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Study on the maximum value of flight distance based on the fractional differential equation for calculating the best path of shot put

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Abstract

From the perspective of physics, this article analyzes the various factors that affect the distance of shot put in the process of actual shot put. Based on the actual situation, it uses fractional calculus and physical formulas to calculate the shot throw distance and to conduct a comprehensive analysis. The experiment confirms that the three factors that affect the shot flying distance are the shot angle, shot speed, and shot height and analyzes the relationship between these three factors and the impact of each univariate on the shot flying distance. The analysis results show that among the three factors, the shot speed has the greatest impact on the flight distance, followed by the shot angle, and the shot height has the least effect; moreover, the optimal shot angle is not a specific value but a general range.

Keywords: fractional differential equation, shot put motion, optimal path, maximum flight distance**AMS 2010 codes:** 12H99

1 Introduction

Shot put in China is the main throwing item in track-and-field teaching and national physical exercise standards. It is one of the key physical education courses in China's sports colleges and universities. It is also an earlier sport that we encountered after entering school. It can exercise people's muscle's explosive power, improve the coordination in the movements of various parts of the body, so that the upper and lower limb muscles can develop symmetrically; improve the central nervous system's regulating function and response speed of action. The heavy shot equipment determines that this sport requires fast movements and high strength. From a physics perspective, shot motion can be regarded as oblique throwing. The flight distance is determined by three

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variables: shot speed, shot angle, and shot height. Therefore, it is necessary to increase the shot. The process of movement must start from increasing the shooting speed, as well as adopting suitable throwing angles and higher shooting heights [1]. At present, the general literature only considers the in-situ throwing when discussing the shot flying distance and does not discuss slipping. Shot put has an initial speed and air resistance, which is like the actual training game. The definition of each mechanical variable is ambiguous, and the mechanical relationship between the shot speed, the shot angle, and the shot height cannot be fully considered. At the same time, on consulting some university textbooks, the discussion of the parabola is only limited. In the ideal case, the initial velocity of the projectile is decomposed into its horizontal and vertical components for consideration. Such a discussion is obviously not comprehensive. In practice, the three are influential and interrelated. This article will comprehensively consider the various factors mentioned above. The effect specifically of the independent mechanical variables that affect the shot flying distance and the degree of influence of each independent mechanical variable on the flying distance of shot put are studied in detail [2].

2 Basic concepts of fractional differential equations

2.1 Proof of the existence of the solution (or zero of $f(x)$) of the equation $f(x) = 0$

Knowing that $f(x)$ is continuous on $[a, b]$ or (a, b) without specifying whether $f(x)$ is differentiable, it is generally proved by the zero-value theorem of continuous functions on closed intervals. Make an original function $F(x)$ of $f(x)$. Prove that $F(x)$ satisfies the conditions of Rolle's theorem and thus obtain the zero proof of $f(x)$.

2.2 The study of the uniqueness of the solution of the equation $f(x) = 0$

Let us look at the theorem that there is uniqueness. The theorem is as follows: if $f(x, y)$ is continuous on R and Lipschitz's condition is satisfied with respect to y , then the equation becomes

$$\frac{dy}{dx} = f(x, y). \quad (1)$$

There is a unique solution $y = \phi(x)$, defined on the interval $|x - x_0| \leq h$, continuous and satisfying the initial condition $\phi(x_0) = y_0$, here

$$h = \min \left(a, \frac{h}{M} \right), \quad M = \max_{(x,y) \in R} |f(x, y)|. \quad (2)$$

Use function monotonicity to prove that $f(x) = 0$ has at the most one real-number solution; we can also use contradiction to prove that $f(x) = 0$ has at the most one real-number solution. The following example illustrates the idea of uniqueness of the above [3].

Proof. Let function $f(x)$ be differentiable on the closed interval $[0, 1]$. For each x on $[0, 1]$, the value of function $f(x)$ is in the open interval $(0, 1)$, and $f'(x) \neq 1$, prove that: in $(0, 1)$, there is only one x such that $f(x) = x$.

Proof. Let $F(x) = f(x) - x$, by the title, know that $F(x)$ is continuous on $[0, 1]$.

Because $0 < f(x) < 1$, $F(0) = f(0) - 0 > 0$, $F(1) = f(1) - 1 < 0$, from the zero-value theorem of the continuous function on the closed interval, we know that at least one point x in $(0, 1)$, i.e., $F(x) = 0$.

Another method: use the proof method to prove that $F(x)$ has at the most one zero in $(0, 1)$. Otherwise, $x_1, x_2 \in (0, 1)$ and $x_1 < x_2$, so that $f(x_1) = x_1, f(x_2) = x_2$; according to the Lagrange median theorem, at least one $x \in (x_1, x_2) \subset (0, 1)$ exists, so that $f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 1$ and the conditions in the question $f'(x) \neq 1$ contradicts. In summary, there is only one Hx in $(0, 1)$, making $f(x) = x$.

Suppose there are $f''(x) \leq 0$, and $f(1) = 2f'(1) = -3$ everywhere on $[1, \infty]$. Prove that there is only one real solution to the equation in $(1, \infty)$.

Proof. $f(x)$ becomes a first-order Taylor expansion at $x = 1$, so

$$f(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(\varepsilon)(x-1)^2 = 2 - 3(x-1) + \frac{1}{2}f''(\varepsilon)(x-1)^2 \quad (3)$$

From the condition $f''(\varepsilon) < 0$ in the problem, $\frac{1}{2}f''(\varepsilon)(x-1)^2 \leq 0$; then, there is $f(x) \leq 2 - 3(x-1) = 5 - 3x$; we can see that when $x_0 > \frac{5}{3}$ is taken, $f(x_0) < 0$, and $f(1) = 2 > 0$; according to Rolle's theorem, $\exists \eta \in (1, \frac{5}{3})$. Let $f(\eta) = 0$ be the equation $f(x) = 0$. When $x > 1$, the equation has a real-number solution. There is $f''(x) \leq 0$ everywhere when the question $x \geq 1$ is set, so $f'(x)$ is monotonically decreasing. Therefore, when $x \geq 1$ is $f''(x) \leq f'(1) = -3$, we know that when $x \geq 1$ is $f'(x)$, the function is strictly monotonically decreasing, so $f(x) = 0$ has at the most one real-number solution. In summary, the equation $f(x) = 0$ has only one real solution in $(1, \infty)$ [4].

2.3 Discussion on the number of solutions of equation $f(x) = 0$

The general steps to discuss the number of equation roots are as follows: first, find the inflection point of $f(x)$ and the nonexistent point of $f'(x) = 0$ to divide the monotonically decreasing interval of $f(x)$; second, find the extreme value (or maximum value) between each monotone; Third, analyze the relative position of the extreme value (or the maximum value) and the x axis. The following example questions apply.

Discuss the solution of equation $a^x = bx$, ($a > 1$). Let $f(x) = a^x - bx$; then, $f'(x) = a^x \ln a - b$. When $b < 0$, $f'(x) > 0$, so we know that $f(x)$ is a monotonically increasing function, and $\lim_{x \rightarrow +\infty} f(x) = +\infty$, $\lim_{x \rightarrow -\infty} f(x) = -\infty$; from Rolle's theorem and monotonicity, we know that there is only one ε in $(-\infty, +\infty)$, so that $f(\varepsilon) = 0$, i.e., $a^\varepsilon - b\varepsilon = 0$, so $a^x = bx$;

When $b > 0$, let $f'(x) = 0$, i.e., $a^x \ln a - b = 0$, we have $x_0 = \log_a \frac{b}{\ln a} = \frac{\ln b - \ln a}{\ln a}$ (inflection point), and because $f''(x_0) = a^{x_0} (\ln a)^2 > 0$, x_0 is the only inflection point of $f(x)$, so

$$f(x_0) = a^{\log_a \frac{b}{\ln a}} - b \log_a \frac{b}{\ln a} = \frac{b}{\ln a} - b \frac{\ln b - \ln a}{\ln a} = \frac{b}{\ln a} \left[1 - \ln b + \ln a \right] = \frac{b}{\ln a} \ln \frac{e \ln a}{b}. \quad (4)$$

The above formula is the maximum value of $f(x)$ on $(-\infty, +\infty)$. Therefore, when $f(x_0) < 0$, $\frac{b}{\ln a} \ln \frac{e \ln a}{b} < 0$, then $\frac{e \ln a}{b} < 1$; so when $b > e \ln a$, the equation $f(x) = 0$ has two real solutions; when $f(x) > 0$ is $\frac{e \ln a}{b} > 1$, i.e., $0 < b < e \ln a$.

The equation has no real roots; when $f(x) = 0$ is $\frac{e \ln a}{b} = 1$, i.e., $b = e \ln a$, the equation has a unique real solution; when $b = 0$, the original equation can be reduced to $a^x = 0$, so the equation has no real solution.

Prove that equation $\ln^x = \frac{x}{e} - \int_0^\pi \sqrt{1 - \cos 2x} dx$ has only two different real solutions in $(0, \infty)$. Prove:

$$\int_0^\pi \sqrt{1 - \cos 2x} dx = \int_0^\pi \sqrt{2 \sin^2 x} dx = \sqrt{2} \int_0^\pi |\sin x| dx = \sqrt{2} \int_0^\pi \sin x dx = \sqrt{2} (-\cos x) \Big|_0^\pi = 2\sqrt{2} \quad (5)$$

make $F(x) = \ln^x + 2\sqrt{2} - \frac{x}{e}$; then $F'(x) = \frac{1}{x} - \frac{1}{e}$; make $F'(x) = 0$, then $x = e$. The following list judges the extreme points, as shown in Table 1.

Table 1 Judging extreme points

x	$(0, e)$	e	(e, ∞)
$F'(x)$	A positive number	0	Negative number
$F(x)$	Increasing	$F(e) = 2\sqrt{2}$ maximum	Decrement

From the table above, we know that $F(x)$ has at the most one zero point in $(0, e)$ and (e, ∞) , and because $x = e$ is the only inflection point of $F(x)$ on $(0, \infty)$, $F(e) = 2\sqrt{2}$ is $F(x)$ in $(0, \infty)$. So it is the maximum value of

$F(x)$ in $(0, \infty)$, and because $F(e) > 0$,

$$\begin{aligned}\lim_{x \rightarrow 0} F(x) &= \lim_{x \rightarrow 0} (\ln x) + 2\sqrt{2} - \frac{x}{e} = -\infty \\ \lim_{x \rightarrow +\infty} F(x) &= +\infty.\end{aligned}\quad (6)$$

It can be known that $F(x)$ has at least one zero in $(0, e)$ and (e, ∞) , so $F(x)$ has only one real solution in $(0, \infty)$, i.e., the equation $\ln x = \frac{x}{e} - \int_0^\pi \sqrt{1 - \cos 2x} dx$ is in $(0, \infty)$. There are only two different solutions.

3 Mechanical parameters affecting the flying distance of shot put

3.1 Flight distance

When researching the sports biomechanics of throwing shot technology, the shot is generally regarded as a mass point to study the mechanical variables of the time of shot release. From a kinematics perspective, since the shot is of high mass but small in volume, spherical and smooth, within a certain error range, the air resistance and the influence caused by the shot rotation can be ignored. The actual shot projectile motion is simplified to a mechanical model as shown in Figure 1.

Fig. 1 Shot parameters (shot speed) and trajectory

Discuss the oblique motion of the mass point. To clearly describe the motion of the shot, simplify Figure 1 and establish the coordinate system with O as the origin, horizontal axis as the X-axis and vertical axis as the Y-axis, as shown in Figure 2.

Let the mass of shot put be m and the shooting speed be v . The shot angle is α . The shot height is $h = h_1 + h_2$ (see Figure 1). The horizontal distance between the throwing circle and the shooting point OA is S_1 , and the horizontal distance between the shot shooting point and the landing point is S_2 ; then the horizontal distance of

the shot put throw is $L = S_1 + S_2$.

$$S_2 = v_0 t \cos \alpha \quad (7)$$

$$y = h + v_0 t \sin \alpha - 1/2 g t^2.$$

When the shot is put on the ground, $y = 0$, so $t = \frac{v_0 \sin \alpha \pm \sqrt{v_0^2 \sin^2 \alpha + 2gh}}{g}$. Since $t > 0$, the above formula takes the plus sign, and so,

$$S_2 = \frac{v_0 \cos \alpha \left(v_0 \sin \alpha + \sqrt{v_0^2 \sin^2 \alpha + 2gh} \right)}{g}. \quad (8)$$

The throw distance can be expressed as

$$L = S_1 + S_2 = S_1 + \frac{v_0^2 \sin 2\alpha}{2g} + \sqrt{\left(\frac{v_0^2 \sin 2\alpha}{2g} \right)^2 + \frac{2h v_0^2 \cos^2 \alpha}{g}}. \quad (9)$$

Because the horizontal distance between S_1 (i.e., OA) shot putter and the shot put point depends on the individual's throwing skills and his/her own physiological conditions (such as different arm lengths between people), the actual shot put distance is

$$L = \frac{v_0^2 \sin 2\alpha}{2g} + \sqrt{\left(\frac{v_0^2 \sin 2\alpha}{2g} \right)^2 + \frac{2h v_0^2 \cos^2 \alpha}{g}}. \quad (10)$$

Fig. 2 The coordinates of the shot put

Fig. 3 The coordinates of the shot motion when considering air resistance

The following considers the resistance of air from the perspective of theoretical mechanics to derive the expression of the shot flying distance.

Suppose that the resistance during the shot motion is $f = -bv$; decompose the motion into the two horizontal and vertical directions to get the following equations:

$$\begin{cases} m \frac{dv_x}{dt} = -bv_x \\ m \frac{dv_y}{dt} = -mg - bv_y. \end{cases} \quad (11)$$

Let $t = 0$, $v_x = v_{x0}$, $v_y = v_{y0}$ be an integral of Eq. (11) to get

$$\begin{cases} \frac{dv_x}{dt} = v_x = v_{x0} e^{-bt/m} \\ \frac{dv_y}{dt} = v_y = \left(\frac{mg}{b} + v_{y0} \right) e^{-bt/m} - \frac{mg}{b}. \end{cases} \quad (12)$$

Integrating again, $t = 0$, $y = 0$, $x = 0$.

So

$$\begin{cases} x = \frac{mv_{x0}}{b} (1 - e^{-bt/m}) \\ y = \left(\frac{m^2 g}{b^2} + \frac{mv_{y0}}{b} \right) (1 - e^{-bt/m}) - \frac{mgt}{b}. \end{cases} \quad (13)$$

So, we have

$$y = \left(\frac{mg}{bv_{x0}} + \frac{v_{y0}}{v_{x0}} \right) x - \frac{m^2 g}{b^2} \ln \left(\frac{mv_{x0}}{v_{x0} - bx} \right) \quad (14)$$

where, $\frac{bx}{mv_{x0}} \ll 1$; and so,

$$y = \frac{v_{y0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2 - \frac{bg}{3mv_{x0}^3}x^3. \quad (15)$$

The air drag coefficient of the shot throw is generally 0.15×10^{-3} . When the shot weight is 6 kg, the shot angle is 36° , the shot speed is 11 km/s, and the shot distance is 2 m, the flight distance is 14.044 m. Substituting the same data into Eq. (2), the flight distance is 14.044 m. The effect of the two parameters on the flight distance due to the presence of air resistance is 0.02 m, which is compared with the total flight distance of 14.04 m. The ratio is 0.001, and this ratio is relatively small; so, the effect of air resistance on the shot flying distance has been ignored in many documents. According to the distance expression of the shot flying movement, the research on shot putting movement often focuses on the independent factors such as shot speed, shot height, and shot angle. The following will first analyze the impact of each variable on the shot flying distance, ignoring air resistance, and then comprehensively analyze the relationship between these variables.

3.2 Shot speed

The release speed refers to the speed at which the shot is shot relative to the ground and is set to v . As shown in Figure 1, suppose the athlete's sliding step to obtain the horizontal initial velocity of the shot relative to the ground is v_0 . After the person exerts force, the shot's velocity relative to the person is v_1 . The shot speed of the shot is $v = v_0 + v_1$, which depends on the shot. The magnitude, direction, distance, and duration of the force can be expressed as

$$v = \frac{Ft}{m} = \sqrt{\frac{2Fs}{m}}. \quad (16)$$

In the above formula, v is the shooting speed, F is the force acting on the shot, t is the time of the force acting, s is the distance of the force acting, and m is the mass of the shot. The condition for the formula to be true is that F is constant and its size depends on the athlete's body preparation level and body position during the exertion stage; the effect of force is reflected by the action time and flight distance of the force [5, 6]. The mechanical characteristics of sports depend on the body's orientation and exercise technology. Possibility is limited by the throwing circle, so it becomes important to increase the force on the shot put. All changes in shot put technology revolve around increasing the force or increasing the time (force distance). The history of development and change also confirms this.

3.3 Shot angle

The shot angle refers to the angle between the shot's shot speed v and the horizontal direction. It is separated from Figure 1 to get the angle α in Figure 4. If v_0 , h , and α are independent of each other, i.e., v_0 , and h are not functions of α . FL is only a unitary function of α , then the best shot angle α can be calculated by the method of extreme value in univariate differentials, i.e., by finding the first derivative of the shot angle α of Eq. (8), when calculating the maximum value of the actual problem, which is

$$\frac{dL}{d\alpha} = 0 \quad (17)$$

which is

$$-\cos 2\alpha = \frac{2v_0^2 \sin 2\alpha \cos 2\alpha - 8gh \sin 2\alpha}{v_0 \sqrt{v_0^2 \sin^2 \alpha \cos^2 \alpha - 8h \cos^2 \alpha}}; \quad (18)$$

$$-2v_0 \cos 2\alpha = \frac{2 \sin \alpha \cos 2\alpha - 4gh \sin 2\alpha}{\sqrt{v_0^2 \sin^2 \alpha + 2gh}}. \quad (19)$$

And then, we get

$$2\sin^2 \alpha (v_0^2 + gh) = v_0^2. \quad (20)$$

Fig. 4 Schematic diagram of each speed vector

Then, the best shot angle can be found as

$$\alpha = \arcsin \frac{v_0}{\sqrt{2(v_0^2 + gh)}}. \quad (21)$$

First, we can see what range of hand angles are constrained according to the actual situation. Table 1 contains the data on world-class shot putters' shot results and shot angles [7, 8].

In traditional sports training theories and textbooks, the best angles for putting shots are set in the range of $38^\circ \sim 42^\circ$, and $<45^\circ$, because the shot point is higher than the landing point. From Table 1, we can see the shooting angle of world-class athletes is roughly in the range of $35^\circ \sim 42^\circ$. Considering the athletes' nervousness and other factors on the playing field, the author believes that the range basically meets the traditional definition. From Table 2 [9], we can see that the reasonable range points out that the hand angle conforms to the principles of human anatomy and mechanics conforms to the motion technology model of the optimal combination of various factors that determine the throwing distance. The shot angle is dynamic.

3.4 Shot height

The shot height refers to the height of the shot from the ground. It depends on the height and arm length of the athlete. As shown in Figure 1, a person's shoulder height is h_1 and arm length is l . For a certain athlete, they are all fixed values [10]. Let β be the angle between the direction of v_1 and the horizontal direction, and it is called the force angle. The shot height is

$$h = h_1 + h_2 = h_1 + l \sin \beta. \quad (22)$$

From Eqs. (8)–(15), it is known that the greater the angle of action, the smaller the shot speed [note $\beta \in (0, \pi/2)$], the greater is the shot height; the larger the shot speed, the smaller the shot angle, and the smaller is the angle of attack [11].

Table 2 Shot put results and shot angles of world-class shot putters

Name	Shot put results, m	Shot angle, °
Li Meisu	20.3	38.69
	20.95	37
	21.76	35.13
Huang Zhihong	20.76	37.75
	21.28	41.3
	21.52	36.9
Slupianek	21.28	41.3
	21.41	40
	22.45	36
Sui Xinmei	21.66	39
Lisovskaya	20.86	42.6
Bayer	21.02	34.1
Gunther	22.23	35.5
Timmerman	21.35	35.8

4 The mutual influence and relationship of various variables on the shot flying distance

The following text uses Eq. (2) to calculate the flight distance to analyze the mutual influence of the three parameters on the flight distance. The calculated specific data are shown in Table 3.

4.1 Relationship between flight distance, shot speed, and shot angle

It can be seen from Table 2 that when the fixed shot height is 2.05 m, the shot distance and shot speed are linearly related, while the shot angle and flight distance are obviously not linearly related. That is, as the shot speed increases, the shot flying distance will increase correspondingly but will not increase indefinitely. This is because the human physiological factors determine that the shot speed will have an extreme value. With the increase of the shot angle, the flight distance image is like a quadratic function image, i.e., the flight distance has a maximum value under the condition that the shot speed and the shot height are constant. Currently, the shot angle is often the best shot angle. Therefore, with a certain shot height, an athlete adjusts the shot angle to increase the shot speed as much as possible to increase the flight distance. At the same time, the effect of

Table 3 Combination modes of shot flying distance, shot speed, shot angle, and shot height

Speed of shot, m/s	Shot height, m	Shot angle, °					
		36	37	38	39	40	41
11	2	14.044	14.102	14.192	14.181	14.2	14.207
	2.05	14.094	14.15	14.237	14.226	14.245	14.25
12	2	16.33	16.409	16.54	16.521	16.554	16.571
	2.05	16.382	16.459	16.586	16.569	16.6	16.616
13	2	18.802	18.904	19.08	19.054	19.102	19.131
	2.05	18.855	18.956	19.129	19.103	19.15	19.177
14	2	21.461	21.589	21.781	21.781	21.845	21.887
	2.05	21.516	21.642	21.831	21.831	21.894	21.935
15	2	24.308	24.464	24.747	24.703	24.785	4.841
	2.05	24.364	24.519	24.797	24.754	24.835	24.89

(Because the influence of the shot angle on the shot flying motion is not the main factor, 1° is used as a unit of measurement when making this table.)

changing the shot angle on the flight distance is much smaller than the effect of changing the shot speed on the flight distance. At the same time, it is also difficult to grasp the optimal shot angle. Therefore, the shot speed is more important in affecting the shot flying distance.

4.2 Relationship between flight distance, shot height, and shot speed

It can be seen from Table 2 that when the shot height is constant, the effect of the shot speed on the flight distance is measured in meters; while when the shot speed is constant, the increase of the flight distance by changing the shot height is only a few tenths of a meter. In other words, the impact of the shooting height on the flying distance is small. If several athletes launch the shot at the same shooting speed and shooting angle, the athlete with a high shooting height will shoot a little longer, but it will not be too obvious.

4.3 Relationship between flight distance, shot height, and shot angle

At a certain speed, as the shot height increases, the flight distance is a positively correlated straight line, i.e., the higher the shot height, the farther the flight distance, but the inclination is relatively small; as the shot angle increases, the flight distance presents concavity and convexity, i.e., nonlinear correlation, which just reflects the best shot angle we often say, but the best shot angle is not a certain value but a general range; as the shot speed increases from top to bottom, the flight distance also gradually increases, and the graph gradually moves to the left. This verifies that with a certain shot height, the shot angle increases as the shot speed increases. To sum up, when the shooting speed is constant, the higher the shooting height, the smaller is the corresponding shooting angle; when the shooting height is constant, the higher the shooting speed, the larger is the corresponding shooting angle.

5 Conclusion

(1) The factors that affect the shot flying distance are shot angle, shot speed, and shot height, but the degree of influence varies. (2) The shooting speed has the greatest effect on the flight distance, so during training, the development of shooting speed is the primary goal; then the shooting angle, and the shooting height has the least impact. (3) The shot speed is linearly related to the flight distance, and the shot angle is nonlinearly related to the flight distance. (4) The best shot angle is not a fixed value, but a range.

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