A Generalization of Archimedes' Theorem on the Area of a Parabolic Segment

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Abstract

Archimedes' well known theorem on the area of a parabolic segment says that this area is $4/3$ of the area of a certain inscribed triangle. In this paper we generalize this theorem to the n -dimensional euclidean space, $n \geq 3$. It appears that the ratio of the volume of an n -dimensional solid bounded by an $(n-1)$ -dimensional hyper-paraboloid and an $(n-1)$ -dimensional hyperplane and the volume of a certain inscribed cone (we analogously repeat Archimedes' procedure) depends only on the dimension of the euclidean space and it equals to $2n/(n+1)$.

Introduction

Let us recall Archimedes' (287-212 BC) theorem on the area of a parabolic segment. Let P be a parabola in the euclidean plane $\mathbb{R}^2$ and let AB be a chord of P . Denote by C' the midpoint of AB . Consider the line ℓ which passes through C' and is parallel to the symmetry line of P . Denote by C the intersection of ℓ with the parabola P , see Figure 1.

The point C is often called the center of the arc AB of the parabola P . Denote by S the parabolic segment bounded by the parabola P and the chord AB . Archimedes' famous result says that

$$\text{Area}(S) = \frac{4}{3} \cdot \text{Area}(\triangle ABC) . \quad (0.1)$$

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Figure 1: Classic version of Archimedes' theorem.

It is amazing that this relationship between the area of the parabolic segment S and the area of the triangle $\triangle ABC$ does not depend on the shape of the parabola P or the chord AB . Of course Archimedes' proof does not involve the integral calculus. In fact he gave two proofs of the above theorem. One of them involves Eudoxus' (408-355 BC) method of exhaustion and another one is based on moving center of masses of certain arcs. For learning Archimedes' original considerations we refer to [3, p. 251], see also [2].

In this paper we generalize Archimedes' theorem to the n -dimensional euclidean space $\mathbb{R}^n$. Let us start with some notations. We denote by $\mathbb{N}$ the set of all positive integers. Let $n \in \mathbb{N}$, $n \geq 3$. For $a_1, \dots, a_{n-1} \in (0; +\infty)$ we set

$$P_{n-1} := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = \sum_{k=1}^{n-1} a_k x_k^2 \right\}. \quad (0.2)$$

Notice that P_{n-1} is an $(n-1)$ -dimensional elliptic hyper-paraboloid. For $b_1, \dots, b_n \in \mathbb{R}$ we consider an $(n-1)$ -dimensional hyperplane H_{n-1} given by

$$H_{n-1} := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = \sum_{k=1}^{n-1} b_k x_k + b_n \right\}. \quad (0.3)$$

Obviously, H_{n-1} is not perpendicular to the hyperplane $\{(x_1, \dots, x_{n-1}, 0) : x_i \in \mathbb{R}, i = 1, \dots, n-1\}$. Notice that $(x_1, \dots, x_n) \in P_{n-1} \cap H_{n-1}$ if and only if

$$x_n = \sum_{k=1}^{n-1} a_k x_k^2 = \sum_{k=1}^{n-1} b_k x_k + b_n,$$

which is equivalent to

$$(x_1, \dots, x_n) \in H_{n-1} \quad \text{and} \quad \sum_{k=1}^{n-1} \left(\sqrt{a_k} x_k - \frac{b_k}{2\sqrt{a_k}} \right)^2 = b_n + \sum_{k=1}^{n-1} \frac{b_k^2}{4a_k}.$$

This means that the intersection $P_{n-1} \cap H_{n-1}$ contains more than one point if and only if

$$b_n + \sum_{k=1}^{n-1} \frac{b_k^2}{4a_k} > 0. \quad (0.4)$$

Moreover, if the condition (0.4) is satisfied, then the number

$$R := \sqrt{b_n + \sum_{k=1}^{n-1} \frac{b_k^2}{4a_k}} \quad (0.5)$$

is well defined and the set

$$E_{n-2} := P_{n-1} \cap H_{n-1} = \left\{ (x_1, \dots, x_n) \in H_{n-1} : \sum_{k=1}^{n-1} \frac{(x_k - b_k/(2a_k))^2}{(R/\sqrt{a_k})^2} = 1 \right\} \quad (0.6)$$

is an $(n-2)$ -dimensional ellipsoid. In what follows we assume that the condition (0.4) is satisfied. We denote by c the center of the ellipsoid E_{n-2} , i.e.

$$c = \left(\frac{b_1}{2a_1}, \frac{b_2}{2a_2}, \dots, \frac{b_{n-1}}{2a_{n-1}}, \sum_{k=1}^{n-1} \frac{b_k^2}{2a_k} + b_n \right),$$

and we denote by $p(c)$ the intersection of the line

$$\ell_1(c) := \left\{ \left(\frac{b_1}{2a_1}, \frac{b_2}{2a_2}, \dots, \frac{b_{n-1}}{2a_{n-1}}, t \right) : t \in \mathbb{R} \right\}$$

with the hyper-paraboloid P_{n-1} . We get

$$p(c) = P_{n-1} \cap \ell_1(c) = \left(\frac{b_1}{2a_1}, \frac{b_2}{2a_2}, \dots, \frac{b_{n-1}}{2a_{n-1}}, \sum_{k=1}^{n-1} \frac{b_k^2}{4a_k} \right).$$

Finally we denote by C_n the convex hull of the union of the point $p(c)$ and the ellipsoid E_{n-2} , i.e.

$$C_n = \text{conv}(E_{n-2} \cup p(c)) := \{\lambda x + (1-\lambda)y : \lambda \in [0, 1], x, y \in E_{n-2} \cup p(c)\}. \quad (0.7)$$

Figure 2: Three dimensional version of Archimedes' theorem.

Notice that C_n is an n -dimensional cone with the vertex at $p(c)$, see Figure 2 for the 3-dimensional case.

In what follows, we refer to C_n as a cone, for short. Comparing to the 2-dimensional case, the role of the parabolic segment plays the set

$$\text{PS}_n := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{k=1}^{n-1} a_k x_k^2 \leq x_n \leq \sum_{k=1}^{n-1} b_k x_k + b_n \right\} \quad (0.8)$$

which is an n -dimensional solid body bounded by the hyper-paraboloid P_{n-1} and the hyperplane H_{n-1} . The role of the triangle $\triangle ABC$ plays the cone C_n . It appears that the ratio of the volume of the hyper-parabolic segment PS_n and the volume of the cone C_n depends only on the dimension of the euclidean space, i.e. depends only on n . More precisely we prove the following result:

$$\text{Vol}_n(\text{PS}_n) = \frac{2n}{n+1} \cdot \text{Vol}_n(C_n).$$

In particular,

$$\text{Vol}_3(\text{PS}_3) = \frac{3}{2} \cdot \text{Vol}_3(C_3).$$

Throughout this paper we write $\text{Vol}_n(E)$ for the n -dimensional Lebesgue measure of a Lebesgue measurable set E , although it would be sufficient to operate on Jordan measurable sets and the Jordan measure.

1 Main results

We will compute multiple integrals by changing variables into spherical n -dimensional coordinates, see [1]. Additionally we will scale along the axes and translate appropriate transformation, in order to integrate comfortably over

convex hulls of ellipsoids. The Jacobian (see [4, 234]) of such a transformation is basically well-known, so a technical Lemma 1.2 is not new.

Let u and v be non-negative integers. In what follows we formally replace the product $\prod_{i=u}^v$ by 1 if $v < u$, regardless what is the expression under the product symbol. We denote by $\mathbb{N}_{u,v}$ the set of all integers k such that $u \leq k \leq v$, if $u \leq v$. We set $\mathbb{N}_{u,v} := \emptyset$, if $u > v$.

Let $m \in \mathbb{N}$, $m \geq 2$, $\theta_1, \dots, \theta_{m-1} \in \mathbb{R}$ and $\theta_m := 0$ (so $\cos(\theta_m) = 1$). Define the matrix $\mathbf{M}_m : \mathbb{N}_{1,m} \times \mathbb{N}_{1,m} \rightarrow \mathbb{R}$ by

$$\mathbf{M}_m(p, q) := \begin{cases} \cos(\theta_p) \prod_{j=1}^{p-1} \sin(\theta_j), & \text{if } p \in \mathbb{N}_{1,m} \text{ and } q = 1, \\ \cos(\theta_p) \cos(\theta_{q-1}) \prod_{\substack{j=1 \\ j \neq q-1}}^{p-1} \sin(\theta_j), & \text{if } p \in \mathbb{N}_{2,m} \text{ and } q \in \mathbb{N}_{2,p}, \\ - \prod_{j=1}^p \sin(\theta_j), & \text{if } p \in \mathbb{N}_{1,m-1} \text{ and } q = p+1, \\ 0, & \text{if } p \in \mathbb{N}_{1,m-2} \text{ and } q \in \mathbb{N}_{p+2,m}. \end{cases}$$

For the convenience of the Reader we give an array form of the matrix $\mathbf{M}_m$:

$$\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 & \dots & 0 \\ \cos(\theta_2) \sin(\theta_1) & \cos(\theta_2) \cos(\theta_1) & -\prod_{j=1}^2 \sin(\theta_j) & 0 & \dots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ \cos(\theta_{m-2}) \prod_{j=1}^{m-3} \sin(\theta_j) & \cos(\theta_{m-2}) \cos(\theta_1) \prod_{\substack{j=1 \\ j \neq 1}}^{m-3} \sin(\theta_j) & \ddots & & \ddots & 0 \\ \cos(\theta_{m-1}) \prod_{j=1}^{m-2} \sin(\theta_j) & \cos(\theta_{m-1}) \cos(\theta_1) \prod_{\substack{j=1 \\ j \neq 1}}^{m-2} \sin(\theta_j) & & \ddots & & -\prod_{j=1}^{m-1} \sin(\theta_j) \\ \prod_{j=1}^{m-1} \sin(\theta_j) & \cos(\theta_1) \prod_{\substack{j=1 \\ j \neq 1}}^{m-1} \sin(\theta_j) & \dots & \dots & \cos(\theta_{m-1}) \prod_{\substack{j=1 \\ j \neq m-1}}^{m-1} \sin(\theta_j) \end{pmatrix}$$

Lemma 1.1. *The determinant of $\mathbf{M}_m$ satisfies the following equality*

$$\det(\mathbf{M}_m) = \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k).$$

Proof. Obviously, $\det(\mathbf{M}_2) = 1$. Assume $m \in \mathbb{N}$, $m \geq 3$. Using Laplace's expansion along the m th column of the matrix $\mathbf{M}_n$, we get

$$\begin{aligned} \det(\mathbf{M}_m) &= (-1)^{2m-1} \left(- \prod_{j=1}^{m-1} \sin(\theta_j) \right) \sin(\theta_{m-1}) \det(\mathbf{M}_{m-1}) \\ &\quad + (-1)^{2m} \left(\cos(\theta_{m-1}) \prod_{\substack{j=1 \\ j \neq m-1}}^{m-1} \sin(\theta_j) \right) \cos(\theta_{m-1}) \det(\mathbf{M}_{m-1}) \\ &= \prod_{j=1}^{m-2} \sin(\theta_j) \det(\mathbf{M}_{m-1}) . \end{aligned}$$

By the mathematical induction, we get the desired formula. $\square$

Let $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, $\beta := (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$. For each $k \in \mathbb{N}_{1,m}$ we set

$$\varphi_k^{(\alpha_k, \beta_k)}(r, \theta_1, \theta_2, \dots, \theta_{m-1}) := \alpha_k r \cos(\theta_k) \prod_{j=1}^{k-1} \sin(\theta_j) + \beta_k ,$$

where $(r, \theta_1, \theta_2, \dots, \theta_{m-1}) \in \mathbb{R}^m$. Recall that $\theta_m = 0$, so $\cos(\theta_m) = 1$. We define the mapping $\Phi_m^{(\alpha, \beta)}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$\Phi_m^{(\alpha, \beta)} := \left(\varphi_1^{(\alpha_1, \beta_1)}, \dots, \varphi_m^{(\alpha_m, \beta_m)} \right) . \quad (1.1)$$

So $\varphi_k^{(\alpha_k, \beta_k)}$ is the k th coordinate function of $\Phi_m^{(\alpha, \beta)}$. The vector α controls a scaling along the axes while the vector β controls a translation. Of course, the mapping $\Phi_m^{(\alpha, \beta)}$ has continuous partial derivatives of all orders.

Lemma 1.2. *The Jacobian $J_{\Phi_m^{(\alpha, \beta)}}$ of the mapping $\Phi_m^{(\alpha, \beta)}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfies the equality*

$$J_{\Phi_m^{(\alpha, \beta)}}(r, \theta_1, \theta_2, \dots, \theta_{m-1}) = r^{m-1} \left(\prod_{k=1}^m \alpha_k \right) \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k),$$

for all $(r, \theta_1, \theta_2, \dots, \theta_{m-1}) \in \mathbb{R}^m$.

Proof. For $k \in \mathbb{N}_{1,m}$ we have

$$\begin{aligned} \frac{\partial \varphi_k^{(\alpha, \beta)}}{\partial r}(r, \theta_1, \theta_2, \dots, \theta_{m-1}) &= \alpha_k \cos(\theta_k) \prod_{j=1}^{k-1} \sin(\theta_j), \\ \frac{\partial \varphi_k^{(\alpha, \beta)}}{\partial \theta_i}(r, \theta_1, \theta_2, \dots, \theta_{m-1}) &= \begin{cases} r \alpha_k \cos(\theta_k) \cos(\theta_i) \prod_{\substack{j=1 \\ j \neq i}}^{k-1} \sin(\theta_j), & \text{if } i \in \mathbb{N}_{1,k-1}, \\ -r \alpha_k \prod_{j=1}^k \sin(\theta_j), & \text{if } i = k, \\ 0, & \text{if } i \in \mathbb{N}_{k+1,m-1}. \end{cases} \end{aligned}$$

Thus, by Lemma 1.1, we get

$$\begin{aligned} J_{\Phi_m^{(\alpha, \beta)}}(r, \theta_1, \theta_2, \dots, \theta_{m-1}) &= r^{m-1} \left(\prod_{k=1}^m \alpha_k \right) \det(\mathbf{M}_m) \\ &= r^{m-1} \left(\prod_{k=1}^m \alpha_k \right) \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k). \end{aligned}$$

□

Remark 1.3. Set $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^m$ and $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^m$. Then the mapping $\Phi_m^{(\mathbf{1}, \mathbf{0})}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ restricted to the set $[0, +\infty) \times \mathbb{R}^{m-1}$ is a transformation of the classical spherical m -dimensional coordinates onto the Cartesian coordinates. In this case

$$J_{\Phi_m^{(\mathbf{1}, \mathbf{0})}}(r, \theta_1, \theta_2, \dots, \theta_{m-1}) = r^{m-1} \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k),$$

for all $(r, \theta_1, \theta_2, \dots, \theta_{m-1}) \in \mathbb{R}^m$ (compare this with the formula given in [1]).

Let $r > 0$. Notice, that the mapping $\Phi_m^{(\mathbf{1}, \mathbf{0})}$ transforms in a non-injective manner the closed m -orthotope $[0; r] \times [0; \pi]^{m-2} \times [0; 2\pi]$ onto the closed ball $\mathbb{B}_m(r) := \{x \in \mathbb{R}^m: \|x\| \leq r\}$ of the radius r (here $\|\cdot\|$ means the euclidean norm). However, the restriction of the mapping $\Phi_m^{(\mathbf{1}, \mathbf{0})}$ to the open m -orthotope

$$\Omega_m(r) := (0, r) \times (0; \pi)^{m-2} \times (0; 2\pi) \quad (1.2)$$

is a diffeomorphism. Moreover

$$\Phi_m^{(\mathbf{1}, \mathbf{0})}(\Omega_m(r)) \subset \mathbb{B}_m(r) \quad \text{and} \quad \text{Vol}_m \left(\Phi_m^{(\mathbf{1}, \mathbf{0})}(\Omega_m(r)) \right) = \text{Vol}_m(\mathbb{B}_m(r)).$$

Theorem 1.4. *Let $a_1, \dots, a_{n-1} \in (0; +\infty)$, $b_1, \dots, b_n \in \mathbb{R}$, and let the condition (0.4) be satisfied. Then*

$$\text{Vol}_n(\text{PS}_n) = \frac{2n}{n+1} \cdot \text{Vol}_n(\text{C}_n),$$

where C_n and PS_n are defined in (0.7) and (0.8), respectively.

Proof. Let us define $\text{E}_{n-1}^* := \text{conv}(\text{E}_{n-2})$. Denote by D_{n-1} the projection of E_{n-1}^* onto the hyperplane $\{(x_1, \dots, x_{n-1}, 0) : x_i \in \mathbb{R}, i = 1, \dots, n-1\}$. By (0.6), we have

$$\text{D}_{n-1} = \left\{ (x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n : \sum_{k=1}^{n-1} \left[\sqrt{a_k} \cdot \left(x_k - \frac{b_k}{2a_k} \right) \right]^2 \leq R^2 \right\},$$

where R is defined as in (0.5). Put $m := n-1$,

$$\alpha := \left(\frac{1}{\sqrt{a_1}}, \dots, \frac{1}{\sqrt{a_{n-1}}} \right) \quad \text{and} \quad \beta := \left(\frac{b_1}{2a_1}, \dots, \frac{b_{n-1}}{2a_{n-1}} \right),$$

and consider the mapping $\Phi_{n-1}^{(\alpha, \beta)} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ defined by (1.1). The restriction of $\Phi_{n-1}^{(\alpha, \beta)}$ to the open $(n-1)$ -orthotope $\Omega_{n-1}(R)$, defined by (1.2), is a diffeomorphism. Moreover,

$$\Phi_{n-1}^{(\alpha, \beta)}(\Omega_{n-1}(R)) \subset \text{D}_{n-1} \quad \text{and} \quad \text{Vol}_{n-1}(\Phi_{n-1}^{(\alpha, \beta)}(\Omega_{n-1}(R))) = \text{Vol}_{n-1}(\text{D}_{n-1}).$$

Hence, by the transformation formula (see [4, 252]) and by Lemma 1.2, we get

$$\begin{aligned} \text{Vol}_n(\text{PS}_n) &= \overbrace{\int \int \dots \int}^{n-1}_{\text{D}_{n-1}} \left(R^2 - \sum_{k=1}^{n-1} a_k \left(x_k - \frac{b_k}{2a_k} \right)^2 \right) dx_1 dx_2 \dots dx_{n-1} \\ &= \overbrace{\int \int \dots \int}^{n-1}_{\Omega_{n-1}(R)} (R^2 - r^2) \left| \text{J}_{\Phi_{n-1}^{(\alpha, \beta)}}(r, \theta_1, \theta_2, \dots, \theta_{n-1}) \right| d\theta_1 d\theta_2 \dots d\theta_{n-2} dr \\ &= \underbrace{\int_0^R \int_0^{2\pi} \int_0^\pi \dots \int_0^\pi}_{n-3} (R^2 - r^2) \left| r^{n-2} \left(\prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} \right) \prod_{k=1}^{n-3} \sin^{n-k-2}(\theta_k) \right| d\theta_1 d\theta_2 \dots d\theta_{n-2} dr \end{aligned}$$

$$\begin{aligned}
&= 2^{n-1} \underbrace{\int_0^R \int_0^{\frac{\pi}{2}} \dots \int_0^{\frac{\pi}{2}}}_{n-2} (R^2 r^{n-2} - r^n) \left(\prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} \right) \prod_{k=1}^{n-3} \sin^{n-k-2}(\theta_k) d\theta_1 d\theta_2 \dots d\theta_{n-2} dr \\
&= 2^{n-1} \cdot \left(\frac{R^{n+1}}{n-1} - \frac{R^{n+1}}{n+1} \right) \cdot \left(\prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} \right) \cdot \frac{\pi}{2} \cdot \prod_{k=1}^{n-3} \int_0^{\frac{\pi}{2}} \sin^{n-k-2}(\theta_k) d\theta_k \\
&= \frac{2^{n-1} \pi R^{n+1}}{n^2 - 1} \cdot \left(\prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} \right) \cdot \prod_{k=1}^{n-3} \frac{\sqrt{\pi} \Gamma\left(\frac{n-k-1}{2}\right)}{2 \Gamma\left(\frac{n-k}{2}\right)} \\
&= \frac{4\pi^{\frac{n-1}{2}} R^{n+1}}{(n^2 - 1) \Gamma\left(\frac{n-1}{2}\right)} \prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} .
\end{aligned}$$

If $n = 3$ then $\underbrace{\int_0^\pi \dots \int_0^\pi}_{n-3}$ just disappear and the result remains valid.

Now we determine $\text{Vol}_n(C_n)$. Since C_n is an n -dimensional cone with base E_{n-1}^* and vertex at $p(c)$. Therefore

$$\text{Vol}_n(C_n) = \frac{1}{n} \text{dist}(p(c); H_{n-1}) \cdot \text{Vol}_{n-1}(E_{n-1}^*) , \quad (1.3)$$

where $\text{dist}(p(c); H_{n-1})$ denotes the euclidean distance between $p(c)$ and H_{n-1} . Since

$$(\text{dist}(p(c); H_{n-1}))^2 = \frac{\left(\sum_{k=1}^{n-1} \frac{b_k^2}{2a_k} - \sum_{k=1}^{n-1} \frac{b_k^2}{4a_k} + b_n \right)^2}{\sum_{k=1}^{n-1} b_k^2 + 1} ,$$

so

$$\text{dist}(p(c); H_{n-1}) = R^2 \left(\sum_{k=1}^{n-1} b_k^2 + 1 \right)^{-\frac{1}{2}} . \quad (1.4)$$

Furthermore,

$$\text{Vol}_{n-1}(E_{n-1}^*) = \overbrace{\int \int \dots \int}_{D_{n-1}} \sqrt{\sum_{k=1}^{n-1} b_k^2 + 1} dx_1 dx_2 \dots dx_{n-1} .$$

Proceeding as above, we get

$$\begin{aligned}
 \text{Vol}_{n-1}(\mathbf{E}_{n-1}^*) &= \\
 &= 2^{n-1} \sqrt{\sum_{k=1}^{n-1} b_k^2 + 1} \int_0^R \underbrace{\int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}}}_{n-2} \left| J_{\Phi_{n-1}^{(\alpha, \beta)}}(r, \theta_1, \theta_2, \dots, \theta_{n-1}) \right| d\theta_1 d\theta_2 \dots d\theta_{n-2} dr \\
 &= 2^{n-1} \sqrt{\sum_{k=1}^{n-1} b_k^2 + 1} \cdot \frac{R^{n-1}}{n-1} \cdot \left(\prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} \right) \cdot \frac{\pi}{2} \cdot \prod_{k=1}^{n-3} \int_0^{\frac{\pi}{2}} \sin^{n-k-2}(\theta_k) d\theta_k \\
 &= \frac{2\pi^{\frac{n-1}{2}} R^{n-1}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \sqrt{\sum_{k=1}^{n-1} b_k^2 + 1} \prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}}.
 \end{aligned}$$

Hence, by (1.3) and (1.4), we have

$$\text{Vol}_n(C_n) = \frac{2\pi^{\frac{n-1}{2}} R^{n+1}}{n(n-1)\Gamma\left(\frac{n-1}{2}\right)} \prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}}.$$

Finally, we get

$$\frac{\text{Vol}_n(\text{PS}_n)}{\text{Vol}_n(C_n)} = \frac{2n}{n+1}.$$

□

Corollary 1.5. *In the 3-dimensional case (see Figure 2) the ratio of the volume of the parabolic segment PS_3 and the volume of the cone C_3 is equal to $3/2$.*

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