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Small amplitude periodic solution of Hopf Bifurcation Theorem for fractional differential equations of balance point in group competitive martial arts

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Abstract

Differential equation modelling was earlier used to discover better and understand various biological phenomena and social problems. We hope to understand the stability of the system and the Hopf bifurcation based on the characteristic roots of the linear system. Because group competitive sports require participants to have certain competitive skills, those who do not have sports skills but want to develop into activities must receive training and specific training. Therefore, based on the research background, the article proposes a time-lag group competitive martial arts activity model with a time lag effect. Through delay differential equation theory and Hopf bifurcation theory, the stability of the equilibrium point and the existence of periodic solutions generated by the Hopf bifurcation caused by the ‘instability’ of the equilibrium point are discussed. Finally, the theoretical results are simulated and verified with the help of MATLAB software.

Keywords: group competitive martial arts activities, balance point, stability, time lag, Hopf bifurcation, numerical simulation**AMS 2010 codes:** 34A08

1 Introduction

Differential equation modelling can be traced back to the early population model of Malthus and the predator–prey model of Lotka and Volterra. These models were once used to discover and understand various biological phenomena and social problems [1]. However, an overly simple model is difficult to accurately reflect the observed complex dynamic behaviours, such as periodic solutions. To this end, the model needs to be continuously improved. One method increases the equation’s dimensionality continuously, but the cost is that it is difficult to estimate the parameters multiplied by actual data. Another way is to consider the time lag effect. The time lag effect is common in real problems. Time lag can correspond to the incubation period, delivery delay and response delay of the disease.

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Moreover, simple time-delay differential systems often contain a wealth of complex dynamic behaviours. Some scholars have studied the gene regulation model of the time lag effect. Some scholars have conducted Hopf bifurcation research on Volterra predator–predator system with time delay [2]. Until the 1960s, the research on delayed differential systems mainly focused on stability, boundedness, asymptotics and equilibrium, periodic solutions, and oscillations of almost periodic solutions. Compared with ordinary differential systems, there is relatively little research on branch theory.

Hall was the first to study the local bifurcation of time-delay differential systems. He studied the existence of the central flow pattern of time-delay differential systems and the Hopf bifurcation theorem. However, Hall's theory is difficult to apply to practical problems. For time-delay differential systems with finite time delays, we hope to understand the stability of the system and the Hopf bifurcation based on the characteristic roots of the linear system. Since the characteristic equation of a linear system is a function of time delay, the characteristic root is also a function of time delay [3].

Moreover, the stability of the singularity will change as the time delay changes so that Hopf branches will occur near some critical values. Part of the time-delay differential equations will alternately appear from stable to unstable and then to stable as the singularity of the time-delay changes. This phenomenon is the so-called stable switching phenomenon.

Some scholars have proposed an ordinary differential equation model for group competitive sports activities [4]. Competitive activities here mean that participants must have certain skills rather than professional competitive sports activities. The article divides the human population into three categories. He set the total number of three types of people to 1, then the ordinary differential equation model of group competitive sports activities is as follows:

$$\begin{cases} \frac{dp_1(t)}{dt} = (r_1 - b)p_1(t) - \alpha p_1(t)p_2(t) + a \\ \frac{dp_2(t)}{dt} = (r_2 - b - \beta)p_2(t) + \alpha p_1(t)p_2(t) \end{cases}$$

where α represents the conversion rate of category *I* individuals into category *II* individuals under the influence of category *II* individuals. β represents the conversion rate of the *II* individual into the *III* individual. It is assumed here that the *III* individual has no influence. a, b stands for birth rate and death rate, respectively. $r_i (i = 1, 2)$, respectively, represents the migration rate of the *I, II* individual.

Because group competitive sports require participants to have certain competitive skills, those who do not have sports skills but want to develop into activities must receive training and specific training. It must take a certain amount of time [5]. The above model does not consider the time lag effect. This paper discusses the stability of the equilibrium point and the existence of periodic solutions generated by the Hopf bifurcation caused by the ‘instability’ of the equilibrium point through the theory of delay differential equations and the Hopf bifurcation theory. Finally, the theoretical results are simulated and verified with the help of MATLAB software.

2 Improved model

Considering that it takes a certain amount of time for an individual who participates in sports skills to transform into an individual who can participate in sports competitions, it is set as in Ref. [6]. Then the improved delay differential equation model can be expressed as:

$$\begin{cases} \frac{dx}{dt} = (r_1 - b)x(t) - \alpha x(t)y(t) + a \\ \frac{dy}{dt} = (r_2 - b - \beta)y(t) + \alpha x(t - \tau)y(t - \tau) \end{cases} \quad (1)$$

where x, y, z represents the total number of individuals of class *I*, class *II*, and class *III* at time t ; α represents the conversion rate of class *I* individuals into class *II* individuals under the influence of class *II* individuals; and β represents the conversion rate of the *II* individual into the *III* type individual. It is assumed here that the

category *III* individuals have no influence. a, b represents the birth rate and death rate, respectively. $r_i (i = 1, 2, 3)$ respectively represents the migration rate of the *I*, *II*, *III* individual [7]. Since the sum of the total number of people in the three groups remains unchanged, only the above-mentioned two-dimensional system composed of *I*, *II* type individuals must be considered.

3 The stability of the equilibrium point and the existence of the Hopf branch

First calculate the balance point of the system. We make the right end of the system (1) zero, that is, $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$. Then the balance points are $E_0(x_0, 0) = \left(\frac{a}{b-r_1}, 0\right)$ and $E_*(x^*, y^*)$, where $x^* = \frac{b-r_2+\beta}{a}$, $y^* = \frac{\alpha a + (r_1-b)(b-r_2+\beta)}{a(b-r_2+\beta)}$.

3.1 Asymptotic stability of boundary equilibrium point E_0

Boundary equilibrium point $E_0(x_0, 0) = \left(\frac{a}{b-r_1}, 0\right)$, because the number of people in each type of group in competitive activities needs to be a positive value [8]. Assuming that system (1) satisfies $H_1: b - r_1 > 0$, the boundary equilibrium point is a non-negative equilibrium point:

$$\begin{cases} \frac{dx}{dt} = (r_1 - b)x(t) - \frac{\alpha a}{b-r_1}y(t) \\ \frac{dy}{dt} = (r_2 - b - \beta)y(t) + \frac{\alpha a}{b-r_1}y(t - \tau) \end{cases} \quad (2)$$

For the convenience of calculation, we make a series of variable substitutions. Assuming $m_1 = r_1 - b$, $m_2 = r_2 - b - \beta$, $m_3 = \frac{\alpha a}{b-r_1}$, then formula (2) can be expressed as:

$$\begin{cases} \frac{dx}{dt} = m_1x(t) - m_3y(t) \\ \frac{dy}{dt} = m_2y(t) + m_3y(t - \tau) \end{cases} \quad (3)$$

The characteristic equation corresponding to system (3) at equilibrium point $E_0(x_0, 0) = \left(\frac{a}{b-r_1}, 0\right)$ is:

$$(\lambda - m_1)(\lambda - m_2 - m_3e^{-\lambda\tau}) = 0 \quad (4)$$

According to the hypothesis H_1 , $\lambda = m_1 = r_1 - b$ is always the negative root of Eq. (4). So let us study the second factor $\lambda - m_2 - m_3e^{-\lambda\tau}$ of the characteristic Eq. (4).

Hypothesis

$$G(\lambda) = \lambda - m_2 - m_3e^{-\lambda\tau} \quad (5)$$

If $m_3 > -m_2$ is true, you can get: $G(0) = -m_2 - m_3 < 0$, $\lim_{\lambda \rightarrow +\infty} G(\lambda) = +\infty$. Therefore $G(\lambda) = 0$ has at least one positive real root. When $m_2 + m_3 = r_2 - b - \beta + \frac{\alpha a}{b-r_1} > 0$, the boundary equilibrium point $E_0(x_0, 0)$ is unstable. If $m_3 < -m_2$ is true, when $\tau = 0$, we know that the system is asymptotically stable at the boundary equilibrium point. Consider the case of $\tau > 0$ below.

Lemma 1. When $\tau > 0$, if $m_3 < -m_2$ holds, the characteristic Eq. (4) has no pure imaginary roots.

Proof. Suppose $\lambda = i\omega (\omega > 0)$ is a pure imaginary root of the characteristic Eq. (4). We substitute $\lambda = i\omega$ into Eq. (4) and separate the real and imaginary parts from obtaining the following equation:

$$\begin{cases} m_2 = -m_3 \cos \omega \tau \\ \omega = -m_3 \sin \omega \tau \end{cases} \quad (6)$$

Add the squares to get

$$\omega^2 = m_3^2 - m_2^2 < 0$$

This contradicts the assumption. There is no pure imaginary root for any $\tau \geq 0$ Eq. (4). In other words, the zero points of the characteristic equation will not appear on the imaginary axis or cross the imaginary axis [9]. Based on the above analysis, when the system satisfies the assumption H_1 , the conclusions that can be obtained for any $\tau \geq 0$ are as follows: a) If $m_3 > -m_2$, $E_0(x_0, 0)$ is unstable; b) If $m_3 < -m_2$, $E_0(x_0, 0)$ is locally asymptotically stable.

3.2 Asymptotic stability of the positive equilibrium point $E_*(x^*, y^*)$

Lemma 2. Assuming that H_2 : $b - r_2 + \beta > 0$, $\alpha a + (r_1 - b)(b - r_2 + \beta) > 0$ holds, then system (1) has the only positive equilibrium point $E_*(x^*, y^*)$: System (1) The linearised system at the positive balance point $E_*(x^*, y^*)$ is: $x'(t) = Ax(t) + Bx(t - \tau)$. Among them: $A = \begin{pmatrix} r_1 - b - ay^* & -xy^* \\ 0 & r_2 - b - \beta \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ ay^* & ax^* \end{pmatrix}$.

According to the above formula, the linearisation system at the positive equilibrium point $E_*(x^*, y^*)$ can be obtained as:

$$\begin{cases} \frac{dx}{dt} = (r_1 - b - ay^*)x(t) - ax^*y(t) \\ \frac{dy}{dt} = ay^*x(t - \tau) + (r_2 - b - \beta)y(t) \\ y(t) + ax^*y(t - \tau) \end{cases} \quad (7)$$

The characteristic equations corresponding to the system (7) can be sorted out:

$$\lambda^2 + (p_0\lambda + p_1)e^{-\lambda\tau} + q_0\lambda + q_1 = 0 \quad (8)$$

Among them: $p_0 = -ax^*$, $q_0 = (b - r_1 + ay^*) - (r_2 - b - \beta)$,
 $p_1 = ax^*(r_1 - b)$, $q_1 = (r_1 - b - ay^*)(r_2 - b - \beta)$.

Lemma 3. Assume that condition H_2 holds and there is H_3 : $q_1 - p_1 = \alpha a - (r_1 - b)(b - r_2 + \beta) < 0$. Then there is τ_0 when τ increases from zero.

Proof. First, prove that the characteristic equation has a pair of purely imaginary roots $\pm i\omega_0$, $\omega_0 > 0$ at the positive equilibrium point E_* .

When $\tau = 0$ is the formula (8), we can get:

$$\lambda^2 + (p_0 + q_0)\lambda + (p_1 + q_1) = 0 \quad (9)$$

For the above condition without time delay, it is known that when $\tau = 0$ is the positive equilibrium point E_* is asymptotically stable [10]. The case of $\tau > 0$ is discussed below. When $\tau > 0$, if $\lambda = i\omega$ ($\omega > 0$) is the pure imaginary root of the characteristic Eq. (8). We substitute it and separate the real part and the imaginary part to get:

$$\begin{cases} \omega^2 - q_1 = p_0\omega \sin \omega\tau + p_1 \cos \omega\tau \\ q_0\omega = p_1 \sin \omega\tau - p_0\omega \cos \omega\tau \end{cases} \quad (10)$$

Eliminate the trigonometric function from Eq. (10) to get:

$$\omega^4 + (q_0^2 - p_0^2 - 2q_1)\omega^2 + (q_1^2 - p_1^2) = 0 \quad (11)$$

In:

$$q_0^2 - p_0^2 - 2q_1 = (r_1 - b - ay^*)^2 > 0 \quad (12)$$

If another H_3 : $q_1 - p_1 = \alpha a - (r_1 - b)(b - r_2 + \beta) < 0$ is established. At the same time, it is known from (12) that Eq. (11) has a unique positive real root ω_0 , then Eq. (8) has a pair of purely imaginary roots $\pm i\omega_0$ that are conjugate to each other when $\tau = \tau_j$, $j = 0, 1, 2, \dots$ is.

According to Eq. (10), we can get:

$$\omega_0 = \sqrt{\frac{1}{2} \left[(p_0^2 - q_0^2 + 2q_1) + \sqrt{(q_0^2 - p_0^2 - 2q_1)^2 - 4(q_1^2 - p_1^2)} \right]} \quad (13)$$

Combining (10)–(12), define

$$\tau_j = \frac{1}{\omega_0} \left(\arccos \frac{\varphi}{\omega} + 2j\pi \right), \quad j = 0, 1, 2, \dots \quad (14)$$

where $\varphi = (p_1 - p_0 q_0) \omega_0^2 - p_1 q_1$, $\tilde{\omega} = p_0^2 \omega_0^2 + p_1^2$. Then $(\tau, \lambda) = (\tau_j, i\omega_0)$ satisfies Eq. (8), with the definition

$$\lambda(\tau) = a(\tau) + i\omega(\tau) \quad (15)$$

It is only necessary to prove that the transversal condition $\left(\frac{d \operatorname{Re} \lambda(\tau)}{d\tau} \Big|_{\tau=\tau_j} \right) > 0$ holds. When $\tau > \tau_j$, there is at least one characteristic root whose real part is > 0 so that the condition of Hopf bifurcation is satisfied and a periodic solution is generated near E_* . Substituting $\lambda = \lambda(\tau)$ into Eq. (8) and deriving the derivation of both sides concerning τ at the same time:

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + p_0 e^{-\lambda\tau} + q_0}{p_0 \lambda^2 e^{-\lambda\tau} + p_1 \lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} \quad (16)$$

Hence $\left(\frac{d \operatorname{Re} \lambda(\tau)}{d\tau} \Big|_{\tau=\tau_j} \right)^{-1} = \operatorname{Re} \left(\frac{2\lambda + p_0 e^{-\lambda\tau} + q_0}{p_0 \lambda^2 e^{-\lambda\tau} + p_1 \lambda e^{-\lambda\tau}} \right)_{\tau=\tau_j} = \frac{MX + NY}{X^2 + Y^2}$. Its $M = p_0 \cos \omega\tau$, $N = 2\omega - p_0 \sin \omega\tau$, $X = -p_0 \cos \omega\tau + p_1 \omega \sin \omega\tau$, $Y = p_0 \omega^2 \sin \omega\tau + p_1 \omega \cos \omega\tau$. According to formula (13), the following formula can be deduced to hold $\frac{MX + NY}{X^2 + Y^2} = \frac{(2\omega^2 - 2q_1 - p_0^2)}{(p_0^2 \omega^2 + p_1^2)} > 0$. This proves that the transversal condition holds. Therefore the Hopf branch occurs at $\omega = \omega_0$, $\tau = \tau_j$.

3.3 Hopf branch of the system

The stability of the time-delay system at $\tau = \tau_j$ changes. Therefore, according to the Hopf bifurcation theory, the time-lag group competitive sports activity model (1) has a Hopf bifurcation at its positive balance point $E_*(x^*, y^*)$ [11].

4 Numerical simulation

We select a set of parameters as follows:

$$a = 0.012, \quad a = 0.38, \quad r_1 - b = 0.21, \quad r_2 - b - \beta = -0.126$$

According to the previous analysis and formula, it is calculated that

$$p_0 = -0.1260, \quad q_0 = 0.1622, \quad p_1 = 0.0265, \quad q_1 = 0.0046$$

$$E_*(x^*, y^*) = (0.3316, 0.6479), \quad \omega^2 = 0.1594; \quad \omega = 0.0254; \quad \tau_0 = 1.5222$$

Drawing through MATLAB software, System (1) obtains the numerical simulation result at time lag $\tau = 1.3 > \tau_0$ and obtains the waveform diagram and the plane phase diagram of the numerical solution of the system (Figures 1 and 2). The numerical simulation results at time lag $\tau = 1.56 > \tau_0$ are the same as above (see Figures 3 and 4).

MATLAB numerical simulation shows that the stability of the positive equilibrium has changed with the change of time delay [12].

Fig. 1 Waveform graph of the percentage of the *I* and *II* population in the total population versus time

Fig. 2 Plane phase diagrams of *I* and *II* groups of people

Fig. 3 Waveform diagram of the percentages of the *I* and *II* groups in the total number of people over time

Fig. 4 Planar phase diagram of the number of people in category *I* and category *II*

5 Conclusion

There are certain critical points in the process of the time delay increasing from 0. At these critical points, the stability of the system equilibrium point always changes from stable to unstable; from zero to increase every time the critical value is passed from the positive equilibrium point. Expenditure periodic solution. Second, a numerical simulation of the time-lag group competitive sports activity model was carried out by MATLAB software. The theoretical results of the above model were verified by selecting a set of parameters.

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