Applied Mathematics and Nonlinear Sciences

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Linear fractional differential equations in bank resource allocation and financial risk management model

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Submission Info

Communicated by Juan Luis García Guirao

Received June 17th 2021

Accepted September 24th 2021

Available online December 30th 2021

Abstract

The advantage of the linear fractional differential equation for bank resource allocation and financial risk management is that it can test random fluctuations in different functional forms. Given this paper is modelling the asset allocation risk model for rural commercial banks, the linear fractional differential equation analysis method is used to make policy recommendations. The research results of this paper show that credit risk is significantly negatively correlated with the bank's resource allocation. The degree of negative correlation between different levels of credit risk and bank resource allocation is different. Appropriate liquidity risk can optimise the bank's resource allocation.

Keywords: linear relationship, fractional differential equation, bank resource allocation, financial risk management**AMS 2010 codes:** 34A08

1 Introduction

Most of the Chinese rural commercial banks are formed through the shareholding system reform of rural credit cooperatives. According to the opinion of the supervisory authority, the existing rural credit cooperatives will be restructured into rural commercial banks, and no new rural credit cooperatives will be formed after that [1]. Rural commercial banks are the main force that serves and support the development of 'agriculture, rural areas, and farmers.' Therefore, China should improve the financial service level of rural commercial banks as soon as possible, stick to its position and strengthen the governance. According to the National Statistical Yearbook data, the total agricultural output value of the western Chinese region in 2020 is 3,458.533 billion yuan, which accounts for 30% of the Chinese total agricultural output and 19% of the western region's annual GDP. The rural population is 182.27 million, which accounts for 13% of the Chinese total rural population and 48% of the total population in the western region. Compared with the central and eastern regions, rural commercial banks in western China are still relatively late in their restructuring and establishment. Their credit

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risk evaluation methods are relatively backward. This article has collected and sorted out the loan data of a rural commercial bank in Sichuan (from now on referred to as KK Bank) in western China from 2018 to 2020 and applied the linear fractional differential equation model for empirical analysis. This completes the evaluation of the credit risk of rural commercial banks in western China. The linear fractional differential equation model is proposed based on the value at risk (VaR) model. Because the linear fractional differential equation model is suitable for small- and medium-sized banks and the calculation result has a single digital expression, it has strong operability and feasibility. This model enriches the research on credit risk evaluation methods of rural commercial banks in western China [2]. This has important practical significance for studying the credit risk evaluation of regional rural commercial banks and promoting the stable and healthy development of rural commercial banks.

2 Analysis of the status quo of credit risk evaluation of rural commercial banks in western China

2.1 Analysis of the status quo of the credit process of rural commercial banks in the western region

This article examines the credit process of nine rural commercial banks in the western region. As shown in Figure 1, the entire credit process can be divided into three parts. That is, before, during and after the loan [3]. The pre-loan part is mainly divided into customer application, customer application acceptance, loan investigation and evaluation. The loan part includes loan review, loan review and approval, loan contract signing and loan issuance. The post-loan part mainly includes management after loan issuance, loan recovery and loan file management.

Fig. 1 The loan approval flow chart of nine rural commercial banks in the western region

2.1.1 Precautions before lending

Good preventive measures in the early stage are an effective way to control credit risk. This is a process of preliminary investigation of customers. As far as Sichuan KK Rural Commercial Bank is concerned, the preliminary investigation of customers is mainly to investigate customers' basic information by filling in questionnaires. This stage belongs to the stage of credit risk identification [4]. At this stage, the customer's basic information, financial status, non-financial status and guarantee capabilities will be investigated. And the preliminary collection, sorting and analysis of customer information. The investigation and evaluation of loan customers and loan review are the keys to whether a loan will be approved successfully. At this stage, customers with greater credit risk can be initially eliminated.

2.1.2 In-loan control

The mid-term loan period requires a further in-depth review of the information of the loan customer. At this stage, most rural commercial banks in western regions used household surveys in rural areas. Commercial banks will check whether the households provide false information by SMEs and their companies and review the information provided utilising on-site investigations. This method has higher labour costs. Based on the investigation and evaluation of loan customers, the bank will eliminate those who do not meet the standards in the preliminary review process [5]. For customers who have passed the preliminary review, their qualifications will be checked again. Banks conduct on-site investigations to determine credit ratings of loan customers and finally form a survey report for review and approval. This stage is also called rating and measuring credit risk.

2.1.3 Post-loan management

Monitoring and management of after loan issuance are also crucial. It mainly includes four stages: the first follow-up inspection, regular and irregular site visits by the account manager, deposit and loan by the archivist, and risk analysis report to the risk department. Among them, account managers' regular and irregular on-site surveys are an important part of the credit risk control of rural commercial banks [6]. After the bank grants a loan to a customer, it needs to continue supervising and investigating to understand or find out whether the customer is speculating. For example, whether the customer transfers the loan for other purposes and whether it is consistent with the use agreed in the loan contract.

2.2 Credit risk evaluation methods of rural commercial banks in the western region

Compared with large commercial banks, rural commercial banks in the western region started late, and credit risk evaluation and management are also relatively backward. Coupled with the shortage of historical data, it will become extremely difficult for credit risk evaluation models to quantify data [7]. The nine rural commercial banks in the western region often adopt subjective judgements and adopt a simplified model to increase assumptions when measuring credit risk. Currently, the credit risk evaluation models of rural commercial banks in western China mainly include expert analysis methods, loan rating methods, financial analysis methods and credit scoring methods.

After combining multiple analysis methods, it is found that only efficient and scientific credit scoring methods can effectively assess credit risk in the market environment of financial innovation reform. Therefore, the evaluation and management of credit risk of rural commercial banks in western China need to adopt modern credit risk evaluation methods [8]. The modern risk evaluation model calculates the loaner's default probability, standard deviation, default loss rate, and distribution and then uses.

3 Linear fractional differential equation model framework

3.1 Linear fractional differential equation

In this part, we suggest some methods for solving linear fractional differential equations.

$${}_t^c D_{0+}^a f(t) + af(t) = 0, \quad t > 0$$

$${}_t^c D_{0+}^a |_{t=0} = C$$

The article uses Laplace inverse transformation to get $f(t) = Ct^{-a}E_{a,a}(-a\sqrt{t})$,

$$F(p) = \frac{C}{p^{\frac{1}{2}+a}}, \quad f(t) = Ct^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})$$

$$C = {}_t^c D_{0+}^{-\frac{1}{2}} f(t) |_{t=0}$$

$${}_t^c D_{0+}^Q f(t) + {}_t^c D_{0+}^q f(t) = h(t), \quad 0 < q < Q < 1 \text{ uses Laplace transform to get } \begin{aligned} (p^Q + p^q)F(p) &= C + H(p) \\ C &= \left({}_t^c D_{0+}^{q-1} f(t) + {}_t^c D_{0+}^{Q-q} f(t) \right) |_{t=0}. \end{aligned}$$

Among them, $F(p) = \frac{C+H(p)}{p^{Q+p^q}} = \frac{C+H(p)}{p^q(p^{Q-q+1})} = (C+H(p)) \frac{p^{-q}}{p^{Q-q+1}}$. Let $a = Q - q$, $\beta = Q$, we use Laplace inverse transformation to get

$$\begin{aligned} f(t) &= CG(t) + \int_0^t G|(t-\tau)h(\tau)d\tau \\ C &= \left({}^c D_{0+}^{q-1} f(t) + {}^c D_{0+}^{Q-1} f(t) \right) |_{t=0} \\ G(t) &= t^{Q-1} E_{Q-q, Q}(-t^{Q-q}) \\ {}^c D_{0+}^Q y(t) - \lambda y(t) &= h(t), \quad t > 0 \\ {}^c D_{0+}^{a-k} y(t) |_{t=0} &= b_k, \quad k = 1, 2, 3, \dots, n \end{aligned}$$

Using Laplace transformation, we can get:

$$p^a Y(p) - \lambda Y(p) = H(p) + \sum_{k=1}^n b_k p^{k-1}$$

$$Y(p) = \frac{H(p)}{p^a - \lambda} + \sum_{k=1}^n b_k \frac{p^{k-1}}{p^a - \lambda}.$$

$$y(t) = \sum_{k=1}^n b_k t^{a-k} E_{a, a-k+1}(\lambda t^a) + \int_0^t (t-\tau)^{a-1} E_{a, a}(\lambda(t-\tau)^a) h(\tau) d\tau.$$

3.2 Basic assumptions

First, it is random whether each loan in the loan portfolio defaults. Second, the probability of default for each loan is very small [9]. The third is that the probability of default between each loan is independent of each other. The occurrence of default events obeys the Poisson distribution $P_i = \lambda e^{-\lambda} / i!$ (i indicates that i debtors defaulted and λ is the average value of the expected default rate).

3.3 Calculation process

The calculation process of the linear fractional differential equation model can be divided into three steps:

The first step is the classification of risk exposure frequency bands. Assuming that a loan portfolio has a total of N loans, the risk exposure scale of the largest loan in this N loan divided by the risk exposure frequency band value L is the risk exposure frequency band classification. We denote the risk exposure frequency band thus divided as $V_1, V_2, V_3, \dots, V_n$ ($n = 1, 2, 3, \dots$). There is a total of n frequency bands, and the amount of risk exposure frequency band corresponding to each frequency band is $L_1, L_2, L_3, \dots, L_n$ and $L_n = L \times n$. Then the risk exposure of each loan is calculated and divided into various frequency bands. Among them, risk exposure = the scale of the lender's risk exposure, default rate and default loss rate.

The second step is the calculation of the default probability of N loan portfolio. After the first frequency band division, we divided N loan into n frequency bands [10]. If the number of loans in each frequency band is N_i , then $N_1, N_2, N_3, \dots, N_n = N$. If the number of default loans in N loan is k , then p is the average default probability. Because the model only considers the two states of default and non-default, and the occurrence of each event is independent of each other, we believe that k obeys the Bernoulli distribution. $K \sim B(N, p)$, then:

$$P(k) = C_N^k p^k (1-p)^{N-k} \quad (1)$$

From mathematical knowledge, it can be known that when N is very large and p is very small, the Bernoulli distribution approximately obeys the Poisson distribution. From this, we can see that we set μ as the expected number of defaults in 1 year, i.e., $\mu = N \times p$. When the debtor's default probability is very small and the number of defaults in the loan portfolio is n , we can get the probability of n default events occurring within a year as:

$$P(n) = \frac{\mu^n e^{-\mu}}{n!} \quad (2)$$

Suppose the probability occurrence function is $F(z)$ and is defined by an auxiliary variable z :

$$F(z) = \sum_{n=0}^{\infty} P(n)z^n \quad (3)$$

So, by formulas (2) and (3), we can wait until the default probability function of a loan portfolio within 1 year is:

$$F(z) = e^{\mu(z-1)} \quad (4)$$

Step 3: Calculation of the loss distribution of N loan portfolio. The expected default loss for each frequency band is:

$$\varphi_j = \varepsilon_j \times \mu_j \quad (1 \leq j \leq m) \quad (5)$$

where ε_j represents the average value of risk exposure in the V_j band. μ_j represents the number of defaults in the V_j band. μ represents the number of defaults expected to occur within a year, namely:

$$\mu = \mu_1 + \mu_2 + \cdots + \mu_j + \cdots + \mu_m \quad (6)$$

From formulas (5) and (6), we get:

$$\mu = \sum_{j=0}^m \mu_j = \sum_{j=1}^m \frac{\varphi_j}{\varepsilon_j} \quad (7)$$

Then turn the number of default events into a loss distribution. This requires the introduction of the probability function $G(z)$ for the occurrence of loss. We use this function to derive the default loss distribution function. The loss is equal to the multiple of the risk exposure, that is, the loss $= L_n = L \times n$. Then we can set the default loss distribution function as:

$$G(z) = \sum_{n=0}^{\infty} P(L_n)z^n \quad (8)$$

Because the various risk exposures in the portfolio are independent of each other, the formula for calculating the probability of the loan portfolio distribution is:

$$G(Z) = \prod_{j=1}^m \exp \{-u_j + u_j Z^{\varepsilon_j}\} = \exp \left(\sum_{j=1}^m u_j + \sum_{j=1}^m u_j Z^{\varepsilon_j} \right) \quad (9)$$

We define the expression $P(z)$ as follows:

$$P(z) = \frac{\sum_{j=1}^m u_j Z^{\varepsilon_j}}{\mu} = \frac{\sum_{j=1}^m \left(\frac{\varphi_j}{\varepsilon_j} \right) Z^{\varepsilon_j}}{\sum_{j=1}^m \left(\frac{\varphi_j}{\varepsilon_j} \right)} \quad (10)$$

From formula (7) and formula (10), we can simplify formula (9) as

$$G(Z) = e^{\mu(P(z)-1)} = F(P(z)) \quad (11)$$

The formula for calculating the probability of portfolio loss is:

$$P(L_i) = \frac{1}{n!} \times \frac{d^n G(z)}{dz^n} \Big|_{z=0} = A_n \quad (12)$$

Which represents the probability that the loss is $(n \times L)$. We expand formula (12) with Taylor's formula to get:

$$A_n = \sum_{j: \varepsilon_j \leq n} \frac{\varphi_j}{n} A_{n-\varepsilon_j}, \quad n = 1, 2, 3 \cdots \quad (13)$$

$$A_0 = e^{-\mu} = e^{-\sum_{j=1}^m \frac{\varphi_j}{\varepsilon_j}} \quad (14)$$

The loss distribution of the loan portfolio can be calculated by formulas (10), (13) and (14).

Step 4: Calculate the economic capital, expected loss and unexpected loss of the loan portfolio. The VaR of the loan portfolio can be calculated given a confidence level in the calculations of the second and third steps. That is the unexpected loss of the loan portfolio. For example, when the confidence level = 95%, we get the maximum loss under the confidence level. VaR (unexpected loss) = maximum loss-expected loss.

4 Sichuan KK Rural Commercial Bank credit risk evaluation

4.1 Credit risk evaluation of all loan samples of Sichuan KK Rural Commercial Bank

4.1.1 Frequency band allocation

This paper selects 1,117 loans from the outstanding loans of Sichuan KK Rural Commercial Bank from 2018 to 2020 as sample data. We analyse the distribution of loan amounts. Following the principle of frequency band allocation, the total loan amount of these 1,117 loans is 112.308 billion yuan. Among them, the loan with the largest risk exposure amounted to 3.28 million yuan. We assume that the unit risk exposure $L = 500,000$ yuan, and then we can divide six frequency bands. It is shown in Table 1.

Table 1 Frequency band distribution of 1,117 loans of Sichuan KK Rural Commercial Bank

Frequency band	Loan balance (ten thousand yuan)	Risk exposure	Number of loans	Average default rate	Expected number of defaults	Expected default loss (ten thousand yuan)
V1	0–50	20	510	4.00%	20.40	408.0
V2	51–100	60	200	3.00%	6.00	360.0
V3	101–150	120	300	2.75%	8.24	989.1
V4	151–200	160	32	2.07%	0.66	106.2
V5	201–250	220	65	2.93%	1.90	418.5
V6	251	310	10	2.53%	0.25	78.6

Fig. 2 Distribution of default loss of loan portfolio under $\alpha = 0.05$ confidence level

4.1.2 Loss distribution

According to the principle of the linear fractional differential equation model, we use the loan data of Sichuan KK Rural Commercial Bank. With the help of MATLAB software, we can calculate the distribution of loan losses. The loss distribution is shown in Figure 2.

4.1.3 Expected loss, extreme loss and unexpected loss (economic capital)

It can be seen from Table 1 that the expected loss of Sichuan KK Rural Commercial Bank is 23.6 million yuan. The linear fractional differential equation model calculates the maximum loss $VaR\alpha$ of the loan portfolio under the confidence level of $\alpha = 0.05 = 1.640$ million yuan. The unanticipated loss (economic capital) $ULE = 1616.4$ million yuan can be obtained by formula (15). The condition $CVaR\alpha = 179783$ million yuan. At this time, the unexpected loss (economic capital) $ULE = 177423$ million yuan.

$$ULE = CVaR - \sum_{j=1}^n \phi_j \quad (15)$$

The total amount of this sample loan of Sichuan KK Rural Commercial Bank is 112.308 billion yuan. At the confidence level of $\alpha = 0.05$, the proportion of $VaR\alpha$ is 1.07%, and the proportion of $CVaR\alpha$ is 1.12%. The average non-performing loan ratio of Sichuan KK Rural Commercial Bank from 2018 to 2019 was 1.17%. It can be seen that the proportion of $VaR\alpha$ is very close to that of $CVaR\alpha$. This shows that the method is feasible. From 2018 to 2020, the average loan loss reserve of Sichuan KK Rural Commercial Bank is 9.454 billion yuan [11]. This shows that the bank's ability to offset risks is better.

4.2 Credit risk evaluation of agricultural loan samples of Sichuan KK Rural Commercial Bank

Among the 1,117 loan samples selected by Sichuan KK Rural Commercial Bank from 2018 to 2020, there are 328 agricultural loans. Sichuan KK Rural Commercial Bank's agricultural loans accounted for 29.7% of total loans from 2018 to 2020. The ratio of agricultural loans to total loans in the sample data is 29.3%, basically in line with the ratio of the overall data.

4.2.1 Frequency band allocation

Among these 328 agricultural loans, the total amount of agricultural loans was 10.57 billion yuan. The agricultural loan with the largest risk exposure amounted to 1.81 million yuan [12]. We assume that the unit risk exposure is $L = 300,000$ yuan; then we can divide six frequency bands as shown in Table 2.

Table 2 Frequency band distribution of agricultural loans sampled by Sichuan KK Rural Commercial Bank

Frequency band	Loan balance (ten thousand yuan)	Risk exposure	Number of loans	Average default rate	Expected number of defaults	Expected default loss (ten thousand yuan)
V1	0–30	20	156	2.72%	4.24	84.80
V2	31–60	40	73	3.07%	2.24	89.60
V3	61–90	70	40	3.50%	1.40	97.90
V4	91–120	100	30	2.82%	0.85	84.60
V5	121–150	130	23	3.03%	0.70	90.60
V6	151	180	6	2.93%	0.18	31.70

4.2.2 Distribution of agricultural loan losses

According to the principle of the linear fractional differential equation model, the loss distribution of agricultural loans can be calculated. The loss distribution is shown in Figure 3.

Fig. 3 Loss distribution of Sichuan KK Rural Commercial Bank's agricultural loan sample

4.2.3 Expected losses, extreme losses and unexpected losses of agricultural loans (economic capital)

From Table 2, it can be seen that the expected loss of Sichuan KK Rural Commercial Bank is 4.791 million yuan. The maximum loss $\text{VaR}\alpha$ of the loan portfolio calculated by the linear fractional differential equation model under the confidence level of $\alpha = 0.05$ is RMB 10.34 million. The unanticipated loss (economic capital) $\text{ULE} = 5.549$ million can be obtained by formula (15). Condition $\text{CVaR}\alpha = 11.78$ million yuan. At this time, the unexpected loss (economic capital) $\text{ULE} = 6.989$ million yuan.

The total agricultural-related sample loan of Sichuan KK Rural Commercial Bank is 10.57 billion yuan. At the confidence level of $\alpha = 0.05$, the proportion of $\text{VaR}\alpha$ is 2.3%, and the proportion of $\text{CVaR}\alpha = 2.75\%$. The average non-performing loan ratio of Sichuan KK Rural Commercial Bank is 1.17% from 2018 to 2020. It can be seen that both the $\text{VaR}\alpha$ and $\text{CVaR}\alpha$ of agricultural loans are higher than the average value of non-performing loan ratios.

5 Empirical discussion

The $\text{VaR}\alpha$ ratio (1.07%) of the sample loans of Sichuan KK Rural Commercial Bank is very close to the ratio of $\text{CVaR}\alpha$ (1.12%). This shows that the method is feasible. Therefore, using the linear fractional differential equation model as a credit risk evaluation model for rural commercial banks in the western region is appropriate. It is effective for the credit risk evaluation results. The proportion of $\text{VaR}\alpha$ in Sichuan KK's agricultural loans is higher than that of $\text{CVaR}\alpha$, which is higher than the proportion of $\text{VaR}\alpha$ and $\text{CVaR}\alpha$ in all loan sample data. The total agricultural-related sample loans of Sichuan KK Rural Commercial Bank amounted to 10.57 billion yuan. At the confidence level of $\alpha = 0.05$, the proportion of $\text{VaR}\alpha$ is 2.3%, and the proportion of $\text{CVaR}\alpha = 2.75\%$. Both the $\text{VaR}\alpha$ and $\text{CVaR}\alpha$ of agricultural loans are higher than the average value of the non-performing loan ratio of 1.17%. In summary, it shows that the credit risk of agricultural loans of rural commercial banks in western China is relatively high and is the main source of credit risk.

6 Conclusion

It is effective to use the linear fractional differential equation model to evaluate the credit risk of rural commercial banks in western China. We need to build a relatively complete credit rating system and speed up the construction of credit rating databases. Rural commercial banks in western China should use financial service consultants as their starting point to improve individualised and diversified financial services. In addition, they should make good use of financial tools and continuously improve the ability and level of rural financial services to serve the real economy. We need to improve the long-term one-way flow of factors between urban and rural areas and promote the deep integration of urban and rural development.

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