price process $\{S_t : t \geq 0\}$ of risk-based assets (stocks) and the price process $\{P_t : t \geq 0\}$ of risk-free assets respectively satisfy:

$$dS_t = S_t(\mu(t)dt + \sigma(t)dB_t) \tag{1}$$

$$dP(t) = P(t)r(t)dt, P_T = 1 \tag{2}$$

Where B_t represents the standard Brownian motion on the complete probability space (Ω, F, P), and $r(t)$, $\mu(t)$, $\sigma(t)$, $\rho(t)$ is a function defined on $[0,\infty) \to R$, which can satisfy the following conditions:

$$\int_0^T \mu(t)dt < \infty \int_0^T r(t)dt < \infty \int_0^T \sigma^2 \text{tdt} < \infty \int_0^T \rho(t)dt < \infty \tag{3}$$

2.2.1 European options that do not pay intermediate dividends

Assuming that Eqs. (1) and (2) are satisfied and the underlying risk asset does not pay dividends within the validity period, then for European option $V_T = f(S_T)$, the price at time is $V_T = f(t, S_T)$. For European call option

$f(S_T) = (S_T - K)_+$, the price at time t is:

$$C(K,t) = e^{\int_t^T (\mu(s)-r(s))ds} S_t N(d_1) - Ke^{-\int_t^T r(s)ds} N(d_2) \tag{4}$$

For European put option $f(S_T) = (K - S_T)_+$, the price at time t is:

$$P(K,t) = Ke^{-\int_t^T \tau(s)ds} N(-d_2) - S_t e^{\int_t^T (\mu(s)-\tau(s))ds} N(-d_1) \tag{5}$$

At time t, the parity relationship between European call and put options can be expressed as:

$$C(K,t) + Ke^{-\int_t^T \tau(s)ds} = S_t + P(K,t) \tag{6}$$

Among them:

$$d_1 = \frac{\ln(\frac{S}{K}) + \int_t^T (\mu(s) + \frac{1}{2}\sigma^2(s))ds}{\sqrt{\int_t^T \sigma^2(s)ds}}$$

$$d_2 = d_1 - \sqrt{\int_t^T \sigma^2(s)ds}$$

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Among them, $C(K,t), P(K,t)$ is the pricing of European call and put options, and $N(\cdot)$ is the probability distribution function of the standard normal distribution. To facilitate the comparison of the difference between the equivalent martingale measure and the actual probability measure, we will prove formulas (4) and (5) from the perspective of the equivalent martingale measure of the stock price process and the actual probability measure, and formula (6) can be simulated [7, 8]. The proof of formula (4), from the nature of the martingale method, we can see that $f(t,S_t) = E^*[F(S_T)e^{-\int_t^T \tau(s)ds}|F_t]$ has equation $f(S_T) = (S_T - K)_+$, where $y = \left(\frac{\int \sigma(s)dBs}{\sqrt{\int_t^T \sigma^2(s)ds}}\right) \sim N(0,1)$ can be set and then expanded according to the expectation formula of the random variable function according to formula (1) to obtain:

$$\begin{aligned} f(t,S_t) &= E^*\left(e^{\int_t^T -\tau(s)ds}(S_t e^{\int_t^T (\mu(s)-\frac{\sigma^2(s)}{2})ds + \int_t^T \sigma(s)dB_s} - K)_+\right) \\ &= E^*\left[(e^{\int_t^T (\mu(s)-\tau(s))ds} S_t e^{\int_t^T \left(-\frac{\sigma^2(s)}{2}ds + y\sqrt{\int_t^T \sigma^2(s)ds}\right)} - Ke^{-\int_t^T \tau(s)ds}) 1_{y+d_2\geq 0}\right] \\ &= \int_{-d_2}^{+\infty} \left(e^{\int_t^T (\mu(s)-\tau(s))ds} S_t e^{\int_t^T (-\frac{\sigma^2(s)}{2})ds + y\sqrt{\int_t^T \sigma^2(s)ds}} - Ke^{\int_t^T \tau(s)ds}\right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &= \int_{-\infty}^{d_2} \left(e^{\int_t^T (\mu(s)-\tau(s))ds} S_t e^{\int_t^T (-\frac{\sigma^2(s)}{2})ds + y\sqrt{\int_t^T \sigma^2(s)ds}} - Ke^{\int_t^T \tau(s)ds}\right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \end{aligned} \tag{7}$$

Then the following formula is obtained by transforming the variable $z = y + \sqrt{\int_t^T \sigma^2(s)ds}$:

$$f(t,S_t) = e^{\int_t^T (\mu(s)-r(s))ds} S_t N(d_1) - Ke^{-\int_t^T r(s)ds} N(d_2) \tag{8}$$

Next, we use the actual probability measure to prove (5), which can be obtained by formula (1):

$$\ln\left(\frac{S_T}{S}\right) \sim N\left[\int_t^T (\mu(s) - \frac{1}{2}\sigma^2(s))ds, \int_t^T \sigma^2(s)ds\right] \tag{9}$$

Let $Y = \frac{S_T}{S}$, then its probability density function is:

$$P(Y) = \frac{1}{\sqrt{2\pi \int_t^T \sigma^2(s)dsY}} \exp\left(-\frac{\ln Y - \int_t^T \mu(s) - \frac{1}{2}\sigma^2(s)ds}{2\int_t^T \sigma^2(s)ds}\right) \tag{10}$$

$$E(P_T) = S\int_{-\infty}^{\infty} \max\left(\frac{K}{S} - Y, 0\right) P(Y)dY = -S\int_{\frac{K}{S}}^{\infty}\left(Y - \frac{K}{S}\right)P(Y)dY \tag{11}$$

$$P_T = \max(K - S_T, 0) = S\max\left(\frac{K}{S} - Y, 0\right) \tag{12}$$

And the European put option price at time t is:

$$P(K,t) = e^{-\int_t^T \tau(s)ds}E(P_T) = e^{-\int_t^T \tau(s)ds}\left[-S\int_{\frac{K}{S}} YP(Y)dY - K\int_{\frac{K}{S}}^{\infty} -P(Y)dY\right] = P_2 - P_1 \tag{13}$$

We divide $P(K,t)$ into two parts to calculate, among which $P(K,t) = P_2 - P_1$, let $V = \ln Y, Y = e^v$, then $dY = e^v dV$ can be obtained, thus:

$$P_1 = \frac{Se^{-\int_t^T \tau(s)ds}Y}{\sqrt{2\pi\int_t^T \sigma^2(s)dsY}}\int_{\frac{K}{S}}^{\infty} e^{-\frac{[V - \int_t^T (\mu(s) - \frac{1}{2}\sigma^2(s)ds]^2}{2\int_t^T \sigma^2(s)ds}} e^V dV \tag{14}$$

$$= \frac{Se^{-\int_t^T \tau(s)ds}Y}{\sqrt{2\pi\int_t^T \sigma^2(s)ds}}\int_{\frac{K}{S}}^{\infty} e^{-\frac{[V - \int_t^T (\mu(s) - \frac{1}{2}\sigma^2(s)ds]^2}{2\int_t^T \sigma^2(s)ds}} e^{\int_t^T \mu(s)ds}dV = \frac{Se^{\int_t^T (\mu(s)-\tau(s))ds}}{\sqrt{2\pi\int_t^T \sigma^2(s)ds}}\int_{\frac{K}{S}}^{\infty} e^{-\frac{[V - \int_t^T (\mu(s) - \frac{1}{2}\sigma^2(s)ds]^2}{2\int_t^T \sigma^2(s)ds}} dV$$

Take again: $l = \frac{\int_t^T (\mu(s) + \frac{1}{2}\sigma^2(s))ds - V}{\sqrt{\int_t^T \sigma^2(s)ds}}$, $dV = -\sqrt{\int_t^T \sigma^2 ds}dl$

Get:

$$P_1 = \frac{Se^{\int_t^T (\mu(s)-\tau(s))ds}}{\sqrt{2\Pi\int_t^T \sigma^2(s)ds}}\int_{d_1}^{-\infty} e^{-\frac{l^2}{2}}\left(\sqrt{\int_t^T \sigma^2(s)ds}\right)dl \tag{15}$$

$$= \frac{Se^{\int_t^T (\mu(s)-\tau(s))ds}}{\sqrt{2\Pi}}\int_{-\infty}^{d_1} -e^{-\frac{l^2}{2}}dl = S_t e^{\int_t^T (\mu(s)-\tau(s))ds}N(-d_1)$$

P_2 can be derived in the same way. Let the variable substitute: $Y = e^v, l = \frac{\int_t^T (\mu(s) - \frac{1}{2}\sigma^2(s))ds - V}{\int_t^T \sigma^2(s)ds}$

Get:

$$P_2 = \frac{Ke^{-\int_t^T \tau(s)ds}}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{l^2}{2}}(-1)dl = \frac{Ke^{-\int_t^T \tau(s)ds}}{\sqrt{2\pi}}\int_{d_2}^{-\infty} e^{-\frac{l^2}{2}}dl = Ke^{-\int_t^T \tau(s)ds}N(-d_2) \tag{16}$$

And because of $P(K,t) = P_2 - P_1$,

$$P(K,t) = Ke^{-\int_t^T \tau(s)ds}N(-d_2) - S_t e^{\int_t^T (\mu(s)-\tau(s))ds}N(-d_1) \tag{17}$$

2.2.2 European options with intermediate dividends

Assuming that the price process of the risk underlying asset stock $\{S_t : t \geq 0\}$ satisfies the formula (1), if dividends are issued at the instantaneous dividend rate $\rho(t)$ at time t, then at time t, the stock price S_t consists of the following two parts: one of which is risk-free [9, 10]. That is, the dividend that can be paid: $S_t(1-e^{-\int_t^T \rho(s)ds})$, the other part contains risk, that is, the present value of the stock value $S_t e^{-\int_t^T \rho(s)ds}$, so the present value of risky assets (stocks) in Theorem 3.1 is $S_t e^{-\int_t^T \rho(s)ds}$, and $S_t e^{-\int_t^T \rho(s)ds}$ can be used to replace S_t in Theorem 3.1 Get the following conclusions:

Assuming that $\{S_t : t \geq 0\}, \{P_t \geq 0\}$ satisfies Eqs. (1) and (2) and the risk asset pays dividends at $\rho(t)$ at time t during the effective period, the European call option pricing and put option pricing and the parity relationship between the two are:

The price of the European call option $f(S_T) = (S_T - K)_+$ at t is:

$$C(K,t) = e^{\int_t^T (\mu(s)-\tau(s)-\rho(s))ds} S_t N(d_1^*) - K e^{-\int_t^T \tau(s)ds} N(d_2^*) \tag{18}$$

European put option $f(S_T) = (K - S_T)_+$, the price at time t is:

$$P(K,t) = K e^{-\int_t^T \tau(s)ds} N(-d_2^*) - S_t e^{\int_t^T (\mu(s)-\tau(s)-\rho(s))ds} N(-d_1^*) \tag{19}$$

The parity relationship between European call options and put options is [4]

$$C(K,t) + K e^{-\int_t^T \tau(s)ds} = S_t e^{-\int_t^T \rho(s)ds} + P(K,t) \tag{20}$$

Among them:

$$\begin{aligned} d_1^* &= \frac{\ln(\frac{S}{K}) + \int_t^T (\mu(s) - \rho(s) + \frac{1}{2}\sigma^2(s))ds}{\sqrt{\int_t^T \sigma^2(s)ds}}, \\ d_2^* &= d_1^* - \sqrt{\int_t^T \sigma^2(s)ds} \end{aligned} \tag{21}$$

In the above formula, if $r(t), \mu(t), \sigma(t), \rho(t)$ is a constant and $r(t) = \mu(t)$ is, then the formula is the Black-Scholes formula under the condition of usual dividend payment.

2.3 Hedging of financial derivatives in a non-risk-neutral sense

The trading strategy $\{a(t), b(t)\}$ is called a self-financing strategy. If the wealth process satisfies:

$$V_t = a(t)S_t + b(t)P_t \tag{22}$$

For European option $V_T = f(S_T)$, to satisfy the hedging, the self-financing strategy $\{a(t), b(t)\}$ should satisfy the following formula:

$$a(t) = \frac{\partial f}{\partial x}(t, S_t), \quad b(t) = \frac{V_t - a(t)S_t}{P_t} \tag{23}$$

In the case of European call options that do not pay dividends, the self-financing parameters are:

$$a(t) = e^{\int_t^T (\mu(s)-r(s))ds} N(d_1), \quad b(t) = -K e^{-\int_0^T r(s)ds} N(d_2) \tag{24}$$

For European put options, the hedging self-financing parameters are:

$$a(t) = -e^{\int_t^T (\mu(s)-r(s))ds} N(-d_1), \quad b(t) = K e^{-\int_0^T r(s)ds} N(-d_2) \tag{25}$$

The hedging strategy of European call options under the condition of paying dividends is:

$$a(t)=e^{\int_t^T(\mu(s)-r(s)-\rho(s))ds}N(d_1^*),\quad b(t)=-Ke^{-\int_0^T r(s)ds}N(d_2^*) \tag{26}$$

In this case, the hedging strategy parameters for European put options are:

$$a(t)=-e^{\int_t^T(\mu(s)-r(s)-\rho(s))ds}N(-d_1^*),\quad b(t)=Ke^{-\int_0^T r(s)ds}N(-d_2^*) \tag{27}$$

Because $\tilde{V}_t=\tilde{f}(t,\tilde{S}_t)$ reflects the discounted value of V_t at time t, according to the derivation of Ito's formula:

$$\tilde{f}(t,\tilde{S}_t)=\tilde{f}(0,\tilde{S}_0)+\int_0^t\frac{\partial\tilde{f}}{\partial x}(u,\tilde{s}_u)d\tilde{s}_u+\int_0^t\frac{\partial\tilde{f}}{\partial x}(u,\tilde{s}_u)du+\int_0^t\frac{1}{2}\frac{\partial^2\tilde{f}}{\partial^2 x}(u,\tilde{s}_u)d\langle\tilde{s},\tilde{s}\rangle_u \tag{28}$$

Under the P^* measurement:

$$d\tilde{S}_t=\sigma(t)\tilde{S}_t dW_t,\quad d\langle\tilde{s},\tilde{s}\rangle_u=\sigma^2(t)\tilde{S}_u du \tag{29}$$

Without paying dividends, its Girsanov formula is:

$$W_t=B_t+\int_0^t\frac{u(s)-r(s)}{\sigma(s)}ds \tag{30}$$

In the case of paying dividends, the Girsanov formula is:

$$W_t=B_t+\int_0^t\frac{u(s)+\rho(s)-r(s)}{\sigma(s)}ds \tag{31}$$

So: $\tilde{f}(t,\tilde{S}_t)=\tilde{f}(0,\tilde{S}_0)+\int_0^t\sigma(t)\frac{\partial\tilde{f}}{\partial x}(u,\tilde{s}_u)\tilde{s}_u dW_u+\int_0^t K_u du$

And because $\tilde{f}(t,\tilde{S}_t)$ is a martingale measure under P^*, $K_u=0$ then:

$$\tilde{f}(t,\tilde{S}_t)=\tilde{f}(0,\tilde{S}_0)+\int_0^t\sigma(t)\frac{\partial\tilde{f}}{\partial x}(u,\tilde{s}_u)\tilde{s}_u dW_u=\tilde{f}(0,\tilde{S}_0)+\int_0^t\frac{\partial\tilde{f}}{\partial x}(u,\tilde{s}_u)\tilde{s}_u d\tilde{s}_u \tag{32}$$

Therefore:

$$a(t)=\frac{\partial\tilde{f}}{\partial x}(t,\tilde{s}_t)=\frac{\partial f}{\partial x}(t,S_t),\quad b(t)=\frac{V_t-a(t)S_t}{P_t} \tag{33}$$

3 Pricing and hedging of financial derivatives in an incomplete market

In an incomplete market, even if there is no arbitrage opportunity in the price system because there are multiple equivalent martingale measures, any financial derivative product may correspond to multiple pricing. For the pricing of financial derivatives to be unique, other pricing methods must be introduced on the condition of the principle of no-arbitrage pricing, resulting in a variety of different approximate pricing methods. Suppose there are $N+1$ assets in the market, of which N are risky assets, and there is another risk-free asset. It can be assumed that $S=(S_t)_{t\in(0,T)}$ is the price process of risky assets, and $S^0\equiv 1$ is risk-free assets. These are all defined in the probability space (Ω,F,P), which is adaptive to the σ-domain $F=(F_t)_{t\in[0,T]}$, and F_t represents the information that can be obtained as of t. A dynamic trading strategy for asset (S^0,S) can be expressed as $(\eta,\theta)=(\eta_t,\theta_t)_{t\in[0,T]}$, where η_t is adapted to $F=(F_t)_{t\in[0,T]}$, and θ_t is a predictable N-dimensional process of $F=(F_t)_{t\in[0,T]}$. For any time t, the value of the asset portfolio corresponding to the trading strategy (η_t,θ_t) is $V_t=\eta_t+\theta_t^T S_t$, and assuming that the cumulative income obtained through the transaction at time t is $G_t(\theta)=\int_0^t\theta_s^T dS_s$, the cumulative cost of the hedging strategy is $C_t=V_t-G_t(\theta)$.

From the definition of self-financing transaction, we know that when the cost process C of the hedging strategy is constant, this transaction strategy can be called self-financing. In an incomplete market, for the unreachable financial derivative asset H, there is no self-financing strategy (η,θ) that can make the value of the hedging strategy equal to the value of the financial derivative asset, that is, $V_T = H_T$ does not exist. In this situation, one solution is to maintain the condition $V_T = H_T$ for the trading strategy (η,θ) and give up the precondition that the trading strategy is self-financing. At this time, the cost process C is not a constant but a random process. The question that needs to be considered now is how to choose an optimal trading strategy that can be hedged, but for different optimal standards, the obtained financial derivatives pricing methods and hedging strategies may also be different. Fonmer et al. used the quadratic expectation as the optimal criterion to choose the trading strategy of financial derivatives hedging under the condition that S is the martingale process, so that the equation of $V_T = H_T$ is established, and $VaR(C)$ reaches the minimum at this time, so it can be called this The method is the minimum risk investment strategy. Later Schweizer extended Former's previous research conclusions to the general semi martingale situation. Another possible situation is that the transaction strategy is still required to be self-financing, so that it is impossible to find a self-financing transaction strategy for unreachable financial derivative assets $C+G_T(\theta) = H_T$ [5], but you can find a self-financing transaction strategy (C,θ) so that $C+G_T(\theta)$ is the most Close to H_T. Glorieux and Rheinlander independently obtained the above-mentioned mean-variance hedging problem investment strategy under the condition that the price of the underlying asset satisfies the condition of continuous semi martingale. In the imperfect market, this chapter introduces the use of projection theory conversion to solve the pricing problem of financial derivatives, and obtains the optimal risk hedging investment strategy of derivatives in which the price of the underlying asset follows the martingale process. At the same time, the projection theory is further expanded, the pricing and hedging strategy of financial derivatives based on the price information of the underlying assets are further studied, and the optimal mixed trading strategy and the approximate pricing method of the derivatives are obtained.

3.1 Basic definitions and assumptions

The investment strategy is called a self-financing strategy, and its value process $V = (V_t)_{t\in[0,T]}$ can be decomposed into the sum of a constant and a random integral about S:

$$V_t = x + \int_0^t \theta_s dS_s \tag{34}$$

The probability measure Q is the equivalent martingale measure of P on (Ω, F_t), if $Q \sim P$, $\frac{dQ}{dP} \in L^2(\Omega, F_t, P)$ and the (discounted) price process S is Q-martingale.

$M(P)_e = \{Q : Q \sim P, \frac{dQ}{dP} \in L^2(\Omega, F_t, P)\}$ represents the set of all martingale measures equivalent to P.

If there is no arbitrage opportunity in a given price system, then $M(P)_e \neq \emptyset$.

3.2 Approximate pricing and optimal hedging strategies for financial derivatives

The requirement of $G_T(\theta)$ to be integrable here is to ensure that $G_T(x,\Theta)$ is a closed set of $L^2(\Omega, F_t, P)$, and to prove that it is a convex set. Define the inner set $\langle x,y\rangle = E(xy); x,y \in L^2(\Omega, F_t, P)$ in $L^2(\Omega, F_t, P)$. The norm $||x|| = \sqrt{E[x^2]}$ can be obtained from the previous conditions, and it is proved that $L^2(\Omega, F_t, P)$ is a Hilbert space under this norm, where H is a closed subspace of $L^2(\Omega, F_t, P)$. Therefore, for any financial derivative product $G_T(x,\Theta)$ with an execution period of T, both can be represented by random variables $L^2(\Omega, F_t, P)$. At the same time, the projection of financial derivatives from H to $G_T(x,\Theta)$ can be regarded as the approximate pricing of H, that is, the approximate pricing of financial derivatives in an incomplete market can be transformed into the following projection problem for solution:

$$\min_{(x,\theta)\in G_T(x,\Theta)} E[H - (x + G_T(\theta))]^2 \tag{35}$$

The following describes the process of solving the problem: For a given price system (S_0, S), where S_0 is a risk-free asset, its price is always equal to 1, and S is a risky asset, and its price process is a continuous semimartingale process. For any $Q \in M(P)_e$, The financial derivative H with the execution period T can be uniquely decomposed into:

$$H = E^Q[H] + G_T(\theta^{Q,H}) + L_T^{Q,H}, a, s \tag{36}$$

Among them: $(L_T^{Q,H})_{0\leq t\leq T}$ is the square integrable martingale, and for any $t \in [0,T]$, there are $E^Q[L_T^{Q,H}, S_t] = 0$ and $E^Q[L_T^{Q,H}] = 0$; $(E^Q[H], \theta^{Q,H}) = 0$ is the self-financing strategy.

$$\min_{(x,\theta)\in G_{T(x,\Theta)}} E[(H-(x+G_T(\theta)))^2] \tag{37}$$

When $P \in M(P)_e$ is P itself is a martingale measure, it can be known from Lemma 4.2.1 that, let $a^{P,H} = E^Q[H] + G_T(\theta^{Q,H})$ then $H = a^{P,H} + L_T^{Q,H}$ for any $a \in G_T(x,\Theta)$:

$$E[(H-a)^2] = E[a^{P,H} + L_T^{Q,H} - a]^2 = E[a^{P,H} - a]^2 + E[(L_T^{Q,H})^2] \tag{38}$$

Obviously, when $a = a^{P,H}$, the minimum target value of the above optimisation problem can be obtained, and then:

$$\min_{(x,\theta)\in G_{T(x,\Theta)}} E[(H-(x+G_T(\theta)))^2] = E[(L_T^{Q,H})^2] \tag{39}$$

When $P \notin M(P)_e$, this situation needs to be transformed into situation 1 to solve the problem. The steps are as follows: First, the original price system needs to be converted and measured, so that the transformed price system is a martingale measure. The situation can be used when the above conditions are met. 1 method to answer [6].

When $H = 0, x = 1$, the optimisation problem (4.3) can be transformed into:

$$\min_{\theta\in\Theta} E[V_T^{1,\theta}]^2 \tag{40}$$

Among them: $V_T^{1,\theta} = 1 + G_T(\theta)$, due to the existing assumptions, $G_T(1,\Theta)$ is a convex closed set, so there is a $\tilde{\theta} \in \Theta$, so that $V_T^{1,\tilde{\theta}} = 1 + G_T(\tilde{\theta})$ is the minimum solution of the above optimisation problem, define the measure $\tilde{P}$:

$$\frac{d\tilde{P}}{dP} = \frac{V_T^{1,\theta}}{E[V_T^{1,\theta}]} \tag{41}$$

Then $\tilde{P}$ is the equivalent martingale measure of the original price system (S_0, S): $\tilde{P} \in M(P)_e$, and $\tilde{P}$ can make the variance $VaR(\frac{dQ}{dP})$ obtain the smallest equivalent martingale measure, which holds for all $Q \in M(P)_e$. Assuming that $V^{1,\tilde{\theta}}$ is a new asset added to the original asset set, the original price system becomes $(V^{1,\tilde{\theta}}, S^0, S)$, and then the price system is transformed with $V^{1,\tilde{\theta}}$ as the basic unit of money, then a contracted price system $(1, \frac{S^0}{V^{1,\tilde{\theta}}}, \frac{S}{V^{1,\tilde{\theta}}})$ can be obtained. Since the new asset $V^{1,\tilde{\theta}}$ can be completely copied from the original asset, the expanded asset collection and its price system will not change the financial derivative asset collection $G_T(x,\Theta)$. For any $g \in G_T(x,\Theta)$, there is a self-financing transaction strategy $(y, \theta^{S^0}, \theta^S)$, so that $g = V_T^{1,\theta}[y + G_T(\theta^{S^0}, \theta^S)]$, among them:

$$G_T(\theta^{S^0}, \theta^S) = \int_0^T \theta^{S\,T}_t d\left(\frac{S}{V^{1,\tilde{\theta}}}\right)_t + \int_0^T \left(\theta^{S^0}\right)_t^T d\left(\frac{S^0}{V^{1,\tilde{\theta}}}\right)_t \tag{42}$$

Define a new measure: $\frac{dW}{dP} = \frac{(V_T^{1,\tilde{\theta}})^2}{E[(V_T^{1,\tilde{\theta}})^2]}$

Then W is the equivalent martingale measure of the contracted price system $(1, \frac{S^0}{V^{1,\tilde{\theta}}}, \frac{S}{V^{1,\tilde{\theta}}})$, from which:

$$E[H-(x+G_T(\theta))]^2 = E[(V_T^{1,\tilde{\theta}})^2] \cdot E^W\left[\left(\frac{H}{V_T^{1,\tilde{\theta}}} - (y + G_T(\theta^{S^0}, \theta^S))\right)^2\right] \tag{43}$$

The conclusion drawn can be known that the initial cost of the optimal hedging of financial derivatives H at this time is:

$$E^W\left[\frac{H}{V_T^{1,\tilde{\theta}}}\right]=\frac{E^P\left[HV_T^{1,\tilde{\theta}}\right]}{E^P\left[(V_T^{1,\tilde{\theta}})^2\right]} \tag{44}$$

3.3 Application of variance approximation theory

In the discussion in the first two sections of this chapter, the approximate pricing of financial derivatives H is based on the existence of a hedging trading strategy. In the actual financial market, only hedging is known. The existence of a trading strategy is far from enough. You must also know the detailed hedging trading strategy so that you can better make an accurate hedging investment strategy according to the changes in the market. The following study uses the variance approximation theory in the discrete state. Find the optimal investment strategy β for the hedging of financial derivatives H in the incomplete market [7].

Assuming that Y and $X_1, X_2, \cdots X_n$ are second-order random variables in the probability space Ω, and assuming that X_i is linearly independent, find $X=\sum\beta_i X_i$ such that:

$$E\left|Y-\sum\beta_i X_i\right|^2=\min \tag{45}$$

Where $E(\cdot)$ represents the expectation, and the above X is called the variance approximation or mean square approximation of Y. The required $X=\sum\beta_i X_i$ is the best approximation of Y on subspace $A\hat{=}span\{X_1,X_2,X_n\}$. From the above conclusions:

$$\beta=(\beta_1,\beta_2,.....\beta_n)^T=G^{-1}(E(Y\bar{X}_1),E(Y\bar{X}_2),.....E(Y\bar{X}_n))^T \tag{46}$$

Among them $G=[E(X_j\bar{X}_i)]_{n\times n}$.

In the discrete case, all financial derivative investments $G(x,\Theta)$ that can be replicated can be fully hedged with n risk assets (stocks) and a risk-free asset (securities) in the market, then $G(x,\Theta)\hat{=}span\{1,S_T^1-S_0^1,...S_T^n-S_0^n\}, S_T^i, S_0^i$, respectively, means that the first stock is in The prices at time T and time 0, according to definition 4.4.1, can be derived, the mean square approximation of any financial derivative H can be expressed as [8]:

$$H=\beta_0+\sum_{i=0}^{n}\beta_i(\beta_T^i-\beta_0^i) \tag{47}$$

So, you can get

$$G=\begin{pmatrix}1 & 1 & ... & 1 & ... & 1\\ 1 & E(S_T^1-S_0^1) & ... & E(S_T^j-S_0^j) & ... & E(S_T^n-S_0^n)\\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & E(S_T^i-S_0^i) & ... & E(S_T^i-S_0^i)(S_T^j-S_0^j) & ... & E(S_T^i-S_0^i)(S_T^n-S_0^n)\\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & E(S_T^n-S_0^n) & ... & E(S_T^n-S_0^n)(S_T^j-S_0^j) & ... & E(S_T^n-S_0^n)(S_T^n-S_0^n)\end{pmatrix} \tag{48}$$

Find the elements in G, according to the martingale process of the discounted price of the stock, we can know:

$$E(S_T^i)=S_0^i e^{\int_0^T\mu(s)ds}, \tag{49}$$
$$(S_T^i)^2=(S_0^i)^2 e^{2\int_0^T\mu(s)ds-\int_0^T\sigma^2(s)ds+2\int_0^T\sigma(s)dB(s)}$$

So, you can get:

$$
\begin{aligned}
E(S_T^i) &= (S_0^i)^2 e^{2\int_0^T \mu(s)ds} \int_0^T \frac{1}{\sqrt{2\pi\int_0^T \sigma^2(s)ds}} e^{-\int_0^T \sigma^2(s)ds+2x} e^{-\frac{x^2}{2\int_0^T \sigma^2(s)ds}} dx \\
&= (S_0^i)^2 e^{2\int_0^T \mu(s)ds} \int_0^T \frac{1}{\sqrt{2\pi\int_0^T \sigma^2(s)ds}} e^{-\frac{x^2-4\int_0^T \sigma^2(s)ds\cdot x+4(\int_0^T \sigma^2(s)ds)^2-2(\int_0^T \sigma^2(s)ds)}{2(\int_0^T \sigma^2(s)ds)}} dx \\
&= (S_0^i)^2 e^{2\int_0^T \mu(s)ds+\int_0^T \sigma^2(s)ds)}
\end{aligned} \tag{50}
$$

Then:

$$
E[S_T^i - S_0^i]^2 = E(S_T^i)^2 - 2E(S_T^i S_0^i) + E(S_0^i)^2 = (S_0^i)^2 e^{2\int_0^T \mu(s)ds+\int_0^T \sigma^2(s)ds} - (S_0^i)^2 e^{\int_0^T \mu(s)ds} - (S_0^i)^2 \tag{51}
$$

You can also get: $E[S_T^i - S_0^i] = S_0^i e^{\int_0^T \mu(s)ds} - S_0^i$, and because S_i, S_j is independent of each other, then:

$$
E(S_T^i - S_0^i)(S_T^j - S_0^j) = E(S_T^i - S_0^i)E(S_T^j - S_0^j) \tag{52}
$$

The calculation of the elements in G has been completely solved. To obtain the mean square approximation of financial derivatives H, $E[H(S_T^i - S_0^i)]$ should also be calculated, because this formula will vary with financial derivatives H, and the results will also change [9]. Discussion of different types and situations:

When the underlying asset of the financial derivative H is one of the risky assets (the first type), then for the call option:

$$
H = (S_T^m - K)_+ \tag{53}
$$

$$
E[H(S_T^i - S_0^i)] = E[(S_T^m - K)_+(S_T^i - S_0^i)] = E[I_A(S_T^m - K)_+(S_T^i - S_0^i)] \tag{54}
$$

Among them:$A = \{w : S_T^m \geq K\}$

Put options: $H = (K - S_T^m)_+$

$$
E\left[H(S_T^i - S_0^i)\right] = E\left[(K - S_T^m)_+(S_T^i - S_0^i)\right] = E\left[I_A(K - S_T^m)_+(S_T^i - S_0^i)\right] \tag{55}
$$

Among them: $A = \{w : S_T^m \leq K\}$

From the calculations of G and $E[H(S_T^i - S_0^i)]$, it can be obtained that the hedging strategy of financial derivatives H in the incomplete market in the discrete state is β [10].

4 Conclusion

The paper introduces the pricing methods and hedging strategies of financial derivatives in the context of non-risk-neutral significance. Through the distribution of the option price process, using the equivalent martingale measure and the actual probability measure, without paying dividends, the generalised European option pricing formula is derived, and the parity between European call options and put options is also obtained. Relationship; then expand the pricing method of financial derivatives that do not pay dividends to financial derivatives that pay continuous dividends. Using Ito's formula again, the specific hedging strategies of European call options and put options are obtained under the conditions of paying dividends and not paying dividends. These results also include traditional European option pricing formulas and hedging strategies in the sense of risk-neutral pricing. The derivation of these conclusions and the choice of strategies have important practical economic significance, especially when the price of the underlying asset fluctuates drastically, and the expected return rate of the underlying asset differs greatly from the risk-free interest rate in the market, which has a wider application economy. value. These results are of great help to the further study of option pricing and hedging issues in the future.

We start by constructing subjective evaluation indicators and objective evaluation indicators based on game theory to balance personal empowerment and objectiveness. Then, the weight set is assigned to determine the complete optimal weight result. Finally, in response to the above theoretical results, suggestions for improving college students' creative ability are proposed: creating a relaxed learning environment for practicing innovative personality development, encouraging students to understand cutting-edge technologies in science and technology, condensing scientific research issues, and proposing innovative ideas and opinions; building a multi-level open development platform, provide a suitable environment for students to practice hands-on; strengthen the ability of teachers of innovation and entrepreneurship, and escort the cultivation and development of students' creative ability. In addition, attention should be paid to cultivating students' innovative ideological awareness and practical ability, strengthening the accumulation of knowledge and ability, and enhancing the transformation of results to implement each training link.

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