Applied Mathematics and Nonlinear Sciences

<https://www.sciendo.com>

Research on the control of quantitative economic management variables under the numerical method based on stochastic ordinary differential equations

Haibo Zhang^{1†}, Ali Alkhathlan², Radwan Kharabsheh³

¹ Jilin Institute of Chemical Technology, Jilin, 132022, China

² Department of Computer Science, Faculty of Computing and Information Technology, King Abdulaziz University, Jeddah, Saudi Arabia

³ Applied Science University, Al Eker, Kingdom of Bahrain

Submission Info

Communicated by Juan Luis García Guirao

Received June 17th 2021

Accepted September 24th 2021

Available online December 27th 2021

Abstract

This paper explores several differential equation models in the economic system and analyses the solution and stability of the differential equation models in order to better reflect the theoretical results in mathematics into reality. From a mathematical point of view, the analysis illustrates the important role of differential equation models in economic systems.

Keywords: differential equation, model, general solution, economy, price

AMS 2010 codes: 34A34

1 Introduction

Differential equation models are extremely effective mathematical methods for solving many practical problems. As an important branch of mathematics, differential equations have been developed over many years, and their solutions and qualitative theories have become more and more perfect. They can be used to obtain solutions (or numerical solutions) of differential equations and provide enough methods to make differential equation modelling very effective and very rich in internal functions [1]. In macroeconomics, people consider the changes of various economic parameters at a finite point in time or between infinite points in time, such as changes over time, how output, consumption levels, wage levels and capital stock levels change, to describe their own changes must resort to differential equations or difference equations.

[†]Corresponding author.

Email address: zhanghaibo1934@126.com

2 Theoretical analysis

2.1 Explanation of terms in economics

Sales promotion: It refers to a professional activity of industrial and commercial enterprises in the economic field to tap potential customers and promote the sale of goods. It refers to the process of a series of promotional methods and activities adopted by industrial and commercial enterprises in a certain business environment for their sales targets. Supply volume: It refers to the volume of commodities that an enterprise is willing to sell and available for sale per unit time under certain price conditions. It is recorded as S . Demand: It refers to the amount of goods that consumers want to buy and have the ability to pay per unit time under certain price conditions [2] and is denoted as D . Equilibrium price: It refers to the price when supply and demand in the market are equal. It is recorded as $\bar{p}$. Induced investment: It refers to the investment caused by the increase in output and is marked as I .

2.2 Related theorems in differential equations

Theorem 1. *If there is a singly connected region in G , the function $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$ does not change its sign and is not always equal to zero on any subdomain in D^* , then the system of equations*

$$\begin{cases} \frac{dx}{dt} = X(x, y) \\ \frac{dy}{dt} = Y(x, y) \end{cases} \quad (1)$$

There is no periodic solution in the domain D^* , let alone any limit cycle.

3 The application of differential equations in economic models

3.1 New product promotion model

3.1.1 Questions

Economists and sociologists have been paying attention to the speed of new products, hoping to establish a mathematical model to describe it and use it to guide production [3].

3.1.2 Model establishment

Suppose a new product is to be launched on the market and the number of new products sold at t is $X(t)$. Assuming that the product is easy to use and has excellent performance, these new products in use actually act as promotional materials, attracting the unpurchased. Assuming that each new product attracts K customers on average per unit time, $X(t)$ satisfies the differential equation:

$$\begin{cases} \frac{dX}{dt} = KX \\ X(0) = X_0 \end{cases} \quad (2)$$

Solution (2):

$$X(t) = X_0 e^{Kt} \quad (3)$$

If $t = 0$ is taken to represent the moment of the birth of a new product, then $X(0) = 0$ can be obtained from $X(t) \equiv 0$ (2). This result is inconsistent with the facts. Because, model (1) only considers the role of physical advertising and ignores the possibility that manufacturers can advertise new products through other means, thereby opening up the market. If the product of X_0 has been put into use through hard work, then the growth of $X(t)$ in the initial stage fits well with the result of Eq. (2). In Eq. (3), if we let $t \rightarrow \infty$, then $X(t) \rightarrow \infty$, which is also inconsistent with the facts. In fact, $X(t)$ should have an upper bound. Suppose the upper bound of demand

is M , the number of households that have not yet used new products is $M - X(t)$. According to the statistical law, $\frac{dX}{dt}$ is $X(M - X)$ proportional to and the proportional coefficient is r , then

$$\frac{dX}{dt} = rX(M - X) \quad (4)$$

Its solution is:

$$X(t) = \frac{M}{1 + ce^{-Mrt}} \quad (5)$$

The curve $X(t)$ becomes a growth curve or logistic curve (see Figure 1).

Fig. 1 Logistic curve

According to Eq. (5), its first and second derivatives can be obtained:

$$X'(t) = \frac{cM^2re^{-Mrt}}{1 + ce^{-Mrt}} \quad (6)$$

$$X''(t) = \frac{cM^3r^2e^{-Mrt}(ce^{-Mrt} - 1)}{(1 + ce^{-Mrt})^2} \quad (7)$$

It is not difficult to see that when $X'(t) > 0$, $X(t)$ increases monotonically, and $ce^{-Mrt} = 1$ is derived from $X''(t_0) = 0$, and $X(t_0) = \frac{M}{2}$ at this time. When $t < t_0$, $X''(t_0) > 0$, that is, $X'(t)$ increases monotonically, which means that when the sales volume is less than half of the maximum demand, the sales speed $X'(t)$ keeps increasing; when $t > t_0$, $X''(t_0) < 0$, $X'(t)$ monotonously decreases, which means that when the sales volume reaches half the demand, ($t = t_0$). The product was the most popular, followed by ($t > t_0$), and the sales rate began to decline. From this survey by many economic experts and product salespeople, the sales curve of many new products is quite close to the logistic curve. Therefore, it is possible to conduct an in-depth analysis of the sales curve characteristics of new products and conclude that new products should be produced in a small amount in the early stage of sales, and at the same time, publicity and advertising should be strengthened. When the new product consumers reach 20–80%, from time to time, new products can be produced in large quantities; when the consumers of new products exceed 80%, then the company should choose the right time to switch production in order to achieve better interests of the company [4].

3.2 The differential equation model of price

3.2.1 Linear differential equation model of supply, demand and price

Price p is the main factor affecting demand D and supply S . Hypothesis

$$\begin{aligned} D &= c - dp \\ S &= -a + bp \end{aligned} \quad (8)$$

Among them, p is the price, and a, b, c and d are normal numbers. The rate of price increase is proportional to the excess demand $D-S$, so there is a mathematical model of price:

$$\frac{dp}{dt} = \alpha(D - S) = \alpha(c - dp + a - bp) = \alpha(c + a - (b + d)p) \quad (9)$$

Among them, $k = \alpha(b + d)$, $h = \alpha(a + c)$. The general solution of Eq. (9) is

$$\begin{aligned} p(t) &= ce^{-kt} + \frac{h}{k} \\ \bar{p} &= \frac{a + c}{b + d} \end{aligned} \quad (10)$$

And when $D = S$, that is, $c - dp = -a + bp$, p is the equilibrium price, so $\bar{p}$ is the equilibrium price; so $p(t) = ce^{-kt} + \bar{p}$.

We see that although $p(t)$ fluctuates, when $t \rightarrow \infty$, $p(t)$ tends to the equilibrium price $\bar{p}$, and the market price at this time tends to stabilise. If supply and demand are constant, but $D > S$, then $\frac{dp}{dt} = \alpha(D - S)$,

$$\begin{aligned} p(t) &= ce^{\alpha(D-S)t} \\ \lim_{t \rightarrow +\infty} p(t) &= +\infty \end{aligned} \quad (11)$$

This is inflation. It is caused by a short supply. In order to stabilise prices, it is necessary to reduce consumption funds, reduce demand or increase the supply of goods. For example, reducing the number of government officials and redundant employees in enterprises can reduce consumption funds. Or the implementation of measures such as the promotion of commercial housing is to increase the supply of goods, which can play a role in restraining price increases [5].

3.2.2 The differential equation model to stabilise prices

Under a certain abnormal economic situation, consumers' shopping psychology is abnormal. The more the price increases, the more they buy, and the faster the price increases, the more they buy, so demand D . The relationship between supply S and the price $p(t)$ is:

$$\begin{cases} D = c_1 - d_1 p + c_2 \frac{dp}{dt} + d_2 \frac{d^2 p}{dt^2} \\ S = -a_1 + b_1 p + a_2 \frac{dp}{dt} + b_2 \frac{d^2 p}{dt^2} \end{cases} \quad (12)$$

$$\frac{dp}{dt} = \alpha(D - S) = \alpha \begin{bmatrix} (d_2 - b_2) \frac{d^2 p}{dt^2} \\ + (c_2 - a_2) \frac{dp}{dt} \\ - (d_1 + b_1) p \\ + (c_1 + a_1) \end{bmatrix} \quad (13)$$

Finished up:

$$\alpha(d_2 - b_2) \frac{d^2 p}{dt^2} + [\alpha(c_2 - a_2) - 1] \frac{dp}{dt} - \alpha(d_1 + b_1) p = -\alpha(c_1 + a_1) \quad (14)$$

Let $\frac{dp}{dt} = q$, then

$$\begin{cases} \frac{dp}{dt} = q \\ \frac{dq}{dt} = \frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)}q + \frac{d_1+b_1}{d_2-b_2}p - \frac{a_1+c_1}{d_2-b_2} \end{cases} \quad (15)$$

The singularity of Eq. (15) is $M(\frac{a_1+c_1}{d_1+b_1}, 0)$, and the characteristic equation is

$$\lambda^2 - \frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)}\lambda - \frac{d_1+b_1}{d_2-b_2} = 0 \quad (16)$$

$$\lambda_{1,2} = \frac{1}{2} \left[\frac{\frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)} \pm \sqrt{\left[\frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)} \right]^2 + \frac{4(d_1+b_1)}{d_2-b_2}}}{2} \right]$$

λ_1 and λ_2 are real numbers with different signs, and the singularity is a saddle point, as shown in Figure 2.

Fig. 2 The differential equation model to stabilise prices

It can be seen from the phase diagram that the right half-plane is divided into two areas by the ray AC, the trajectory starting from above AC, when $t \rightarrow \infty$, $p(t) \rightarrow \infty$. The trajectory starts from below AC, and when $t \rightarrow \infty$, $p(t) \rightarrow 0$ due to consumers psychological abnormalities and unstable prices. Taking the ray AC as the dividing line, when the price state is above AC, inflation will be triggered, and the government intervenes in the market and forcibly lowers prices, causing the price state to fall below the AC ray, and prices will continue to fall again the consequences of depression. At this time $c_2 < 0$, $d_2 < 0$, and it should be supplemented by publicity and guidance to correct the abnormal shopping psychology of consumers. When consumers return to their normal shopping mentality, they are unwilling to shop when prices are increased or accelerated. On the other hand, when prices are increased or accelerated, manufacturers believe that they are profitable and increase supply. At this time, $a_2 > 0$, $b_2 > 0$. So in the characteristic root expression

$$\frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)} < 0 \quad (17)$$

$$\frac{4(d_1+b_1)}{d_2-b_2} < 0$$

λ_1 and λ_2 are two negative real numbers or conjugate complex numbers with negative real parts, M is a stable node or focus, and

$$\frac{\partial}{\partial p}(q) + \frac{\partial}{\partial q} \left(\frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)}q + \frac{d_1+b_1}{d_2-b_2}p - \frac{a_1+c_1}{d_2-b_2} \right) = \frac{1-\alpha(c_2-a_2)}{\alpha(d_2-b_2)} < 0 \quad (18)$$

Therefore, it is known from Theorem 1 that there is no closed track, that is, prices will not oscillate periodically and can only tend to a stable value $\frac{a_1+c_1}{d_1+b_1}$ as time increases, resulting in a better market situation [6].

3.3 The differential equation model of economic adjustment

3.3.1 Questions

If the output value fluctuates too much over time, it will cause instability of economic life and social life, and economic adjustment must be made to make it develop in a relatively healthy manner [7].

3.3.2 Model establishment

Suppose $Y(t)$ is the output value, $I(t)$ is induced investment, A is a spontaneous investment, $C(t)$ is consumption, where $C(t) = cY(t)$ is, and suppose

$$\begin{aligned}\frac{dI}{dt} &= -k \left(I - v \frac{dY}{dt} \right) \\ \frac{dY}{dt} &= -\lambda (Y - C - I - A)\end{aligned}\quad (19)$$

Then

$$\frac{dY}{dt} = -\lambda Y + \lambda [(1-s)Y + I + A] \quad (20)$$

where $1-s=c$ is obtained from (13)

$$I = \frac{1}{\lambda} \frac{dY}{dt} + sY - A \quad (21)$$

Derivation from (20)

$$\frac{dI}{dt} = \frac{1}{\lambda} \frac{d^2Y}{dt^2} + s \frac{dY}{dt} \quad (22)$$

From (21)

$$\begin{aligned}\frac{1}{\lambda} \frac{d^2Y}{dt^2} + s \frac{dY}{dt} &= -k \left(\frac{1}{\lambda} \frac{dY}{dt} + sY - A \right) + kv \frac{dY}{dt} \\ \frac{d^2Y}{dt^2} + (\lambda s + k - k\lambda v) \frac{dY}{dt} + ks\lambda Y &= k\lambda A\end{aligned}\quad (23)$$

Converted into an equivalent system of equations

$$\begin{cases} \frac{dY}{dt} = u \\ \frac{du}{dt} = (k\lambda v - k - \lambda s)u - ks\lambda Y + k\lambda A \end{cases} \quad (24)$$

The singularity is $P_0 = (Y_0, u_0) = (\frac{A}{s}, 0)$. The characteristic equation is

$$\begin{aligned}\begin{vmatrix} 0-\mu & 1 \\ -ks\lambda & (k\lambda v - k - \lambda s) - \mu \end{vmatrix} &= 0 \\ \mu^2 - (k\lambda v - k - \lambda s)\mu + ks\lambda &= 0 \\ \mu_{1,2} &= \frac{1}{2} \left[\frac{(k\lambda v - k - \lambda s) \pm \sqrt{(k\lambda v - k - \lambda s)^2 - 4ks\lambda}}{\sqrt{(k\lambda v - k - \lambda s)^2 - 4ks\lambda}} \right]\end{aligned}\quad (25)$$

Remember $\Delta = (k\lambda v - k - \lambda s)^2 - 4ks\lambda$.

- A. If $k\lambda v - k - \lambda s = 0$, μ_1, μ_2 are pure imaginary numbers, then (Y_0, u_0) is the centre, as shown in Figure 3. At this time, $Y(t)$ and $\frac{dY}{dt}$ change periodically, and $Y(t)$ vibrates up and down $Y_1 = \frac{A}{s}$; from (25) we know this time

$$\frac{d^2Y}{dt^2} + ks\lambda Y = k\lambda A \quad (26)$$

The general solution of (26) is

$$Y(t) = c_1 \cos\left(\sqrt{ks\lambda}t + \phi\right) + \frac{A}{s} \quad (27)$$

c_1 and ϕ are determined by initial values, and the period of $Y(t)$ is $\frac{2\pi}{\sqrt{ks\lambda}}$, $k\lambda v - k - \lambda s = 0$. It is a critical situation.

Fig. 3 Differential equation model of economic adjustment

- B. If $k\lambda v - k - \lambda s > 0$, $\Delta > 0$, μ_1, μ_2 are all positive real numbers, and (Y_0, u_0) is an unstable node, as shown in Figure 4.

Fig. 4 Differential equation model of economic adjustment under unstable nodes

- C. If $k\lambda v - k - \lambda s > 0$, $\Delta = 0$, then $\mu_1 = \mu_2 > 0$, then (Y_0, u_0) is an unstable node, as shown in Figure 5.

Fig. 5 Schematic diagrams of the differential structure model of economic adjustment under unstable nodes

- D. If $k\lambda v - k - \lambda s > 0$, $\Delta < 0$, then μ_1, μ_2 is a conjugate complex number with positive real part, and (Y_0, u_0) is an unstable focus, as shown in Figure 6.

Fig. 6 Differential structure model of economic adjustment under unstable focus

- E. If $k\lambda v - k - \lambda s < 0$, $\Delta > 0$ then μ_1, μ_2 are all negative real numbers, and (Y_0, u_0) is a stable node, as shown in Figure 7.

Fig. 7 Differential structure model of economic adjustment under stable nodes

- F. If $k\lambda v - k - \lambda s < 0$, $\Delta = 0$, then $\mu_1 = \mu_2 < 0$, then (Y_0, u_0) is a stable node, see Figure 8.

Fig. 8 Model under stable node

- G. If $k\lambda v - k - \lambda s < 0$, $\Delta < 0$, then μ_1, μ_2 is a conjugate complex number with negative real part, and (Y_0, u_0) is a stable focus, as shown in Figure 9.

In order to be economically stable, $k\lambda v - k - \lambda s \leq 0$ must be increased, s must be increased and $s = 1 - c$

Fig. 9 Model in stable focus

must be decreased. c , c is the consumption coefficient, so it is necessary to compress consumption, reduce the proportion of consumption funds or decrease v , v , which is the investment coefficient, that is, control. The speed of inducing investment does not make the induced investment grow too fast due to the stimulation of increased production [8].

4 Summary

This article mainly makes a simple exploration of the application of differential equations in economic systems from the perspective of mathematics, concretely analyses specific problems in the economic system [9] and makes corresponding assumptions and simplifications for establishing appropriate differential equation models. The listed equations are solved according to the laws and methods in the field of differential equations. The results obtained were described and analysed, and finally reflected in economic reality [10].

Conflict of interest.

The authors declare no conflicts of interest.

References

- [1] Chow, S. M., Lu, Z., Sherwood, A., & Zhu, H. Fitting nonlinear ordinary differential equation models with random effects and unknown initial conditions using the stochastic approximation expectation–maximisation (saem) algorithm. *Psychometrika*, 2016.81(1):pp. 102-134.
- [2] S. F. Wu, J. Liu, L. Z. Liu. Modeling method of internet public information data mining based on probabilistic topic model, *The Journal of Supercomputing*, 2019.75(2):pp. 5882–5897.
- [3] Crooks, E. C. M., & Hilhorst, D. Self-similar fast-reaction limits for reaction-diffusion systems on unbounded domains. *Journal of Differential Equations*, 2016.261(3):pp. 2210-2250.
- [4] Liebovitch, L. S., Coleman, P. T., Bechhofer, A., Colon, C., & Wang, Q. Complexity analysis of sustainable peace: mathematical models and data science measurements. *New Journal of Physics*, 2019.21(7):pp. 73-78.
- [5] S. F. Wu, Q. Zhang, W. T. Chen, J. Liu, L. Z. Liiu, Research on trend prediction of internet user intention understanding and public intelligence mining based on fractional differential method, *Chaos, Solitons and Fractals*, 2019.128:pp. 331-338.
- [6] Dunsby, P. K. S., & Luongo, O. On the theory and applications of modern cosmography. *International Journal of Geometric Methods in Modern Physics*, 2016.13(03):pp. 702-702.
- [7] Plesa, T., Vejchodsky, Tomá, & Erban, R. Chemical reaction systems with a homoclinic bifurcation: an inverse problem. *Journal of Mathematical Chemistry*, 2016.54(10):pp. 1884-1915.
- [8] Damanik, D., Gorodetski, A., & Yessen, W. The fibonacci hamiltonian. *Inventiones mathematicae*, 2016.206(3):pp. 629-692.
- [9] Lijun Li, Yaru Wang, Xiaoyue Li, Tourists Forecast Lanzhou Based on the Baolan High-Speed Railway by the Arima Model, *Applied Mathematics and Nonlinear Sciences*, 2020.5(1):pp. 55-60.
- [10] Tao Li, Wenyin Yang, Solution to Chance Constrained Programming Problem in Swap Trailer Transport Organisation

based on Improved Simulated Annealing Algorithm., *Applied Mathematics and Nonlinear Sciences*, 2020.5(1):pp. 47-54.