Some integral inequalities for a polynomial with zeros outside the unit disk

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Abstract. The goal of this paper is to generalize and refine some previous inequalities between the L^p -norms of the s^{th} derivative and of the polynomial itself, in the case when the zeros are outside of the open unit disk.

1 Introduction

Let $\mathbb{P}_n$ be the class of polynomials $P(z) := \sum_{v=0}^n a_v z^v$ of degree n and $P^{(s)}(z)$ is its s^{th} derivative. For $P \in \mathbb{P}_n$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1)$$

and for every $r \geq 1$,

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \quad (2)$$

The inequality (1) is a classical result of Bernstein [10], where as the inequality (2) is due to Zygmund [14] who proved it for all trigonometric polynomials of

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degree n and not only for those of the form $P(e^{i\theta})$. Arestov [1] proved that (2) remains true for $0 < r < 1$ as well. If we let $r \rightarrow \infty$ in (2) we get (1).

The above two inequalities (1) and (2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then (1) and (2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \quad (3)$$

and

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n C_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad (4)$$

where

$$C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\gamma}|^r d\gamma \right\}^{\frac{-1}{r}}. \quad (5)$$

The inequality (3) was conjectured by Erdős and later verified by Lax [9], where as (4) was proved by De-Bruijn [6] for $r \geq 1$. Further, Rahman and Schmeisser [12] have shown that (4) holds for $0 < r < 1$ as well. If we let $r \rightarrow \infty$ in inequality (4), we get (3).

In the literature, there already exists various refinements and generalisations of (3) and (4), for example see Aziz [2], Aziz and Dawood [3], Mir and Baba [11], Zireh [13] etc.

2 Main results

In this paper, we shall use a parameter β and obtain certain generalisations and refinements of inequalities (3) and (4).

Theorem 1 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $1 \leq s \leq n$ and $\gamma \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \quad (6)$$

$$\leq n(n-1)\dots(n-s+1) E_\gamma \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}},$$

where

$$E_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma d\alpha \right\}^{\frac{-1}{\gamma}}. \quad (7)$$

Instead of proving Theorem 1, we prove the following more general result which includes not only Theorem 1 as a special case, but also leads to several interesting generalisations and refinements of (3) and (4).

Theorem 2 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta, \delta \in \mathbb{C}$ with $|\beta| \leq 1, |\delta| \leq 1, 1 \leq s \leq n$ and $\gamma \geq 1$,

$$\begin{aligned} & \left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right. \right. \\ & \quad \left. \left. + \frac{\delta m n(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right)^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ & \leq n(n-1)\dots(n-s+1) E_\gamma \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \end{aligned} \quad (8)$$

where $m = \min_{|z|=1} |P(z)|$ and E_γ is defined by (7).

Now we present and discuss some consequences of these results. First, we point out that inequalities involving polynomials in the Chebyshev norm on the unit circle in the complex plane are a special case of the polynomial inequalities involving the integral norm. For example if we let $\gamma \rightarrow \infty$ in (6), noting that $E_\gamma \rightarrow \frac{1}{2}$ we get the following result.

Corollary 1 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\begin{aligned} & \max_{|z|=1} \left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| \\ & \leq \frac{n(n-1)\dots(n-s+1)}{2} \left(1 + \frac{|\beta|}{2^{s-1}} \right) \max_{|z|=1} |P(z)|. \end{aligned} \quad (9)$$

If we take $s = 1$ in (9), we get the following generalization of (3).

Corollary 2 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$,

$$\max_{|z|=1} \left| zP'(z) + \frac{\beta n}{2} P(z) \right| \leq \frac{n}{2} (1 + |\beta|) \max_{|z|=1} |P(z)|. \quad (10)$$

Remark 1 For $\beta = 0$, (10) reduces to (3). It should be noted that inequality (10) can also be obtained by simply applying the triangle inequality to the left hand side of it and estimating the first of the resulting terms directly by inequality (3).

Next, we show that Theorem 2 implies other inequalities in the Chebyshev norm on the unit circle of a polynomial. If we let $\gamma \rightarrow \infty$ in (8) and choose the argument of δ suitably with $|\delta| = 1$, we get the following result.

Corollary 3 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\begin{aligned} \max_{|z|=1} \left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| \\ \leq \frac{n(n-1)\dots(n-s+1)}{2} \left\{ \left(1 + \frac{|\beta|}{2^{s-1}} \right) \max_{|z|=1} |P(z)| \right. \\ \left. - \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) m \right\}, \end{aligned} \quad (11)$$

where $m = \min_{|z|=1} |P(z)|$.

Taking $s = 1$ in inequality (11), we get the following refinement of (10).

Corollary 4 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$,

$$\max_{|z|=1} \left| zP'(z) + \frac{\beta n}{2} P(z) \right| \leq \frac{n}{2} \left\{ (1 + |\beta|) \max_{|z|=1} |P(z)| - \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right) m \right\}, \quad (12)$$

where $m = \min_{|z|=1} |P(z)|$.

When $\beta = 0$, the above inequality (12) recovers a result of Aziz and Dawood [1]. Several other interesting results easily follow from Theorem 2 and here, we mention a few of these. Taking $\beta = 0$ in (8), we immediately get the following result.

Corollary 5 c2.5 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, $1 \leq s \leq n$ and $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\delta m n(n-1)\dots(n-s+1)}{2} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n(n-1)\dots(n-s+1) E_\gamma \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \quad (13)$$

where $m = \min_{|z|=1} |P(z)|$ and E_γ is defined by (7).

For $s = 1$ and $\delta = 0$, inequality (13) reduces to inequality (4).

Letting $\gamma \rightarrow \infty$ in (13) and choosing the argument of δ with $\delta = 1$, we get the following interesting generalization of a result of Aziz and Dawood [1].

Corollary 6 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $1 \leq s \leq n$, we have

$$\max_{|z|=1} |P^{(s)}(z)| \leq \frac{n(n-1)\dots(n-s+1)}{2} \left(\max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right). \quad (14)$$

For the proof of Theorem 2, we need the following lemmas.

3 Lemmas

Lemma 1 Let $F \in \mathbb{P}_n$ and $F(z)$ has all its zeros in $|z| \leq 1$. If $P(z)$ is a polynomial of degree n such that

$$|P(z)| \leq |F(z)| \text{ for } |z| = 1,$$

then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| \leq \left| z^s F^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} F(z) \right| \text{ for } |z| \geq 1.$$

The above lemma is due to Hans and Lal [8].

By applying Lemma 1 to polynomials $P(z)$ and $z^n \min_{|z|=1} |P(z)|$, we get the following result.

Lemma 2 If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$,

$$\left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| \geq n(n-1)\dots(n-s+1)|z|^n \left| 1 + \frac{\beta}{2^s} \right| \min_{|z|=1} |P(z)| \text{ for } |z| \geq 1.$$

Lemma 3 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$ and $|z| = 1$,

$$\begin{aligned} & \left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| \\ & \leq \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(z) \right| \\ & \quad - n(n-1)\dots(n-s+1) \left\{ \left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right\} m, \end{aligned}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ and $m = \min_{|z|=1} |P(z)|$.

Proof of Lemma 3.3. If $P(z)$ has a zero on $|z| = 1$, then $m = 0$ and the result follows by Lemma 1. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < 1$ and so $m > 0$, we have $|\lambda m| < |P(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. It follows by Rouché's theorem that the polynomial $G(z) = P(z) - \lambda m$ has no zeros in $|z| < 1$. Therefore, the polynomial $H(z) = z^n \overline{G(\frac{1}{\bar{z}})} = Q(z) - \bar{\lambda} m z^n$ will have all its zeros in $|z| \leq 1$. Also $|G(z)| = |H(z)|$ for $|z| = 1$. On applying Lemma 1, we get for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$ and $|z| \geq 1$,

$$\begin{aligned} & \left| z^s G^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} G(z) \right| \\ & \leq \left| z^s H^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} H(z) \right|. \end{aligned}$$

Equivalently

$$\begin{aligned} & \left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} (P(z) - \lambda m) \right| \\ & \leq \left| \left(z^s Q^{(s)}(z) - \bar{\lambda} m n(n-1)\dots(n-s+1) z^n \right) \right| \end{aligned}$$

$$+ \frac{\beta n(n-1)\dots(n-s+1)}{2^s} (Q(z) - \bar{\lambda} m z^n) \Big|.$$

This implies that

$$\begin{aligned} & \left| \left(z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right) \right. \\ & \quad \left. - \frac{\beta n(n-1)\dots(n-s+1)}{2^s} \lambda m \right| \\ & \leq \left| \left(z^s Q^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(z) \right) \right. \\ & \quad \left. - \bar{\lambda} m n(n-1)\dots(n-s+1) z^n \left(1 + \frac{\beta}{2^s} \right) \right|. \end{aligned} \quad (15)$$

Since $Q(z)$ has all its zeros in $|z| \leq 1$, therefore, by Lemma 3.2 we have for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned} & \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(z) \right| \\ & \geq n(n-1)\dots(n-s+1) |z|^n \left| 1 + \frac{\beta}{2^s} \right| \min_{|z|=1} |Q(z)| \\ & = n(n-1)\dots(n-s+1) |z|^n \left| 1 + \frac{\beta}{2^s} \right| m. \end{aligned} \quad (16)$$

Now choosing a suitable argument of λ in the left-hand side of (15), in view of (16), we get for $|z| = 1$,

$$\begin{aligned} & \left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| - |\lambda| n(n-1)\dots(n-s+1) \left| \frac{\beta}{2^s} \right| m \\ & \leq \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(z) \right| \\ & \quad - n(n-1)\dots(n-s+1) \left| 1 + \frac{\beta}{2^s} \right| |\lambda| m. \end{aligned}$$

Equivalently

$$\left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right|$$

$$\leq \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(z) \right| \\ - n(n-1)\dots(n-s+1) \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) |\lambda| m.$$

Letting $|\lambda| \rightarrow 1$, we get for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1$,

$$\left| z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(z) \right| \\ \leq \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(z) \right| \\ - n(n-1)\dots(n-s+1) \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) m,$$

which completes the proof of Lemma 3

The following lemma is due to Aziz and Rather [5].

Lemma 4 *If A, B, C are non-negative real numbers such that $B + C \leq A$, then for every real number α ,*

$$|(A - C)e^{i\alpha} + (B + C)| \leq |Ae^{i\alpha} + B|.$$

Lemma 5 *If $P \in \mathbb{P}_n$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for each $\alpha, 0 \leq \alpha < 2\pi$ and $\gamma > 0$, we have*

$$\int_0^{2\pi} \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta})|^\gamma d\theta d\alpha \leq 2\pi n^\gamma \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta.$$

The above lemma is due to Aziz and Rather [4].

4 Proof of the Theorem

Proof of Theorem 2.2. Since $P \in \mathbb{P}_n$, $P(z) \neq 0$ in $|z| < 1$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, therefore, for each $\alpha, 0 \leq \alpha < 2\pi$, $F(z) = P(z) + e^{i\alpha} Q(z) \in \mathbb{P}_n$ and we have

$$F^{(s)}(z) = P^{(s)}(z) + e^{i\alpha} Q^{(s)}(z),$$

which is clearly a polynomial of degree $n - s$, $1 \leq s \leq n$. By the repeated application of inequality (2), we have for each $\gamma \geq 1$,

$$\begin{aligned}
 & \int_0^{2\pi} |P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta})|^\gamma d\theta \\
 & \leq (n - s + 1)^\gamma \int_0^{2\pi} |P^{(s-1)}(e^{i\theta}) + e^{i\alpha} Q^{(s-1)}(e^{i\theta})|^\gamma d\theta \\
 & \vdots \\
 & \leq (n - s + 1)^\gamma (n - s + 2)^\gamma \dots (n - 1)^\gamma \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta})|^\gamma d\theta.
 \end{aligned} \tag{17}$$

Integrating (17) with respect to α on $[0, 2\pi]$ and using Lemma 4 we get

$$\begin{aligned}
 & \int_0^{2\pi} \int_0^{2\pi} |P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta})|^\gamma d\theta d\alpha \\
 & \leq 2\pi (n - s + 1)^\gamma (n - s + 2)^\gamma \dots (n - 1)^\gamma n^\gamma \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta.
 \end{aligned} \tag{18}$$

Now by Lemma 3 for each θ , $0 \leq \theta < 2\pi$, $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\begin{aligned}
 & \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \\
 & \quad + \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \\
 & \leq \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \\
 & \quad - \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right).
 \end{aligned} \tag{19}$$

Taking

$$A = \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right|,$$

$$B = \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right|$$

and

$$C = \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right)$$

in Lemma 3.4, so that by (19),

$$B + C \leq A - C \leq A,$$

we get for every real α ,

$$\begin{aligned} & \left| \left\{ \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \right. \right. \\ & \quad \left. \left. + \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right\} \right. \\ & \quad \left. + e^{i\alpha} \left\{ \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \right. \right. \\ & \quad \left. \left. - \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right\} \right| \\ & \leq \left| \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \right. \\ & \quad \left. + e^{i\alpha} \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \right|. \end{aligned}$$

This implies for each $\gamma \geq 1$,

$$\begin{aligned} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^\gamma & \leq \int_0^{2\pi} \left| \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \right. \\ & \quad \left. + e^{i\alpha} \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \right|^\gamma d\theta, \end{aligned} \quad (20)$$

where

$$F(\theta) = \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \\ + \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right),$$

and

$$G(\theta) = \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \\ - \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right).$$

Integrating both sides of (20) with respect to α from 0 to 2π , we get for each $\gamma \geq 1$,

$$\int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\alpha} G(\theta)|^\gamma d\theta d\alpha \\ \leq \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \right. \\ \left. + e^{i\alpha} \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \right|^\gamma d\alpha \right\} d\theta \\ = \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| \left(e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right) \right. \right. \\ \left. \left. + e^{i\alpha} \left(e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right) \right|^\gamma d\alpha \right\} d\theta \\ = \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| e^{is\theta} \left(P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta}) \right) \right. \right. \\ \left. \left. + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} \left(P(e^{i\theta}) + e^{i\alpha} Q(e^{i\theta}) \right) \right|^\gamma d\alpha \right\} d\theta.$$

Therefore, it follows by Minkowski's inequality that for $\gamma \geq 1$,

$$\begin{aligned}
 & \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^\gamma d\theta d\alpha \right\}^{\frac{1}{\gamma}} \\
 & \leq \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| e^{is\theta} \left(P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta}) \right) \right. \right. \\
 & \quad \left. \left. + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} \left(P(e^{i\theta}) + e^{i\alpha} Q(e^{i\theta}) \right) \right|^\gamma d\theta d\alpha \right\}^{\frac{1}{\gamma}} \quad (21) \\
 & \leq \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta}) \right|^\gamma d\theta d\alpha \right\}^{\frac{1}{\gamma}} \\
 & \quad + \frac{|\beta| n(n-1)\dots(n-s+1)}{2^s} \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) + e^{i\alpha} Q(e^{i\theta}) \right|^\gamma d\theta d\alpha \right\}^{\frac{1}{\gamma}}.
 \end{aligned}$$

For $0 \leq \theta < 2\pi$, it can be easily verified that

$$nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) = e^{i(n-1)\theta} \overline{Q'(e^{i\theta})}$$

and

$$nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) = e^{i(n-1)\theta} \overline{P'(e^{i\theta})}.$$

Hence

$$\begin{aligned}
 nP(e^{i\theta}) + e^{i\alpha} nQ(e^{i\theta}) &= e^{i\theta} P'(e^{i\theta}) + e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} \\
 &\quad + e^{i\alpha} (e^{i\theta} Q'(e^{i\theta}) + e^{i(n-1)\theta} \overline{P'(e^{i\theta})}),
 \end{aligned}$$

which gives

$$\begin{aligned}
 n|P(e^{i\theta}) + e^{i\alpha} Q(e^{i\theta})| &\leq |P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta})| + |\overline{Q'(e^{i\theta})} + e^{i\alpha} \overline{P'(e^{i\theta})}| \\
 &= 2|P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta})|. \quad (22)
 \end{aligned}$$

Using (18), (22) and Lemma in (21), we get for every $\gamma \geq 1, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\left\{ \int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^\gamma d\theta d\alpha \right\}^{\frac{1}{\gamma}} \leq (2\pi)^{\frac{1}{\gamma}} n(n-1)\dots(n-s+1) \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}. \quad (23)$$

Now for every real α and $t \geq 1$, it is easy to verify that

$$|t + e^{i\alpha}| \geq |1 + e^{i\alpha}|.$$

Observe that for every $\gamma \geq 1$ and $a, b \in \mathbb{C}$ such that $|b| \geq |a|$, we have

$$\int_0^{2\pi} |a + be^{i\alpha}|^\gamma d\alpha \geq |a|^\gamma \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma d\alpha. \quad (24)$$

Indeed, if $a = 0$, the above inequality (24) is obvious. In case of $a \neq 0$, we get

$$\begin{aligned} \int_0^{2\pi} \left| 1 + e^{i\alpha} \frac{b}{a} \right| d\alpha &= \int_0^{2\pi} \left| 1 + e^{i\alpha} \left| \frac{b}{a} \right| \right| d\alpha = \int_0^{2\pi} \left| \left| \frac{b}{a} \right| + e^{i\alpha} \right| d\alpha \\ &\geq \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma d\alpha. \end{aligned}$$

If we take

$$a = F(\theta),$$

$$b = G(\theta),$$

because $|b| \geq |a|$ from (19), we get from (24) that

$$\int_0^{2\pi} |F(\theta) + e^{i\alpha} G(\theta)|^\gamma d\alpha \geq |F(\theta)|^\gamma \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma d\alpha. \quad (25)$$

Integrating both sides of (25) with respect to θ from 0 to 2π , we get from (23), that

$$\begin{aligned} & \left\{ \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma d\alpha \int_0^{2\pi} \left(\left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \right. \\ & \quad \left. + \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right)^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ & \leq (2\pi)^{\frac{1}{\gamma}} n(n-1)\dots(n-s+1) \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}. \end{aligned} \quad (26)$$

Now using the fact that for every $\delta \in \mathbb{C}$ with $|\delta| \leq 1$,

$$\begin{aligned} & \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right. \\ & \quad \left. + \frac{\delta mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right| \leq \left| e^{is\theta} P^{(s)}(e^{i\theta}) \right. \\ & \quad \left. + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \\ & \quad + \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right), \end{aligned}$$

we get from (26) that for every $\gamma \geq 1$,

$$\begin{aligned} & \left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right. \right. \\ & \quad \left. \left. + \frac{\delta mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ & \leq \frac{(2\pi)^{\frac{1}{\gamma}} n(n-1)\dots(n-s+1)}{\left\{ \int_0^{2\pi} |1 + e^{i\alpha}|^\gamma d\alpha \right\}^{\frac{1}{\gamma}}} \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \end{aligned}$$

which is equivalent to (8) and this completes the proof of Theorem 2

Note. Given a polynomial $P(z) = \sum_{v=0}^n \alpha_v z^v \in \mathbb{P}_n$, we associate with it the polynomial

$$\bar{P}(z) := \overline{P(\bar{z})} = \sum_{v=0}^n \bar{\alpha}_v z^v$$

and

$$Q(z) := z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = \sum_{v=0}^n \bar{\alpha}_{n-v} z^v.$$

If $P(z) \equiv \alpha Q(z)$, where $|\alpha| = 1$, then $P(z)$ is said to be self-inversive.

It was shown by Dewan and Govil [7] that the inequality (4) is still valid if the condition that $P(z) \neq 0$ in $|z| < 1$ is replaced by the condition that $P(z) = \alpha Q(z)$, $|\alpha| = 1$. Here we present the following result for self-inversive polynomials.

Proposition. If $P \in \mathbb{P}_n$ is self-inverse, then for every $\beta \in \mathbb{C}$, $1 \leq s \leq n$ and $\gamma \geq 1$,

$$\begin{aligned} & \left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ & \leq n(n-1)\dots(n-s+1) E_\gamma \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \end{aligned}$$

where E_γ is defined by (7).

Proof. Since $P \in \mathbb{P}_n$ is a self-inversive polynomial, therefore, $P(z) \equiv \alpha Q(z)$, where $|\alpha| = 1$ and $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} \in \mathbb{P}_n$. This implies for every $\beta \in \mathbb{C}$ and $1 \leq s \leq n$,

$$\begin{aligned} & \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \\ & = \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right|, \end{aligned}$$

for all $z \in \mathbb{C}$ so that

$$\frac{\left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right|}{\left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right|} = 1.$$

Now proceeding similarly as in the proof of Theorem 2, the preposition follows.

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