Orthogonal decomposition of the sum-symmetry model using the two-parameters sum-symmetry model for ordinal square contingency tables

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SUMMARY

Studies have been carried out on decomposing a model with symmetric structure using a model with asymmetric structure. In the existing decomposition theorem, the sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and concordance-discordance models hold. However, this existing decomposition theorem does not satisfy the asymptotic equivalence for the test statistic, namely that the value of the likelihood ratio chi-squared statistic of the sum-symmetry model is asymptotically equivalent to the sum of those of the decomposed models. To address this issue, this study introduces a new decomposition theorem in which the sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and weighted global-sum-symmetry models hold. The proposed decomposition theorem satisfies the asymptotic equivalence for the test statistic—the value of the likelihood ratio chi-squared statistic of the sum-symmetry model is asymptotically equivalent to the sum of those of the two-parameters sum-symmetry, global symmetry and weighted global-sum-symmetry models.

Key words: asymptotic equivalence, asymmetry, orthogonality, symmetry, test statistic

1. Introduction

Square contingency tables with the same row and column ordinal classifications are obtained by means of cross-classified matched pairs of ordinal categorical variables. In the analysis of square contingency tables, a model with symmetric structure or a model with asymmetric structure is used.

This is because the null association (independence) model does not generally hold for square contingency tables.

The symmetry model (Bowker, 1948) is the origin of a model with symmetric structure for square contingency tables. Since then, various models with symmetric or asymmetric structure have been considered. For example, as models with symmetric structure, studies have proposed the quasi-symmetry model (Causinus, 1965), the marginal homogeneity model (Stuart, 1955), and the sum-symmetry model (Yamamoto et al., 2013); as models with asymmetric structure, studies have proposed the conditional symmetry model (McCullagh, 1978), the linear diagonals-parameter symmetry model (Agresti, 1983), the two-parameters symmetry model (Tomizawa, 1987), the conditional sum-symmetry model (Yamamoto et al., 2013), the exponential sum-symmetry model (Yamamoto et al., 2016), and the two-parameters sum-symmetry model (Yamamoto et al., 2016).

This study focuses on decomposing a model with symmetric structure using a model with asymmetric structure. The decomposition of a model is useful for searching for the reason why the model does not fit the data. Concerning the decomposition of a model with symmetric structure using a model with asymmetric structure, Read (1977) gave a decomposition theorem—the symmetry model holds if and only if both the conditional symmetry and global symmetry models hold—and showed the asymptotic equivalence for the test statistic: the value of the likelihood ratio chi-squared statistic for the symmetry model is equal to the sum of those for the conditional symmetry and global symmetry models. Using this decomposition theorem, we can see whether the fact that the symmetry model does not hold is caused by the conditional symmetry model or the global symmetry model.

Yamamoto et al. (2013) gave a decomposition theorem—the sum-symmetry model holds if and only if both the conditional sum-symmetry and global symmetry models hold—and showed the asymptotic equivalence for the test statistic: the value of the likelihood ratio chi-squared statistic for the sum-symmetry model is equal to the sum of those for the conditional sum-symmetry and global symmetry models.

Yamamoto et al. (2016) gave a decomposition theorem—the sum-symmetry model holds if and only if both the exponential sum-symmetry and global symmetry models hold. Ando (2021) pointed out that this decomposition does not satisfy the asymptotic equivalence for the test statistic. To address this issue, Ando (2021) gave an alternative decomposition the-

orem satisfying the asymptotic equivalence for the test statistic—the sum-symmetry model holds if and only if both the exponential sum-symmetry and weighted global sum-symmetry models hold.

Moreover, Yamamoto et al. (2016) gave a decomposition theorem—the sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and concordance-discordance models hold. We note that this decomposition also fails to satisfy the asymptotic equivalence for the test statistic. To address this issue, this study provides an alternative decomposition theorem satisfying the asymptotic equivalence for the test statistic—the sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and weighted global sum-symmetry models hold.

The rest of this paper is organized as follows. Section 2 provides the alternative decomposition theorem satisfying the asymptotic equivalence for the test statistic. Section 3 demonstrates the utility of the proposed decomposition theorem by applying it to real data. We close with concluding remarks in section 4.

2. Orthogonal decomposition of the sum-symmetry model

Consider $R \times R$ square contingency tables. Let π_{ij} denote the probability that an observation will fall in the (i, j) th cell of the table for $i = 1, \dots, R; j = 1, \dots, R$. Bowker (1948) proposed the symmetry model defined by

$$\pi_{ij} = \pi_{ji} \quad \text{for } i \neq j.$$

This model indicates the symmetric structure of the cell probabilities with respect to the main-diagonal cells of the table. The quasi-symmetry model (Causinus, 1965) and the marginal homogeneity model (Stuart, 1955) indicate the symmetric structure of the odds ratios and marginal probabilities, respectively.

As a model with an asymmetric structure of the cell probabilities with respect to the main-diagonal cells of the table, we consider the following formula:

$$\pi_{ij} = \delta_{ij} \pi_{ji} \quad \text{for } i < j.$$

This formula includes some models, for example: (i) the model with $\delta_{ij} = 1$, for all $i < j$, is identical to the symmetry model, (ii) the model with $\delta_{ij} = \gamma$,

for all $i < j$, is identical to the conditional symmetry model (McCullagh, 1978), (iii) the model with $\delta_{ij} = \delta^{j-i}$, for all $i < j$, is identical to the linear diagonals-parameter symmetry model (Agresti, 1983).

Tomizawa (1987) considered the two-parameters symmetry model defined by

$$\pi_{ij} = \gamma \delta^{j-i} \pi_{ji} \quad \text{for } i < j.$$

This model has the following characteristics: (i) this model with $\gamma = 1$ and $\delta^{j-i} = 1$, for all $i < j$, is identical to the symmetry model, (ii) this model with $\delta^{j-i} = 1$, for all $i < j$, is identical to the conditional symmetry model, and (iii) this model with $\gamma = 1$ is identical to the linear diagonals-parameter symmetry model.

Let

$$A_t = \sum_{(i,j) \in a_t} \pi_{ij} \quad \text{and} \quad B_t = \sum_{(i,j) \in b_t} \pi_{ij} \quad \text{for } t = 3, 4, \dots, 2R-1,$$

where

$$a_t = \{(i, j) \mid i + j = t, i < j\} \quad \text{and} \quad b_t = \{(i, j) \mid i + j = t, i > j\}.$$

Yamamoto et al. (2013) proposed the sum-symmetry and conditional sum-symmetry models defined by

$$A_t = B_t \quad \text{and} \quad A_t = \Gamma B_t \quad \text{for } t = 3, 4, \dots, 2R-1,$$

respectively. Yamamoto et al. (2016) proposed the exponential sum-symmetry and two-parameters sum-symmetry models defined by

$$A_t = \Delta^{t-2} B_t \quad \text{and} \quad A_t = \Gamma \Delta^{t-2} B_t \quad \text{for } t = 3, 4, \dots, 2R-1,$$

respectively. The two-parameters sum-symmetry model has the following characteristics: (i) the model with $\Gamma = 1$ and $\Delta^{t-2} = 1$, for all $t = 3, 4, \dots, 2R-1$, is identical to the sum-symmetry model, (ii) the model with $\Delta^{t-2} = 1$, for all $t = 3, 4, \dots, 2R-1$, is identical to the conditional sum-symmetry model, and (iii) the model with $\Gamma = 1$ is identical to the exponential sum-symmetry model.

The global symmetry and concordance-discordance models are defined by

$$\sum_{t=3}^{2R-1} A_t = \sum_{t=3}^{2R-1} B_t \quad \text{and} \quad 2 \sum_{s=3}^{2R-2} \sum_{t=s+1}^{2R-1} A_s B_t = 2 \sum_{s=3}^{2R-2} \sum_{t=s+1}^{2R-1} B_s A_t,$$

respectively. Yamamoto et al. (2016) gave the decomposition theorem—the sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and concordance-discordance models hold; however, this decomposition does not satisfy the asymptotic equivalence for the test statistic. To address this issue, we propose a new decomposition theorem of the sum-symmetry model using the two-parameters sum-symmetry model. Ando (2021) introduced the weighted global sum-symmetry model defined by

$$\sum_{t=3}^{2R-1} (t-2)A_t = \sum_{t=3}^{2R-1} (t-2)B_t.$$

We obtain the following lemma.

Lemma 2.1. *The sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and weighted global sum-symmetry models hold.*

Proof. The necessary condition is clear: If the sum-symmetry model holds, then all of the two-parameters sum-symmetry, global symmetry, and weighted global sum-symmetry models hold. We need to establish that the sufficient condition also holds: If all of the two-parameters sum-symmetry, global symmetry, and weighted global sum-symmetry models hold, then the sum-symmetry model holds.

Since both the two-parameters sum-symmetry and global symmetry models hold, we obtain the following equality:

$$\Gamma = \frac{\sum_{t=3}^{2R-1} B_t}{\sum_{t=3}^{2R-1} \Delta^{t-2} B_t}. \quad (1)$$

On the other hand, since both the two-parameters sum-symmetry and weighted global sum-symmetry models hold, we obtain the following equality:

$$\Gamma = \frac{\sum_{t=3}^{2R-1} (t-2)B_t}{\sum_{t=3}^{2R-1} (t-2)\Delta^{t-2} B_t}. \quad (2)$$

From the equalities (1) and (2), we obtain the following equality:

$$\begin{aligned}
& \left(\sum_{s=3}^{2R-1} B_s \right) \left(\sum_{t=3}^{2R-1} (t-2) \Delta^{t-2} B_t \right) = \left(\sum_{t=3}^{2R-1} \Delta^{t-2} B_t \right) \left(\sum_{s=3}^{2R-1} (s-2) B_s \right) \\
& \iff \sum_{t=3}^{2R-1} \left[\left(\sum_{s=3}^{2R-1} (t-s) B_s \right) \Delta^{t-2} B_t \right] = 0 \\
& \iff \sum_{t=3}^{2R-1} \Delta^{t-2} B_t C_t = 0
\end{aligned} \tag{3}$$

It must be noted that $C_t = \sum_{s=3}^{2R-1} (t-s) B_s$. Since $\sum_{s=3}^{2R-1} \sum_{t=3}^{2R-1} (t-s) B_s B_t = 0$, the equality (3) is also expressed as follows:

$$(\Delta - 1) \left[\sum_{t=4}^{2R-1} (B_{2R-1} C_{2R-1} + \cdots + B_t C_t) \Delta^{t-3} \right] = 0.$$

For $t = 4, \dots, 2R-1$,

$$\begin{aligned}
B_{2R-1} C_{2R-1} + \cdots + B_t C_t &= \sum_{k=t}^{2R-1} \sum_{s=3}^{2R-1} (k-s) B_k B_s \\
&= \sum_{k=t}^{2R-1} \sum_{s=3}^{t-1} (k-s) B_k B_s + \sum_{k=t}^{2R-1} \sum_{s=t}^{2R-1} (k-s) B_k B_s.
\end{aligned} \tag{4}$$

The first term on the right-hand side of the equality (4) is positive, and the second term is zero. Therefore, we obtain $\Delta = 1$. Additionally, from the equalities (1) and (2), we obtain $\Gamma = 1$. From aforementioned result, we obtain $\Gamma = 1$ and $\Delta = 1$ (i.e., $A_t = B_t$ for $t = 3, \dots, 2R-1$). The proof is complete. $\square$

The numbers of degrees of freedom for testing the goodness-of-fit of the sum-symmetry, conditional sum-symmetry, exponential sum-symmetry, two-parameters sum-symmetry, global symmetry, and weighted global sum-symmetry models are $2R-3$, $2R-4$, $2R-4$, $2R-5$, 1, and 1, respectively. It must be noted that the number of degrees of freedom for the sum-symmetry model is equal to the sum of those for the two-parameters sum-symmetry, global symmetry and weighted sum-symmetry models.

Each f_{ij} denotes the observed frequency in the (i, j) cell of the table $(i = 1, \dots, R; j = 1, \dots, R)$, with a sample size $N (= \sum \sum f_{ij})$. We assume multinomial sampling over the cells of the square contingency table. In other

words, the observed frequencies $\{f_{ij}\}$ have a multinomial distribution with parameters that are cell probabilities $\{\pi_{ij}\}$.

Each model can be tested for goodness-of-fit by means of, for example, the likelihood ratio chi-squared statistic with the corresponding degrees of freedom. The likelihood ratio chi-squared statistic of the model M is given by

$$G^2(M) = 2 \sum_{i=1}^R \sum_{j=1}^R f_{ij} \log \left(\frac{f_{ij}}{\hat{e}_{ij}} \right),$$

where $\hat{e}_{ij}$ is the maximum likelihood estimate of the expected frequency e_{ij} under the model M.

Assume that the model M_1 holds if and only if both the models M_2 and M_3 hold and the following asymptotic equivalence holds:

$$G^2(M_1) \simeq G^2(M_2) + G^2(M_3), \quad (5)$$

where the number of degrees of freedom for the model M_1 is equal to the sum of those for the models M_2 and M_3 . Darroch and Silvey (1963) stated that (i) when the expression (5) holds, if both the models M_2 and M_3 are accepted (at the α significance level) with high probability, then the model M_1 would be accepted. However, (ii) when the expression (5) does not hold, it will most likely give rise to an incompatible situation wherein both the models M_2 and M_3 are accepted with high probability, but the M_1 model is rejected with high probability. This has been exemplified by Darroch and Silvey (1963) and Tahata et al. (2011). Therefore, it is preferable that the decomposition theorem satisfy the expression (5). We shall refer to the decomposition of a model satisfying the expression (5) as orthogonal decomposition.

We need to consider modifying Lemma 2.1; this is because Lemma 2.1 does not satisfy the expression (5). We consider a model that simultaneously satisfies both the global symmetry and weighted global sum-symmetry models, defined by

$$\sum_{t=3}^{2R-1} A_t = \sum_{t=3}^{2R-1} B_t \quad \text{and} \quad \sum_{t=3}^{2R-1} (t-2)A_t = \sum_{t=3}^{2R-1} (t-2)B_t.$$

We shall refer to this model as the double global symmetry model. The number of degrees of freedom for testing the goodness-of-fit of the double global sum-symmetry model is 2. From Lemma 2.1, we obtain the following corollary.

Corollary 2.1. *The sum-symmetry model holds if and only if both the two-parameters sum-symmetry and double global symmetry models hold.*

We obtain the following theorem.

Theorem 2.1. *The following asymptotic equivalence holds:*

$$G^2(\text{SS}) \simeq G^2(\text{TSS}) + G^2(\text{DGS}),$$

where $G^2(\text{SS})$, $G^2(\text{TSS})$ and $G^2(\text{DGS})$ are the likelihood ratio chi-squared statistics of the sum-symmetry, two-parameters sum-symmetry and double global symmetry models, respectively.

Proof. Let $\boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \dots, \pi_{1R}, \dots, \pi_{R1}, \pi_{R2}, \dots, \pi_{RR})^T$, and let $\mathbf{A}^T$ denote the transpose of the matrix (or vector) $\mathbf{A}$. Under the two-parameters sum-symmetry model,

$$\Gamma = \left(\frac{A_3}{B_3} \right)^2 \frac{B_4}{A_4} \quad \text{and} \quad \Delta = \frac{B_3 A_4}{A_3 B_4}.$$

Therefore, the two-parameters sum-symmetry model is expressed as follows:

$$\mathbf{h}_1(\boldsymbol{\pi}) = \mathbf{0}_{2R-5},$$

where

$$\mathbf{h}_1(\boldsymbol{\pi}) = (h_{1,5}(\boldsymbol{\pi}), h_{1,6}(\boldsymbol{\pi}), \dots, h_{1,2R-1}(\boldsymbol{\pi}))^T$$

with

$$h_{1,t}(\boldsymbol{\pi}) = (A_3)^{t-4} (B_4)^{t-3} A_t - (B_3)^{t-4} (A_4)^{t-3} B_t,$$

and $\mathbf{0}_d$ is an $d \times 1$ vector, with all components being zero.

The double global symmetry model is expressed as

$$\mathbf{h}_2(\boldsymbol{\pi}) = \mathbf{0}_2,$$

where

$$\mathbf{h}_2(\boldsymbol{\pi}) = (h_{2,1}(\boldsymbol{\pi}), h_{2,2}(\boldsymbol{\pi}))^T$$

with

$$\begin{aligned} h_{2,1}(\boldsymbol{\pi}) &= \sum_{t=3}^{2R-1} A_t - \sum_{t=3}^{2R-1} B_t, \\ h_{2,2}(\boldsymbol{\pi}) &= \sum_{t=3}^{2R-1} (t-2) A_t - \sum_{t=3}^{2R-1} (t-2) B_t. \end{aligned}$$

From Corollary 2.1, the sum-symmetry model is expressed as

$$\mathbf{h}_3(\boldsymbol{\pi}) = \left(\mathbf{h}_1(\boldsymbol{\pi})^T, \mathbf{h}_2(\boldsymbol{\pi})^T \right)^T = \mathbf{0}_{2R-3}.$$

Let $\mathbf{H}_s(\boldsymbol{\pi})$ ($s = 1, 2, 3$) denote the matrix of partial derivatives of $\mathbf{h}_s(\boldsymbol{\pi})$ with respect to $\boldsymbol{\pi}$ (i.e., $\mathbf{H}_s(\boldsymbol{\pi}) = \partial \mathbf{h}_s(\boldsymbol{\pi}) / \partial \boldsymbol{\pi}^T$). Let $\boldsymbol{\Sigma}(\boldsymbol{\pi})$ be $\text{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi} \boldsymbol{\pi}^T$, where $\text{diag}(\boldsymbol{\pi})$ is a diagonal matrix, with the i th component of $\boldsymbol{\pi}$ as the i th diagonal element. We denote $\mathbf{p}$ as $\boldsymbol{\pi}$, with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$, where $p_{ij} = f_{ij}/N$. Using the delta method, $\sqrt{N}(\mathbf{h}_3(\mathbf{p}) - \mathbf{h}_3(\boldsymbol{\pi}))$ has an asymptotically normal distribution, with mean $\mathbf{0}_{2R-3}$ and covariance matrix

$$\mathbf{H}_3(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_3(\boldsymbol{\pi})^T = \begin{bmatrix} \mathbf{H}_1(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_1(\boldsymbol{\pi})^T & \mathbf{H}_1(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_2(\boldsymbol{\pi})^T \\ \mathbf{H}_2(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_1(\boldsymbol{\pi})^T & \mathbf{H}_2(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_2(\boldsymbol{\pi})^T \end{bmatrix}.$$

All the elements of $\mathbf{H}_1(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_2(\boldsymbol{\pi})^T$ are equal to 0, under the sum-symmetry model. This is because we obtain the following equalities:

$$\frac{\partial h_{1,t}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}^T} \text{diag}(\boldsymbol{\pi}) \frac{\partial h_{2,1}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = 0 \quad \text{for } t = 5, \dots, 2R-1,$$

$$\frac{\partial h_{1,t}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}^T} \text{diag}(\boldsymbol{\pi}) \frac{\partial h_{2,2}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = 0 \quad \text{for } t = 5, \dots, 2R-1,$$

$$\boldsymbol{\pi}^T \frac{\partial h_{2,1}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_{t=3}^{2R-1} A_t - \sum_{t=3}^{2R-1} B_t,$$

$$\boldsymbol{\pi}^T \frac{\partial h_{2,2}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_{t=3}^{2R-1} (t-2) A_t - \sum_{t=3}^{2R-1} (t-2) B_t.$$

Therefore, we obtain

$$\begin{aligned} & \mathbf{h}_3(\boldsymbol{\pi})^T [\mathbf{H}_3(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_3(\boldsymbol{\pi})^T]^{-1} \mathbf{h}_3(\boldsymbol{\pi}) \\ &= \mathbf{h}_1(\boldsymbol{\pi})^T [\mathbf{H}_1(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_1(\boldsymbol{\pi})^T]^{-1} \mathbf{h}_1(\boldsymbol{\pi}) \\ &+ \mathbf{h}_2(\boldsymbol{\pi})^T [\mathbf{H}_2(\boldsymbol{\pi}) \boldsymbol{\Sigma}(\boldsymbol{\pi}) \mathbf{H}_2(\boldsymbol{\pi})^T]^{-1} \mathbf{h}_2(\boldsymbol{\pi}). \end{aligned}$$

Since the Wald statistic is asymptotically equivalent to the likelihood ratio statistic—see, for example, Rao (1973, Sec. 6e. 3), Darroch and Silvey (1963), and Aitchison (1962)—we obtain Theorem 2.1. The proof is complete. $\square$

From Theorem 2.1, we see that while the Corollary 2.1 satisfies the expression (5), the decomposition theorem of Yamamoto et al. (2016) does not satisfy the expression (5). Thus, the decomposition of Theorem 2.1 is an orthogonal decomposition.

We consider a model that simultaneously satisfies both the global symmetry and concordance-discordance models, defined by

$$\sum_{t=3}^{2R-1} A_t = \sum_{t=3}^{2R-1} B_t \quad \text{and} \quad 2 \sum_{s=3}^{2R-2} \sum_{t=s+1}^{2R-1} A_s B_t = 2 \sum_{s=3}^{2R-2} \sum_{t=s+1}^{2R-1} B_s A_t.$$

We shall refer to this model as the GSCD model. In the decomposition theorem of Yamamoto et al. (2016), even if we replace both the global symmetry and concordance-discordance models with the GSCD model, it still does not satisfy the expression (5).

3. Application to data

We consider the dataset in Table 1, taken from Tomizawa (1985). In Table 2, we denote the sum-symmetry, two-parameters sum-symmetry, global symmetry, weighted global sum-symmetry, double global symmetry and concordance-discordance models as the SS, TPSS, GS, WGSS, DGS and CD models, respectively. Table 2 gives the values of G^2 , for each of the sum-symmetry, two-parameters sum-symmetry, global symmetry, weighted global sum-symmetry, double global symmetry, concordance-discordance and GSCD models. This table shows that the two-parameters sum-symmetry and concordance-discordance models fit well, but the other models fit poorly. Under the two-parameters sum-symmetry model, the maximum likelihood estimates of the parameters Γ and Δ are 0.894 and 0.972, respectively. Thus, the maximum likelihood estimates of $\Gamma\Delta^{t-2}$, for all $t = 3, 4, 5, 6, 7$, are less than 1. Therefore, a university student's left eye grade is estimated to be better than his/her right eye grade. If the double global symmetry model fits well for the dataset in Table 1, then we can make the interpretation that a university student's left eye grade is estimated to be equivalent to his/her right eye grade.

We want to detect the cause of the fact that the sum-symmetry model does not fit the dataset in Table 1. From Lemma 2.1, we can infer that the sum-symmetry model does not hold for the dataset in Table 1 because of the global symmetry and weighted global sum-symmetry models rather than because of the two-parameters sum-symmetry model. From the decomposition

Table 1. Dataset of unaided vision data of 4,746 university students in Japan; source Tomizawa (1985).

Right eye grades	Left eye grades				Total
	(1)	(2)	(3)	(4)	
Highest (1)	1291	130	40	22	1483
Second (2)	149	221	114	23	507
Third (3)	64	124	660	185	1033
Lowest (4)	20	25	249	1429	1723
Total	1524	500	1063	1659	4746

Table 2. Values of the likelihood ratio chi-squared statistic (G^2) for the models applied to the data of Table 1.

Models	Degrees of freedom	G^2
SS	5	16.668*
TSS	3	4.086
GS	1	11.976*
WGSS	1	11.867*
DGS	2	12.584*
CD	1	0.648
GSCD	2	12.625*

The symbol * implies significance at the 5% level.

theorem of Yamamoto et al. (2016), we can infer that the sum-symmetry model does not hold because of the global symmetry model rather than because of the two-parameters sum-symmetry and concordance-discordance models.

In this context, we point out that (i) the value of the likelihood ratio chi-squared statistic of the sum-symmetry model is nearly equal to the sum of those of the two-parameters sum-symmetry and double global symmetry models, (ii) the value of the likelihood ratio chi-squared statistic of the sum-symmetry model is slightly greater than the sum of those of the two-parameters sum-symmetry and GSCD models, and (iii) the value of the likelihood ratio chi-squared statistic of the sum-symmetry model is greater than the sum of those of the two-parameters sum-symmetry, global symmetry and weighted global sum-symmetry models.

4. Concluding remarks

This study has established an alternative decomposition theorem in which the sum-symmetry model holds if and only if all of the two-parameters sum-symmetry, global symmetry and weighted global sum-symmetry models hold (i.e., Lemma 2.1 or Corollary 2.1), and has shown that the value of the likelihood ratio chi-squared statistic of the sum-symmetry model is asymptotically equivalent to the sum of those of the two-parameters sum-symmetry and double global symmetry models (i.e., Theorem 2.1).

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