INTEREST RATE MODELLING IN THE PRESENCE OF DISCONTINUITIES AND ITS SENSITIVITIES

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Abstract

Interest rate paths experience discontinuities in the presence of certain factors. Much of the work on interest rate modelling has no consideration for effects of such unexpected occurrences in real life. A good risk manager needs to have a better model that considers possibility of unexpected occurrences. In this paper, we discuss step by step extension of Vasicek model to both jump model and jump-diffusion model using Itô's formula as the major tool. We also derive the greeks 'delta' and 'vega' that measure sensitivity of the interest rate with respect to both changes in its initial interest rate and volatility in an interbank rate.

Keywords: Interest Rate, Vasicek model, Compound Poisson process, Jump-diffusion process, Itô formula

MSC2020: 60J75, 60J60, 91G15

1.0 Introduction

A good model should consider behaviours inherent in the system under consideration. A given system behaves abnormally because of certain circumstances. Factors such as change in government policies, inflation, natural disasters, presence of pandemic (e.g. corona virus), etc contribute to spikes in the interest rates. To solve the problem of getting a better model that takes care of discontinuities in options, Merton (1976) obtained a pricing formula for underlying stock returns produced by both continuous and discontinuous (jump) processes. He further highlighted the extension of the formula to the pricing of corporate liabilities. Kou (2002) derived a double exponential jump-diffusion model for the valuation of options and emphasised on the importance of jump-diffusion model for pricing assets in financial engineering. Later, many researchers have contributed to the relevance of the jump-diffusion model in different areas. Buraschi and Jiltsov (2007) observed that interest rates exhibit continuous and discontinuous behaviours, as a result, they suggested obtaining solutions of term structure model under jump-diffusion processes. Berhane *et al.* (2019) derived a better model for Ethiopian commodity prices to minimise risks caused by jumps using the jump-diffusion model. Carr and Mayo (2007) proposed numerical pricing of options with partial integral differential equation when density functions of jump processes have polynomials, Gaussian and exponential. Feng and Linetsky (2008) discussed a computational technique involving partial integro differential equation for options' pricing in jump-diffusion model. Bates (1996) proposed an approach for stochastic volatility model driven by the model. Chang and Wang (2020) presented option pricing driven by double stochastic volatility model in conjunction with stochastic interest rate model and double exponential jumps and stochastic intensity. Lau *et al.* (2019) presented an approach to estimate market parameters modelled using a certain jump-diffusion model. Novat *et al.* (2019) discussed Merton's jump-diffusion model in stock markets of three East African countries. Salmi and Toivanen (2011) derived iterative steps for valuing American option using the model. Wang *et al.* (2017) proposed pricing of vulnerable American option under jump-diffusion process, whereas Jiahui *et al.* (2019) obtained a closed form evaluation formula for that of European option having credit and jump risks under inadequate information. Perelló *et al.* (2020) focused on stochastic interest rate modelling for assessing future with ramification in climate change, while Verschuren (2020) developed a coherent structure on how to include certain interest rates in a single curve stochastic term structure model and did comparison with multiple curve analogue.

In this paper, we shall employ the jump-diffusion model in the modelling of interest rate consisting of an interbank offered rate by modifying Vasicek short rate model and obtain expression for the greeks 'delta' and 'vega' of the interest rate using integration by parts technique of Malliavin calculus as given by Bavouzet-Morel and Messaoud (2006). Wu and Liang (2018) studied the Vasicek model with mixed-exponential jumps and its applications in finance and insurance, by first obtaining the explicit form of Laplace transform of the model distribution through the aid of piecewise deterministic Markov process theory and the martingale theory. Rao (2021) discussed asymptotic properties of the maximum likelihood estimator of unknown parameters in a fractional Vasicek model driven by a fractional Brownian motion, while Guo (2021) evaluated the benefit of bias-correction method in enhancing the Vasicek model for market risk management. In the next section, we discuss the important mathematical tools needed for the success of the paper.

2.0 Experimental

In this section, we discuss important mathematical tools required for the success of the work.

The Vasicek model (1977): The model has the property of mean-reversion and interest rates are known to revert back to certain mean after sometime. Hence, it is necessary that a good model of interest rates should be able to take care of its mean-reversion feature. The dynamics for the Vasicek short rate model is given by

$$dr_t = \alpha(\rho - r_t)dt + \sigma dW_t, \quad (2.1)$$

where r_t is the interrate rate at time t ; α , ρ and σ represent speed of mean-reversion, long-term mean rate and volatility of the interest rate, respectively, while W_t denotes the Wiener process.

Itô's formula for jump-diffusion process (Cont and Tankov, 2004): Let $Y = Y_t$ be a diffusion process with jumps at time t defined as

$$Y_t = Y_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta Y_i \quad (2.2)$$

where a_s and σ_s are non-anticipating processes with $E\left[\int_0^t \sigma_s^2 dt\right] < \infty$, respectively, and $0 \leq s \leq t$.

$$f(t, Y_t) = f(0, Y_0) + \int_0^t \left[\frac{\partial f(s, Y_s)}{\partial s} + a_s \frac{\partial f(s, Y_s)}{\partial y} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f(s, Y_s)}{\partial y^2} \right] ds + \int_0^t \frac{\partial f(s, Y_s)}{\partial y} dW_s + \sum_{i \geq 1, \tau_i \leq t} [f(Y_{\tau_i-} + \Delta Y_i) - f(Y_{\tau_i-})].$$

Then, for each $C^{1,2}$ function $f: [0, T] \rightarrow R$, the process $f(t, Y_t)$ can be written as

$$(2.3)$$

Itô's formula for jump process

Suppose that X_t is a jump process having values in R given as

$$Y_t = \int_0^t a_s ds + \sum_{i=1}^{N_t} \Delta Y_i \quad (2.4)$$

where a_s is a non-anticipating left continuous with right limit process, $\Delta Y_i = Y(T_i) - Y(T_{i-1})$ denote the i th jump size and N_t represents the random number of jumps between 0 and t . Let $(T_n)_{n \geq 1}$ be the jump times of Y_t , then the following holds:

$$\begin{aligned} f(Y_T) &= f(Y_0) + \int_0^T a_t \frac{\partial f(Y_t)}{\partial y} dt + \sum_{\{i, T_i \leq t\}} f(Y_{T_i-} + \Delta Y_i) - f(Y_{T_i-}) \\ &= f(Y_0) + \int_0^T a_t \frac{\partial f(Y_{t-})}{\partial y} dt \\ &\quad + \sum_{\substack{\Delta X_t \neq 0 \\ 0 \leq t \leq T}} [f(Y_{T_i-} + \Delta Y_i) - f(X_{t-})] \end{aligned} \quad (2.5)$$

where the sum covers the random jumps.

The Malliavin Calculus

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space where $\Omega, \mathcal{F}$ and $\mathbb{P}$ denote sample space, set of filtration and the probability measure. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with density function $\check{f}_n$ and assume that G_n has moments of any order for $n \geq 1$ such that G_n is absolutely continuous and continuously differentiable in R with $\lim_{n \rightarrow \infty} |y|^n \check{f}_n(y) = 0$ and let $\partial_y \ln(\check{f}_n(y)) = \frac{\partial_y \check{f}_n(y)}{\check{f}_n(y)}$ has at most polynomial growth. $f \in C^k(R^n)$ represents the function $f: R^n \rightarrow R$ that are k times continuously differentiable with derivatives of f and f permitting at most polynomial growth. Following Bavouzet-Morel and Messaoud (2006), we have the definitions:

- A random variable $F = f(G_1, \dots, G_n)$ where $f: R^n \rightarrow R$ is a measurable function for some $n \in \mathbb{N}$, is called a *simple functional*. $S_{(n,k)}$ denotes the space of simple functional.
- A sequence of random variables $U = (U_i, i \leq n)$ is called a *simple process of length n* provided $U_i = u_i(G_1(w), \dots, G_n(w))$. The space of simple processes of length n is denoted by $P_{(n,k)}$.
- The *Malliavin derivative* is defined as $D: S_{(n,1)} \rightarrow P_{(n,0)}$ such that if $F = f(G_1, \dots, G_n)$ then

$$D_i F = \left(\frac{\partial f}{\partial x_i} \right) (G_1(w), \dots, G_n(w)). \quad (2.6)$$

- The *Skorohod integral operator* defined by $\delta: P_{(n,1)} \rightarrow S_{(n,0)}$ is such that

$$\delta(U) = - \sum_{i=1}^n (D_i U_i + \varphi_i(G_i) U_i) \quad (2.7)$$

where $\varphi_i(y) = \frac{\partial \ln \check{f}_i(y)}{\partial y}$ if $\check{f}_i(y) > 0$; otherwise, it is equal to 0.

- The *Ornstein-Uhlenbeck operator* defined as $L: S_{(n,2)} \rightarrow S_{(n,0)}$ is a Malliavin derivative operator on the Skorohod integral, that is,

$$LF = - \sum_{i=1}^n [D_i D_i F + \varphi_i(G_i) D_i F], \quad i = 1, \dots, n. \quad (2.8)$$

- Let $F = (F^1, \dots, F^d)$, $F^i \in S_{(n,1)}$. Then, the *Malliavin covariance matrix* is defined as

$$\begin{aligned} M^{ij}(F) &= \langle DF^i, DF^j \rangle \\ &= \sum_{p=1}^n (\partial_p f^i \partial_p f^j)(G_1, \dots, G_n). \end{aligned} \quad (2.9)$$

The Malliavin integration by parts theorem. Suppose that $F = (F^1, \dots, F^d) \in S_{(n,2)}^d$, $B \in S_{(n,1)}$. Let $M(F)$ be an invertible matrix with its inverse denoted by $M^{-1}(F)$. Let $E[(\det M^{-1}(F))^4] < \infty$. Then, for any smooth function $\Phi: R^d \rightarrow R$, any $i = 1, \dots, d$,

$$E[\partial_i \Phi(F) B] = E[\Phi(F) H^i(F, B)]$$

where

$$\begin{aligned} H^i(F, B) &= \sum_{j=1}^d B(M^{-1})^{ji}(F) L F^j - (M^{-1})^{ji}(F) \langle DF, DB \rangle \\ &\quad - B \langle DF^j, D(M^{-1})^{ji}(F) \rangle. \end{aligned} \quad (2.10)$$

3.0 Results and Discussion

In this section, we derive the modified Vasicek model driven by pure jump and jump-diffusion processes to settle different scenarios with the aid of Bavouzet-Morel and Messaoud (2006). The interest rate paths experience spikes as can be seen in figure 1 showing Nigerian interbank offered rates data between January 2007 and July 2020 obtained from the website of the Central Bank of Nigeria.

Fig. 1: Interest rate paths.

Theorem 3.1. Let r_t be an interest rate at time t and let the short rate dynamics of the interest rate be given by

$$dr_t = \alpha(\rho - r_t)dt + \sigma dW_t + dJ_t \quad (3.1)$$

where α, ρ, σ and W_t represent speed of mean-reversion, long-term mean rate, volatility and Wiener process of the interest rate, respectively, while J_t denotes compound Poisson process that can be written as

$$J_t = \sum_{i=1}^{N_t} \Delta r_i \quad (3.2)$$

where N_t is a Poisson process that counts the number of jumps to time t and $\Delta r_i = r_{T_i} - r_{T_{i-1}}$ represents jump sizes that occur at jump times T_i . Then, $r_t = r(t)$ is given by

$$\begin{aligned} r_t &= r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \\ &\quad + \sum_{i=1}^{N_t} \Delta r_i e^{-\alpha(t-T_i)}. \end{aligned} \quad (3.3)$$

Proof. Using Itô's formula, let $f = r_t e^{\alpha t}$, $\frac{\partial f}{\partial s} = \alpha r_t e^{\alpha t}$, $\frac{\partial f}{\partial r} = e^{\alpha t}$ and $\frac{\partial^2 f}{\partial r^2} = 0$. Then,

$$r_t e^{\alpha t} = r_0 + \int_0^t (\alpha r_s e^{\alpha s} + (\alpha(\rho - r_s)) e^{\alpha s}) ds + \int_0^t \sigma e^{\alpha s} dW_s$$

$$\begin{aligned}
& + \sum_{i \geq 1, T_i \leq t} [r_t e^{\alpha t} + \Delta r_i e^{\alpha T_i} - r_t e^{\alpha t}] \\
& = r_0 + \int_0^t \alpha \rho e^{\alpha s} ds + \int_0^t \sigma e^{\alpha t} dW_s + \sum_{i \geq 1, T_i \leq t} \Delta r_i e^{\alpha T_i} \\
& = r_0 + \rho(1 - e^{-\alpha t}) + \sigma \int_0^t e^{\alpha t} dW_s + \sum_{i=1}^{N_t} \Delta r_i e^{\alpha T_i}.
\end{aligned}$$

Whence,

$$r_t = r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s + \sum_{i=1}^{N_t} \Delta r_i e^{-\alpha(t-T_i)}.$$

Hence, equation (3.3) is obtained by substituting $n = N_t$.

Remark. If there is no jump, the original Vasicek model can be recovered as shown in the corollary below.

Corollary 3.2. Let r_t be a jump-diffusion interest rate model with zero jumps. Then,

$$r_t = r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t}) + \int_0^t e^{-\alpha(t-s)} dW_s \quad (3.4)$$

which can be written as

$$r_t = r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t}) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \quad (3.5)$$

Proof. From equation (3.3), we have that $N_t = 0$ which implies that the jump sum is equal to zero whenever there is no jump. Thus, it follows that equation (3.4) is valid.

Furthermore, the expectation of r_t in equation (3.4) is given by

$$E[r_t] = r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t})$$

while the variance is given by

$$\begin{aligned}
\text{Var}[r_t] &= E[r_t - E[r_t]]^2 = E\left[\sigma^2 e^{-2\alpha t} \int_0^t e^{2\alpha s} ds\right] \\
&= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).
\end{aligned}$$

Summing the expectation and variance of the interest rate r_t , we obtain equation (3.5).

Theorem 3.3. The extended Vasicek model driven by pure jump process is given by the dynamics

$$dr_t = \alpha(\rho - r_t)dt + \sigma dJ_t \quad (3.6)$$

and its solution is given by

$$r_t = r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t}) + \sigma \sum_{i=1}^n \Delta r_i e^{-\alpha(t-T_i)}. \quad (3.7)$$

Proof. Using Itô's formula in Section 2.3, we let $f(r_t) = r_t e^{\alpha t}$ and $\frac{\partial f}{\partial r} = e^{\alpha t}$. Thus,

$$\begin{aligned}
r_t e^{\alpha t} &= r_0 + \int_0^t (\alpha r_s e^{\alpha s} + \alpha(\rho - r_s) e^{\alpha s}) ds \\
&\quad + \sigma \sum_{i \geq 1, T_i \leq t} [r_t e^{\alpha t} + \Delta r_i e^{\alpha T_i} - r_t e^{\alpha t}] \\
&= r_0 + \rho(e^{\alpha t} - 1) + \sigma \sum_{i \geq 1, T_i \leq t} [r_t e^{\alpha t} + \Delta r_i e^{\alpha T_i} - r_t e^{\alpha t}] \\
&= r_0 + \rho(e^{\alpha t} - 1) + \sigma \sum_{i=1}^{N_t} \Delta r_i e^{\alpha T_i} \\
r_t &= r_0 e^{-\alpha t} + \rho(1 - e^{-\alpha t}) + \sigma \sum_{i=1}^{N_t} \Delta r_i e^{\alpha(T_i - t)}.
\end{aligned}$$

Sensitivities computation

In the following, $E[\cdot]$ denotes expected payoff function, $\Phi(r_T) = \max(r_T - K, 0)$ is the payoff function of the interest rate, K is the strike

price and T is the maturity time.

European call price on r_T is given by

$$V = e^{-r_0 T} E[\Phi(r_T)]. \quad (3.8)$$

Furthermore $F = r_T$.

From equation (3.3), we adopt the interest rate given by

$$\begin{aligned}
r_T &= r_0 e^{-\alpha T} + \rho(1 - e^{-\alpha T}) + \sigma \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \\
&\quad + \sum_{i=1}^n \Delta r_i e^{-\alpha(T-T_i)} \quad (3.9)
\end{aligned}$$

where

$$W(t_k) - W(t_{k-1}) = \sqrt{t_k - t_{k-1}} Z_k. \quad (3.10)$$

The Malliavin derivative on r_T is given by

$$DF = Dr_T = Dr_T^W + Dr_T^{\Delta_i}. \quad (3.11)$$

We obtain Dr_T^W as

$$Dr_T^W = \sigma \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}}. \quad (3.12)$$

We obtain $Dr_T^{\Delta_i}$ as

$$Dr_T^{\Delta_i} = \sigma \sum_{i=1}^n e^{-\alpha(T-T_i)}. \quad (3.13)$$

The Malliavin covariance matrix on r_T is given by

$$M(F) = M^W(F) + M^{\Delta_i}(F)$$

where

$$M^W(F) = \sigma^2 \sum_{k=1}^m e^{-2\alpha(T-t_k)} (t_k - t_{k-1}). \quad (3.14)$$

and

$$M^{\Delta_i}(F) = \sum_{i=1}^n e^{-2\alpha(T-T_i)}. \quad (3.15)$$

The inverse Malliavin covariance matrix is given by

$$M^{-1}(F) = M^{-1}(r_T) = M_W^{-1}(r_T) + M_{\Delta_i}^{-1}(r_T).$$

From equations (3.13) and (3.14), we have

$$M_W^{-1}(r_T) = \frac{1}{\sigma^2} \sum_{k=1}^m \frac{e^{2\alpha(T-t_k)}}{t_k - t_{k-1}} \quad (3.16)$$

and

$$M_{\Delta_i}^{-1}(r_T) = \sum_{i=1}^n e^{2\alpha(T-T_i)}. \quad (3.17)$$

The Malliavin derivative on the inverse Malliavin covariance matrix is given by

$$DM_W^{-1} = 0 = DM_{\Delta_i}^{-1}. \quad (3.18)$$

The Ornstein-Uhlenbeck operator on r_T is given by

$$LF = Lr_T = -[DDr_T + \varphi Dr_T] = Lr_T^W + Lr_T^{\Delta_i}. \quad (3.19)$$

From equations (3.8) and (3.15), the Malliavin derivative $DDr_T = 0$ since

$$DDr_T^W = 0 = DDr_T^{\Delta_i}. \quad (3.20)$$

$$\varphi = \frac{\partial \ln \tilde{f}(z)}{\partial z} = \frac{\partial \ln \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \right)}{\partial z} = -z. \quad (3.21)$$

Thus,

$$\begin{aligned}
& Lr_T^W \\
&= \sigma \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \quad (3.22)
\end{aligned}$$

and

$$= \sum_{i=1}^n e^{-\alpha(T-T_i)} \Delta_i. \quad (3.23)$$

Derivation of the greek 'delta' of the Interest Rate

The greek 'delta' measures the sensitivity of the interest rate with respect to change in its initial value. It is given by

$$\begin{aligned} \text{delta} = \Delta &= \frac{\partial V}{\partial r_0} = \frac{\partial}{\partial r_0} e^{-r_0 T} E[\Phi(r_T)] \\ &= -T e^{-r_0 T} E[\Phi(r_T)] + e^{-r_0 T} E\left[\Phi'(r_T) \frac{\partial r_T}{\partial r_0}\right] \\ &= -T e^{-r_0 T} E[\Phi(r_T)] + e^{-r_0 T} E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial r_0}\right)\right] \\ &= e^{-r_0 T} (-T E[\Phi(r_T)] + E[\Phi(r_T) H(r_T, B_{r_0})]) \end{aligned}$$

where

$$B = B_{r_0} = \frac{\partial r_T}{\partial r_0} = e^{-\alpha T} \quad (3.24)$$

and

$$H(r_T, B_{r_0}) = H(F, B) = BM^{-1}(F)LF - B\langle DF, DM^{-1}(F) \rangle - M^{-1}\langle DF, DG \rangle$$

is the Malliavin weight.

We proceed to obtain the Malliavin weight.

Malliavin derivative on $B = B_{r_0}$ satisfies

$$DB = DB_{r_0} = 0. \quad (3.25)$$

Theorem 3.4. Let the interest rate $r_T = r(T)$ be as given above, then the Malliavin weight for the greek 'delta' is given by

$$\begin{aligned} H(F, B) &= \frac{e^{-\alpha T}}{\sigma} \sum_{k=1}^m \frac{e^{\alpha(T-t_k)}}{\sqrt{t_k - t_{k-1}}} Z_k \\ &\quad + e^{-\alpha T} \sum_{i=1}^n e^{\alpha(T-T_i)} \Delta_i. \end{aligned} \quad (3.26)$$

Proof.

The Malliavin weight

$$H(F, B) = H^W(F, B) + H^{\Delta_i}(F, B).$$

We proceed to obtain $H^W(F, B)$.

Thus,

$$H^W(F, B) = BM^{-1}(F)LF - M^{-1}(F)\langle DF, DB \rangle - B\langle DM^{-1}(F), DF \rangle.$$

From equations (3.17) and (3.25), $DM^{-1} = 0$ and $DB = 0$, respectively. This implies that

$$M^{-1}(F)\langle DF, DB \rangle = B\langle DM^{-1}(F), DF \rangle = 0.$$

Thus, from equations (3.24), (3.12) and (3.22), we obtain

$$\begin{aligned} H^W(F, B) &= BM^{-1}(F)LF \\ &= e^{-\alpha T} \\ &\quad \cdot \frac{1}{\sigma^2} \sum_{k=1}^m \frac{e^{2\alpha(T-t_k)}}{t_k - t_{k-1}} \cdot \sigma e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \\ \therefore H^W(F, B) &= \frac{e^{-\alpha T}}{\sigma} \sum_{k=1}^m \frac{e^{\alpha(T-t_k)}}{\sqrt{t_k - t_{k-1}}} Z_k. \end{aligned}$$

We now derive $H^{\Delta_i}(F, B)$,

$$H(F, B) = BM^{-1}(F)LF - M^{-1}(F)\langle DF, DB \rangle - B\langle DM^{-1}(F), DF \rangle.$$

But from equations (3.17) and (3.25),

$$DM^{-1} = 0 = DB.$$

This implies that

$$\begin{aligned} H^{\Delta_i}(F, B) &= BM^{-1}(F)LF = e^{-\alpha T} \cdot \frac{1}{\sigma^2} \sum_{i=1}^n e^{2\alpha(T-T_i)} \cdot e^{\alpha(T-T_i)} \Delta_i \\ \therefore H^{\Delta_i}(F, B) &= e^{-\alpha T} \sum_{i=1}^n e^{\alpha(T-T_i)} \Delta_i. \end{aligned}$$

From equations (3.20) and (3.21), the Malliavin weight with respect to

the greek 'delta' becomes

$$\begin{aligned} H(F, B) &= H^W(F, B) + H^{\Delta_i}(F, B) \\ &= \frac{e^{-\alpha T}}{\sigma} \sum_{k=1}^m \frac{e^{\alpha(T-t_k)}}{\sqrt{t_k - t_{k-1}}} Z_k + e^{-\alpha T} \sum_{i=1}^n e^{\alpha(T-T_i)} \Delta_i, \end{aligned}$$

Hence, the result follows.

Derivation of the greek 'vega' of the Interest Rate

The greek 'vega' measures the sensitivity of the interest rate with respect to change in its volatility. It is given by

$$\begin{aligned} \text{vega} = \vartheta &= \frac{\partial V}{\partial \sigma} = \frac{\partial}{\partial \sigma} e^{-r_0 T} E[\Phi(r_T)] \\ &= e^{-r_0 T} E\left[\Phi(r_T) H\left(r_T, \frac{\partial r_T}{\partial \sigma}\right)\right] \\ &= e^{-r_0 T} E[\Phi(r_T) H(r_T, B_\sigma)] \end{aligned} \quad (3.27)$$

where $H(r_T, B_\sigma)$ is the Malliavin weight with respect to the greek 'vega'.

From equation (3.9),

$$\begin{aligned} B_\sigma &= \frac{\partial r_T}{\partial \sigma} \\ &= \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k. \end{aligned} \quad (3.28)$$

The Malliavin derivative of B_σ gives

$$\begin{aligned} DB_\sigma &= \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}}. \end{aligned} \quad (3.29)$$

Theorem 3.5. Let r_T be given by equation (3.9). Then, the Malliavin weight for the greek 'vega' is given by

$$\begin{aligned} H(F, B_\sigma) &= \frac{1}{\sigma} \sum_{k=1}^m (Z_k - 1) \\ &\quad + \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} \left(Z_k \sum_{i=1}^n e^{\alpha(T-T_i)} \Delta_i \right. \\ &\quad \left. - \sum_{i=1}^n e^{\alpha(T-T_i)} \right). \end{aligned} \quad (3.30)$$

Proof. The Malliavin weight is defined as

$$\begin{aligned} H(F, B_\sigma) &= B_\sigma M^{-1}(F)LF - M^{-1}(F)\langle DF, DB_\sigma \rangle - B_\sigma\langle DM^{-1}(F), DF \rangle \\ &= H^W(F, B_\sigma) + H^{\Delta_i}(F, B_\sigma). \\ H(F, B) &= H(F, B_\sigma) = B_\sigma M^{-1}(F)LF - M^{-1}(F)\langle DF, DB_\sigma \rangle \\ &\quad - B_\sigma\langle DM^{-1}(F), DF \rangle. \end{aligned}$$

From equation (3.17),

$$DM^{-1}(F) = DM^{-1}(r_T) = 0.$$

Thus,

$$\begin{aligned} H^W(F, B_\sigma) &= B_\sigma M^{-1}(F)LF - M^{-1}(F)\langle DF, DB_\sigma \rangle - B_\sigma\langle DM^{-1}(F), DF \rangle \\ &= B_\sigma M^{-1}(F)LF - M^{-1}(F)\langle DF, DB_\sigma \rangle. \end{aligned}$$

From equations (3.16), (3.15), (3.22), (3.10) and (3.29), we obtain

$$\begin{aligned} H^W(F, B_\sigma) &= \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \\ &\quad \cdot \frac{1}{\sigma^2} \sum_{k=1}^m \frac{e^{2\alpha(T-t_k)}}{t_k - t_{k-1}} \cdot \sigma \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \\ &\quad - \frac{1}{\sigma^2} \sum_{k=1}^m \frac{e^{2\alpha(T-t_k)}}{t_k - t_{k-1}} \cdot \sigma \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} \cdot \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}}. \end{aligned}$$

Hence,

$$H^W(F, B_\sigma) = \sum_{k=1}^m \frac{1}{\sigma} Z_k^2 - \frac{1}{\sigma} = \frac{1}{\sigma} \sum_{k=1}^m (Z_k^2 - 1). \quad (3.31)$$

Furthermore, we have for the jump part,

$$H^{\Delta_i}(F, B_\sigma) = B_\sigma M^{-1}(F)LF - M^{-1}(F)\langle DF, DB_\sigma \rangle - B_\sigma \langle DM^{-1}(F), DF \rangle \\ = B_\sigma M^{-1}(F)LF - M^{-1}(F)\langle DF, DB_\sigma \rangle.$$

From equations (3.24), (3.14), (3.17), (3.10) and (3.29), we get

$$H^{\Delta_i}(F, B_\sigma) = \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \cdot \sum_{i=1}^n e^{2\alpha(T-T_i)} \cdot e^{-\alpha(T-T_i)} \Delta_i \\ - \sum_{i=1}^n e^{2\alpha(T-T_i)} \cdot e^{-\alpha(T-T_i)} \cdot \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} \\ = \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} Z_k \sum_{i=1}^n e^{\alpha(T-T_i)} \Delta_i \\ - \sum_{i=1}^n e^{\alpha(T-T_i)} \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}}.$$

Thus,

$$H^{\Delta_i}(F, B_\sigma) = \sum_{k=1}^m e^{-\alpha(T-t_k)} \sqrt{t_k - t_{k-1}} \left(Z_k \sum_{i=1}^n e^{\alpha(T-T_i)} \Delta_i \right. \\ \left. - \sum_{i=1}^n e^{\alpha(T-T_i)} \right). \quad (3.32)$$

Combining equations (3.31) and (3.) gives the desired result.

Using the Nigerian interbank offered rate monthly data between January 2007 and July 2020 obtained from the website of Central Bank of Nigeria, we estimated the parameter values as $\alpha = 96.2777$ which stands for speed of mean reversion; $\rho = 0.1317$ implies that the long-term mean rate; $\sigma = 0.7168$ which implies that the volatility of the interest rate is 0.7168, while the initial interest rate is 0.1317. Hence, the dynamics of the interbank interest rate considering equation (3.1) becomes

$$dr_t = \alpha(\rho - r_t)dt + \sigma dW_t + dJ_t \\ = 96.27(0.1317 - r_t)dt + 0.7168dW_t + dJ_t.$$

The greeks become

$$\text{delta} = \Delta = e^{-0.1317T} (-TE[\Phi(r_T)] + E[\Phi(r_T)H(r_T, B_{r_0})]),$$

where its Malliavin weight in equation (3.26) is given by

$$H(r_T, B_{r_0}) = \frac{e^{-96.2777T}}{0.7168} \sum_{k=1}^m \frac{e^{96.28(T-t_k)}}{\sqrt{t_k - t_{k-1}}} Z_k + e^{-96.28T} \sum_{i=1}^n e^{96.28(T-T_i)} \Delta_i;$$

and from equations (3.27) and (3.30), respectively, we have

$$\text{vega} = \vartheta = e^{-r_0 T} E[\Phi(r_T)H(r_T, B_\sigma)]$$

where

$$H(r_T, B_\sigma) \\ = \frac{1}{0.7168} \sum_{k=1}^m (Z_k - 1) \\ + \sum_{k=1}^m e^{-96.28(T-t_k)} \sqrt{t_k - t_{k-1}} \left(Z_k \sum_{i=1}^n e^{96.28(T-T_i)} \Delta_i - \sum_{i=1}^n e^{96.28(T-T_i)} \right).$$

The value of the greek 'delta' lies between -1 and 1 after computation. A delta of 0.4 implies that the interest rate will rise by #0.40 if the underlying interest rate rises by #1. The greek 'vega' determines the worth of the interest rate whenever there is 1% move in its volatility. Hence, the greeks help in monitoring derivatives in order to minimize risks.

Conclusion

Having a good interest rate model is essential for any economy since interest rates serve as basis for many transactions and decisions. The need to inculcate jumps in a model cannot be overemphasized. In this

paper, we have discussed a modified version of the existing Vasicek model to include jumps which is inherent in interest rate paths. We also derived the greeks 'delta' and 'vega' that measure the sensitivities of the interest rates to changes in the initial interest rate and volatility, respectively. These greeks are very useful in minimizing risks.

Declaration of Conflict of Interests

The authors declare no conflict of interests

Authors Contribution

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