Acceptable-and-attractive Approximate Solution of a Continuous Non-Cooperative Game on a Product of Sinusoidal Strategy Functional Spaces

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Abstract. A problem of solving a continuous noncooperative game is considered, where the player's pure strategies are sinusoidal functions of time. In order to reduce issues of practical computability, certainty, and realizability, a method of solving the game approximately is presented. The method is based on mapping the product of the functional spaces into a hyperparallelepiped of the players' phase lags. The hyperparallelepiped is then substituted with a hypercubic grid due to a uniform sampling. Thus, the initial game is mapped into a finite one, in which the players' payoff matrices are hypercubic. The approximation is an iterative procedure. The number of intervals along the player's phase lag is gradually increased, and the respective finite games are solved until an acceptable solution of the finite game becomes sufficiently close to the same-type solutions at the preceding iterations. The sufficient closeness implies that the player's strategies at the succeeding iterations should be not farther from each other than at the preceding iterations. In a more feasible form, it implies that the respective distance polylines are required to be decreasing on average once they are smoothed with respective polynomials of degree 2, where the parabolas must be having positive coefficients at the squared variable.

Keywords: game theory, payoff functional, sinusoidal strategy, continuous game, finite approximation, attractive situation.

1. Introduction

Under conditions of predominant and ever-growing demands or/and queries, mathematical modeling by the game-theory approach is a powerful method to simulate and predict redistribution of ever-limited resources. Continuous noncooperative games (CNCGs) model interactions of subjects (players or persons) possessing continuums of their pure strategies [20, 13]. Such games, for instance, are typical for economic interaction processes, where the player may use short-term time-varying strategies [1, 8, 2]. Besides, many strategically-

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controlled processes of growth-and-decrease procedures [2, 12, 19], accumulation-and-reduction development [24], birth-and-decay evolution involve such strategies also [22, 12].

A practical obstacle of CNCGs is that finding and practicing a solution in mixed strategies is almost intractable even for the case when the players act within finite-dimensional Euclidean subspaces [21, 25, 18]. A tiny class of CNCGs can be solved in pure strategies. Nevertheless, a CNCG may have multiple solutions in pure strategies, so the problem of solution uniqueness, which can be thought of as the problem of the single solution selection, is still plausible [7].

Furthermore, the solution of a game-theory problem can be defined in several ways using various criteria of equilibrium, profitability, payoff symmetry, stability, etc. Thus, even if the solution is unique, it is not guaranteed to be simultaneously profitable and symmetric [21, 3, 25]. Eventually, an exact solution in pure strategies is not always determinable as a rigorous search of analytic solutions is possible only in special classes of CNCGs [20, 21, 13]. All these issues can be troubleshot by converting the game continuity into finiteness: finite games (especially, bimatrix and trimatrix games) are always solvable [20, 21, 17] (at least, a rigorous search of analytic solutions in pure strategies is realizable by sweeping through all the pure strategy situations).

2. Motivation

The strategy is called pure due to it can be realized by just involving a single act, whichever its duration is. When the player's pure strategy is a time-varying function, its set of pure strategies is a functional space. As of 2020, CNCGs defined on functional spaces are studied weakly, indeed. Besides, peculiarities of finite approximation of such games are not studied and reported at all.

Due to that many processes of growth-and-decrease competitive activity have distinct features of seasonality, the strategy has a sinusoidal component [8, 4, 10, 22, 24]. So, the strategy can be a sinusoidal function of time describing a short-term trend of growth-and-decrease activity within a competition environment [21, 3, 18, 1]. As the strategy is defined on a short time interval, the length of this interval is usually not longer than the sinusoidal component period [9, 6]. So, a CNCG is a model of a growth-and-decrease competitive interaction process considered on just that short interval. A whole process, which is a stack of a multitude of the short intervals, is modeled as a series of such CNCGs [10, 23].

The series of the games is required to be solved without delay. The solution should be unambiguous. This can be provided by finite approximation of CNCGs, wherein the players' sets of pure strategies are sampled. Obviously, the sampling cannot be arbitrary and should be additionally substantiated.

3. Goals and tasks to be fulfilled

The sinusoidal pure strategy is determined at least by one parameter (coefficient) which varies through some interval. Usually, this parameter is the phase lag. Without losing generality, the amplitudes of the players' sinusoidal functions can be equal (this can be presumed owing to standardization). Their periods (frequencies) can be equal as well. Due

to above reasons, the goal is to develop a method of finite approximation of CNCGs whose players' payoff functionals are defined on a product of sinusoidal strategy functional spaces. For achieving the described goal, the four following tasks are to be fulfilled:

1. To formalize a CNCG, in which the player's payoff functional is defined on a product of sinusoidal strategy functional spaces. In such a game, the set of the player's pure strategies is a continuum of sinusoidal functions of time, which differ only in the phase lag. The payoff functional is an integral over the short time interval, on which the pure strategy is defined. The payoff functional is assumed to be continuous.

2. To formalize a method of finite approximation. The conditions which allow sampling the players' sets of pure strategies should be explicitly stated.

3. To give an illustrative example of how the finite approximation is fulfilled.

4. To discuss applicability and significance of the method, wherein the importance of this contribution and an outlook for a further research should be described.

4. CNCG

In the above-mentioned CNCG of N players, $N \in \mathbb{N} \setminus \{1\}$, each of the players uses short-term time-varying sinusoidal strategies determined by the phase lag. The pure strategy is valid on interval $[t_1; t_2]$ by $t_2 > t_1$, so pure strategies of the player belong to a functional space of sinusoidal functions of time:

$$S[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2].$$

Denote the phase lag of the n -th player by α_n , $n = \overline{1, N}$. The range of the phase lag in the sinusoidal function cannot be greater than 2π , so

$$\alpha_n \in [\alpha_n^{(\min)}; \alpha_n^{(\max)}] \subset [0; 2\pi] \text{ by } \alpha_n^{(\max)} > \alpha_n^{(\min)}. \quad (1)$$

Thus, the set of pure strategies of the n -th player is

$$\begin{aligned} X_n = \{x_n(t) = \sin(\omega t + \alpha_n), t \in [t_1; t_2]: \alpha_n \in [\alpha_n^{(\min)}; \alpha_n^{(\max)}] \subset [0; 2\pi]\} \subset \\ \subset S[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \end{aligned} \quad (2)$$

by $\omega > 0$.

The n -th player's payoff in situation $\{x_m(t)\}_{m=1}^N$ is $K_n(\{x_m(t)\}_{m=1}^N)$, $n = \overline{1, N}$. It is presumed that this payoff is an integral functional:

$$K_n(\{x_m(t)\}_{m=1}^N) = \int_{t_1}^{t_2} f_n(\{x_m(t)\}_{m=1}^N) dt, \quad (3)$$

where $f_n(\{x_m(t)\}_{m=1}^N)$ is an algebraic function of N sinusoidal functions $\{x_m(t)\}_{m=1}^N$

defined everywhere on $[t_1; t_2]$. Therefore, the CNCG

$$\left\langle \{X_n\}_{n=1}^N, \left\{ K_n \left(\{x_m(t)\}_{m=1}^N \right) \right\}_{n=1}^N \right\rangle \quad (4)$$

is defined on product

$$\bigtimes_{n=1}^N X_n \subset \bigtimes_{n=1}^N S[t_1; t_2] \subset \bigtimes_{n=1}^N \mathbb{L}_2[t_1; t_2] \quad (5)$$

of sinusoidal strategy functional spaces (2).

5. Acceptable solutions

Since a series of CNCGs (4) on product (5) is to be solved without delay, the only acceptable solutions are equilibrium or/and efficient situations in pure strategies. Such situations are defined similarly to those in games on finite-dimensional Euclidean subspaces [20, 21, 25].

Definition 1. Situation $\{x_m^*(t)\}_{m=1}^N$ in CNCG (4) on product (5) by conditions (1) — (3) is an equilibrium situation in pure strategies if inequalities

$$K_n \left(\left(\{x_m^*(t)\}_{m=1}^N \setminus \{x_n^*(t)\} \right) \cup x_n(t) \right) \leq K_n \left(\{x_m^*(t)\}_{m=1}^N \right) \quad \forall x_n(t) \in X_n \text{ for } n = \overline{1, N} \quad (6)$$

are simultaneously true.

Definition 2. Situation $\{x_m^{**}(t)\}_{m=1}^N$ in CNCG (4) on product (5) by conditions (1) — (3) is an efficient situation in pure strategies if inequalities

$$K_n \left(\{x_m^{**}(t)\}_{m=1}^N \right) \leq K_n \left(\{x_m(t)\}_{m=1}^N \right) \\ \text{by } \exists g \in \{\overline{1, N}\} \text{ such that } K_g \left(\{x_m^{**}(t)\}_{m=1}^N \right) < K_g \left(\{x_m(t)\}_{m=1}^N \right) \quad (7)$$

are impossible for any $x_n(t) \in X_n$ for $n = \overline{1, N}$.

It is well known that the CNCG even with two players can have the empty set of equilibria in pure strategies [20], let alone CNCGs with three or more players. Moreover, an efficient situation can be too asymmetric, when it is profitable for a subset of players and unacceptably unprofitable for another subset of players. In such cases, the game does not have an acceptable solution because a subset of players cannot accept it, although the pooled payoff (the sum of players' payoffs) is maximal or close to a maximum (in an efficient situation). Then the concepts of ε -equilibrium and ε -efficiency [20, 21] can be applied to equilibrate a situation and to smooth payoff asymmetries.

Definition 3. Situation $\{x_m^{*(\varepsilon)}(t)\}_{m=1}^N$ in CNCG (4) on product (5) by conditions (1) — (3) is an ε -equilibrium situation in pure strategies for some $\varepsilon > 0$ if inequalities

$$\begin{aligned} K_n \left(\left(\{x_m^{*(\varepsilon)}(t)\}_{m=1}^N \setminus \{x_n^{*(\varepsilon)}(t)\} \cup x_n(t) \right) \right) &\leq \\ &\leq K_n \left(\{x_m^{*(\varepsilon)}(t)\}_{m=1}^N \right) + \varepsilon \quad \forall x_n(t) \in X_n \text{ for } n = \overline{1, N} \end{aligned} \quad (8)$$

are simultaneously true.

Definition 4. Situation $\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N$ in CNCG (4) on product (5) by conditions (1) — (3) is an ε -efficient situation in pure strategies for some $\varepsilon > 0$ if inequalities

$$\begin{aligned} K_n \left(\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N \right) + \varepsilon &\leq K_n \left(\{x_m(t)\}_{m=1}^N \right) \\ \text{by } \exists g \in \{\overline{1, N}\} \text{ such that } K_g \left(\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N \right) + \varepsilon &< K_g \left(\{x_m(t)\}_{m=1}^N \right) \end{aligned} \quad (9)$$

are impossible for any $x_n(t) \in X_n$ for $n = \overline{1, N}$.

The best case is when a situation is simultaneously equilibrium (by Definition 1) and efficient (by Definition 2). If this is impossible, then the most preferable is an efficient situation in which the pooled payoff is maximal. However, if the payoffs are unacceptably asymmetric, then the best consequent is to find such ε for which a situation is simultaneously equilibrium (by Definition 3) and efficient (by Definition 4). This approach relates to the method of solving games approximately by providing concessions [14, 15]. Eventually, a payoff asymmetry may be smoothed by compensations from the players whose payoffs are unacceptably greater [17, 18, 5]. Hence, the acceptability of the game solution is determined by following properties (sorted in priority descending order):

1. Simultaneous equilibrium (by Definition 1) and efficiency (by Definition 2).
2. Closeness to efficiency (by Definition 4), which is close to equilibrium (by Definition 3).
3. Closeness to efficiency (by Definition 4), if the exact efficient situation (by Definition 2) either does not exist or is too asymmetric.
4. Closeness to equilibrium (by Definition 3), if the exact equilibrium situation (by Definition 1) either does not exist or is unprofitable (for one player or, more, up to $N - 1$ players).

In the list above, property #3 implies that there are no equilibria, and ε -equilibria are either unprofitable or unacceptably asymmetric. Property #4 implies that every efficient situation is unacceptably asymmetric and ε -efficient situations are unprofitable as well.

6. The finite approximation as a mapping into a finite game

In CNCG (4) on product (5) by conditions (1) — (3), the pure strategy of the player is factually determined by the phase lag. Although this is a formal controller, and the factual

strategic control is done by enabling real-world resources, the phase lag represents the control result. Therefore, this game can be thought of as it is defined, instead of product (5) of sinusoidal strategy functional spaces (2), on N -dimensional hyperparallelepiped

$$\bigtimes_{n=1}^N [\alpha_n^{(\min)}; \alpha_n^{(\max)}] \subset \bigtimes_{n=1}^N [0; 2\pi] \subset \mathbb{R}^N. \quad (10)$$

This hyperparallelepiped is easily sampled by using a number of equal intervals along each dimension. Denote this number by S , $S \in \mathbb{N} \setminus \{1\}$. Then

$$\begin{aligned} A_n &= \left\{ \alpha_n^{(\min)} + (s-1) \cdot \frac{\alpha_n^{(\max)} - \alpha_n^{(\min)}}{S} \right\}_{s=1}^{S+1} = \\ &= \{ \alpha_n^{(s)} \}_{s=1}^{S+1} \subset [\alpha_n^{(\min)}; \alpha_n^{(\max)}] \subset [0; 2\pi] \quad \forall n = \overline{1, N}. \end{aligned} \quad (11)$$

So, hyperparallelepiped (10) is substituted with hypercubic grid $\bigtimes_{n=1}^N A_n$. Set (11) leads to a finite set

$$\begin{aligned} X_n^{(A_n)} &= \{ x_n(t) = \sin(\omega t + \alpha_n), t \in [t_1; t_2] : \alpha_n \in A_n \subset [\alpha_n^{(\min)}; \alpha_n^{(\max)}] \subset [0; 2\pi] \} = \\ &= \{ x_{ns}(t) = \sin(\omega t + \alpha_n^{(s)}) \}_{s=1}^{S+1} \subset X_n \subset S[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \end{aligned} \quad (12)$$

of pure strategies (sinusoidal functions of time) of the n -th player, where

$$x_{n1}(t) = \sin(\omega t + \alpha_n^{(\min)}), \quad x_{n, S+1}(t) = \sin(\omega t + \alpha_n^{(\max)}).$$

Hence, product (5) of sinusoidal strategy functional spaces (2) is mapped into product

$$\bigtimes_{n=1}^N X_n^{(A_n)} \subset \bigtimes_{n=1}^N X_n \subset \bigtimes_{n=1}^N S[t_1; t_2] \subset \bigtimes_{n=1}^N \mathbb{L}_2[t_1; t_2]. \quad (13)$$

Subsequently, CNCG (4) on product (5) by conditions (1) — (3) is mapped into a finite

$$\bigtimes_{n=1}^N (S+1)\text{-game}$$

$$\left\langle \{ X_n^{(A_n)} \}_{n=1}^N, \left\{ K_n \left(\{ x_m(t) \}_{m=1}^N \right) \right\}_{n=1}^N \right\rangle \text{ by } x_n(t) \in X_n^{(A_n)} \text{ for } n = \overline{1, N} \quad (14)$$

defined on product (13).

To perform an appropriate approximation via the sampling, number S is selected so that none of S^N sub-hyperparallelepipeds

$$\begin{aligned} \bigtimes_{n=1}^N [\alpha_n^{(i_m)}; \alpha_n^{(i_m+1)}] \subset \bigtimes_{n=1}^N [\alpha_n^{(\min)}; \alpha_n^{(\max)}] \subset \bigtimes_{n=1}^N [0; 2\pi] \subset \mathbb{R}^N \\ \text{by } i_m = \overline{1, S} \text{ and } m = \overline{1, N} \end{aligned} \quad (15)$$

would contain significant specificities of payoff functionals (3) $\forall n = \overline{1, N}$. In fact, such specificities are extremals of these functionals.

Theorem 1. In CNCG (4) on product (5) by conditions (1) — (3), the player's payoff functional achieves its maximal and minimal values on any closed sub-hyperparallelepiped of the phase lags.

Proof. Every $f_n(\{x_m(t)\}_{m=1}^N)$ ($n = \overline{1, N}$) is an algebraic function of sinusoidal functions $\{x_m(t)\}_{m=1}^N$ defined everywhere on $[t_1; t_2]$. Therefore, the integral in every functional (3) achieves some maximal and minimal values on any closed sub-hyperparallelepiped of the phase lags. $\square$

Thus, Theorem 1 allows controlling extremals of payoff functionals (3) $\forall n = \overline{1, N}$ by the phase lags. Moreover, it is quite obvious that Theorem 1 is straightforwardly expanded on closed sub-hyperparallelepipeds (15) for any number S . Consequently, if inequalities

$$\begin{aligned} \bigtimes_{n=1}^N \max_{[\alpha_n^{(i_m)}; \alpha_n^{(i_m+1)}]} K_j(\{x_m(t)\}_{m=1}^N) - \bigtimes_{n=1}^N \min_{[\alpha_n^{(i_m)}; \alpha_n^{(i_m+1)}]} K_j(\{x_m(t)\}_{m=1}^N) = \\ = \bigtimes_{n=1}^N \max_{[\alpha_n^{(i_m)}; \alpha_n^{(i_m+1)}]} \int_{t_1}^{t_2} f_j(\{x_m(t)\}_{m=1}^N) dt - \bigtimes_{n=1}^N \min_{[\alpha_n^{(i_m)}; \alpha_n^{(i_m+1)}]} \int_{t_1}^{t_2} f_j(\{x_m(t)\}_{m=1}^N) dt \leq \mu \\ \forall i_m = \overline{1, S} \text{ and } \forall m = \overline{1, N} \text{ and } \forall j = \overline{1, N} \end{aligned} \quad (16)$$

are simultaneously true for some sufficiently small $\mu > 0$, then those μ -variations can be ignored. Thus, for the properly selected S and μ , CNCG (4) on product (5) by conditions (1) — (3) can be mapped into and approximated by finite game (14). The quality of the approximation can be comprehended by the following assertions.

Theorem 2. If $\{x_m^*(t)\}_{m=1}^N$ is an equilibrium in CNCG (4) on product (5) by conditions (1) — (3), where functionals (3) $\forall n = \overline{1, N}$ are continuous, conditions (16) hold for some S and μ ,

$$\begin{aligned} x_m^*(t) = \sin(\omega t + \alpha_m^*) \text{ by } \alpha_m^* \in [\alpha_m^{(h_m)}; \alpha_m^{(h_m+1)}] \\ \text{for } h_m \in \{\overline{1, S}\} \text{ and } \forall m = \overline{1, N}, \end{aligned} \quad (17)$$

then every situation $\{x_m^{*(e)}(t)\}_{m=1}^N$ for which

$$x_m^{*(\varepsilon)}(t) = \sin(\omega t + \alpha_m^{*(\varepsilon)}) \text{ by } \alpha_m^{*(\varepsilon)} \in [\alpha_m^{(h_m)}; \alpha_m^{(h_m+1)}] \\ \text{for } h_m \in \{1, \overline{S}\} \text{ and } \forall m = \overline{1, N}, \quad (18)$$

is an ε -equilibrium for some $\varepsilon > 0$. As number S is increased, the value of ε can be made smaller.

Proof. Whichever integer S and the corresponding μ are, the value of ε always can be chosen such that inequalities (8) be true for every situation composed of strategies (18) by (17). It is obvious that, owing to the continuity of the functionals, increasing number S allows decreasing the value of μ , which provides ε -equilibria to be closer to the equilibrium composed of strategies (17). $\square$

Theorem 3. If $\{x_m^{**}(t)\}_{m=1}^N$ is an efficient situation in CNCG (4) on product (5) by conditions (1) — (3), where functionals (3) $\forall n = \overline{1, N}$ are continuous, conditions (16) hold for some S and μ ,

$$x_m^{**}(t) = \sin(\omega t + \alpha_m^{**}) \text{ by } \alpha_m^{**} \in [\alpha_m^{(h_m)}; \alpha_m^{(h_m+1)}] \\ \text{for } h_m \in \{1, \overline{S}\} \text{ and } \forall m = \overline{1, N}, \quad (19)$$

then every situation $\{x_m^{**(\varepsilon)}(t)\}_{m=1}^N$ for which

$$x_m^{**(\varepsilon)}(t) = \sin(\omega t + \alpha_m^{**(\varepsilon)}) \text{ by } \alpha_m^{**(\varepsilon)} \in [\alpha_m^{(h_m)}; \alpha_m^{(h_m+1)}] \\ \text{for } h_m \in \{1, \overline{S}\} \text{ and } \forall m = \overline{1, N}, \quad (20)$$

is an ε -efficient situation for some $\varepsilon > 0$. As number S is increased, the value of ε can be made smaller.

Proof. Whichever integer S and the corresponding μ are, value ε always can be chosen such that inequalities (9) be true for every situation composed of strategies (20) by (19). It is obvious that, owing to the continuity of the functionals, increasing number S allows decreasing the value of μ , which provides ε -efficient situations to be closer to the efficient situation composed of strategies (19). $\square$

Hence, starting with some integer S , the finite approximation routine is as follows:

1. Build and solve a finite $\bigtimes_{n=1}^N (S+1)$ -game (14).
2. Find an acceptable solution in this game.
3. Stop if the acceptable solution (whether it is an equilibrium, efficient, ε -equilibrium, or ε -efficient situation) by the last iteration does not differ much from the acceptable solutions (of the same type) by a few preceding iterations; otherwise, go to the next item.
4. Increase integer S by some increment and go to item #1.

At the very start of the routine, the increment is not necessary to be 1. Moreover, the number of equal intervals along each dimension can be increased intensely if the difference between the acceptable solutions decreases slowly. The difference can be considered via the functional closeness. Thus, if

$$\left\{x_n^{<l>*}(t)\right\}_{n=1}^N = \left\{\sin(\omega t + \alpha_n^{<l>*})\right\}_{n=1}^N \in \bigotimes_{n=1}^N X_n^{(A_n)} \subset \bigotimes_{n=1}^N X_n \quad (21)$$

is an acceptable solution at the l -th iteration, then the conditions of the sufficient closeness to the solutions at the preceding and succeeding iterations are as follows:

$$\sqrt{\int_{t_1}^{t_2} \left(x_n^{<l-1>*}(t) - x_n^{<l>*}(t)\right)^2 dt} \geq \sqrt{\int_{t_1}^{t_2} \left(x_n^{<l>*}(t) - x_n^{<l+1>*}(t)\right)^2 dt} \quad \forall n = \overline{1, N} \quad (22)$$

and

$$\max_{t \in [t_1; t_2]} \left| x_n^{<l-1>*}(t) - x_n^{<l>*}(t) \right| \geq \max_{t \in [t_1; t_2]} \left| x_n^{<l>*}(t) - x_n^{<l+1>*}(t) \right| \quad \forall n = \overline{1, N} \quad (23)$$

by $l = 2, 3, 4, \dots$ So, if inequalities

$$\begin{aligned} & \sqrt{\int_{t_1}^{t_2} \left(\sin(\omega t + \alpha_n^{<l-1>*}) - \sin(\omega t + \alpha_n^{<l>*})\right)^2 dt} \geq \\ & \geq \sqrt{\int_{t_1}^{t_2} \left(\sin(\omega t + \alpha_n^{<l>*}) - \sin(\omega t + \alpha_n^{<l+1>*})\right)^2 dt} \quad \forall n = \overline{1, N} \end{aligned} \quad (24)$$

and

$$\begin{aligned} & \max_{t \in [t_1; t_2]} \left| \sin(\omega t + \alpha_n^{<l-1>*}) - \sin(\omega t + \alpha_n^{<l>*}) \right| \geq \\ & \geq \max_{t \in [t_1; t_2]} \left| \sin(\omega t + \alpha_n^{<l>*}) - \sin(\omega t + \alpha_n^{<l+1>*}) \right| \quad \forall n = \overline{1, N} \end{aligned} \quad (25)$$

by $l = 2, 3, 4, \dots$ hold for at least three iterations, the approximation procedure can be stopped. Clearly, the closeness strengthens if inequalities (24) and (25) hold strictly. However, inequalities (24) and (25) may not hold for a wide range of iterations, so it is better to require that polylines

$$\lambda_n(l) = \sqrt{\int_{t_1}^{t_2} \left(\sin(\omega t + \alpha_n^{<l>*}) - \sin(\omega t + \alpha_n^{<l+1>*})\right)^2 dt} \quad \forall n = \overline{1, N} \quad (26)$$

and

$$\eta_n(l) = \max_{t \in [t_1; t_2]} \left| \sin(\omega t + \alpha_n^{<l>*}) - \sin(\omega t + \alpha_n^{<l+1>*}) \right| \quad \forall n = \overline{1, N} \quad (27)$$

by $l = 1, 2, 3, \dots$ be decreasing on average. Herein, term “on average” implies that, in the case when inequalities (24) and (25) do not hold, polylines (26) and (27) are smoothed (approximated) with the respective polynomials of degree 2 (the parabolas must be having positive coefficients at the squared variable). These conditions allow sampling the players’ sets of pure strategies without missing better situations.

7. An illustrative example of the finite approximation

In practice, the finite approximation may bring a great many of possible solutions even in CNCGs with more than just two players. To select the best one (which would be acceptable according to the abovementioned list of properties of the game solution acceptability), apart from operating on (22) — (27), it is about to analyze profitability of all possible solutions through a range of iterations. The range is taken based on common sense. Consider an example in which the seasonality develops through time $t \in [0; 5]$ with a phase rate of $\omega = 1$ (e. g., in economics, this is a really short time interval measured in days or weeks). The sets of pure strategies of the players are

$$\begin{aligned} X_n &= \{x_n(t) = \sin(t + \alpha_n), t \in [0; 5] : \alpha_n \in [0; 2\pi]\} \subset \\ &\subset S[0; 5] \subset \mathbb{L}_2[0; 5] \quad \text{for } n = \overline{1, 3}. \end{aligned} \quad (28)$$

The payoff functionals are:

$$K_1(x_1(t), x_2(t), x_3(t)) = \int_0^5 \left(\frac{6x_1 + 3x_2}{x_2 + x_3 + 4} + 1 \right) dt, \quad (29)$$

$$K_2(x_1(t), x_2(t), x_3(t)) = \int_0^5 \left(3x_2 - \frac{x_1 + 2x_3}{5 - x_2} + 1 \right) dt, \quad (30)$$

$$K_3(x_1(t), x_2(t), x_3(t)) = \int_0^5 \left(\frac{4(x_3 + x_1x_2)}{2x_1 - x_2 - 6} + 1 \right) dt. \quad (31)$$

Consequently, this CNCG can be thought of as it is defined on cube

$$[0; 2\pi] \times [0; 2\pi] \times [0; 2\pi] \subset \mathbb{R}^3. \quad (32)$$

The example is going to be a clarification of how to organize and complete the finite approximation in the general case.

Functions $\{x_m(t)\}_{m=1}^3$ are sinusoidal and their values are within interval $[-1; 1]$. This implies that the denominators under integrals (29) — (31) are always positive (for the first

and second players) or negative (for the third player). Thus, it is easy to see that functionals (29) — (31) are continuous. Therefore, Theorem 2 and Theorem 3 ensure a possibility of the finite approximation. For making this, the number of intervals is gradually increased from 5 up to 36, i. e. $S = \overline{5, 36}$ and the respective $6 \times 6 \times 6$ -game, $7 \times 7 \times 7$ -game, ..., $37 \times 37 \times 37$ -game are built and solved. Each of these games has equilibria: it can be 1, 2, or 3 equilibrium situations (Figure 1). The efficient situations exist also. Moreover, their number grows dramatically (Figure 2).

Figure 3 shows the aggregate of all equilibria and the cloud of efficient situations (associated with just the phase lags) found by $S = \overline{5, 36}$. The aggregate is shown in

Figure 1. The number of equilibrium situations versus the number of intervals

Figure 2. The number of efficient situations versus the number of intervals

Figure 4 as a zoom-in on cube (32), where the efficient equilibria are covered with squared marks, and the mark is shown bigger as the number of intervals increases. This three-dimensional plot along with its two-dimensional projections in Figure 5 allow seeing that there are no regularities in the positions of the equilibria (even without considering those two seen separated whose first player's phase lag is 0). Polyline (26) for those equilibrium-and-efficient situations in Figure 4 whose pooled payoff is maximal are shown in Figure 6, whence the insufficiency of the closeness is clearly visible. Such an inference is also confirmed by polyline (27) for those equilibrium-and-efficient situations in Figure 4 whose pooled payoff is maximal (Figure 7). Surely, it is worth noting that the difference between the respective polyline (26) and (27) is quite insignificant. Anyway,

Figure 3. All 57 equilibria (circle-marked) and the cloud of 9015 efficient situations (dot-marked) visualized on cube (32) in the 32 solved finite (trimatrix) games

the convergence of the respective polylines (26) and (27) on average either does not exist or is very slow. Of course, the latter cannot give an acceptable solution.

Nevertheless, the profitability of all possible solutions still can be analyzed. The pooled payoff in the 9015 efficient situations (the dot-marked cloud in Figure 3) is shown in Figure 8. The pooled payoff in the 57 equilibria is shown on the same plot with squared marks (the locations of the equilibria are tied to the ends of the 32 sequences of efficient situations). The plot allows seeing that the equilibrium pooled payoff is less than the

Figure 4. All 57 equilibria (circle-marked), where the efficient equilibria are covered with squared marks

maximal efficient pooled payoff. However, the efficient pooled payoff drops dramatically between neighboring maxima (along with the major drops, the minor drops are easily seen in Figure 8 as well). At the same time, the equilibrium pooled payoff is relatively “stable” (Figure 9). Moreover, all the equilibrium pooled payoffs are closer to the maxima of the efficient pooled payoffs than to their minima (the major drops). Consequently, the players are unlikely to aim at “unstable” (with respect to the pooled payoff) efficient situations, because a miss on an efficient situation will cost more than aiming at the equilibria.

Figure 5. Two-dimensional projections of the three-dimensional plot of the 57 equilibria in Figure 4

Figure 6. Polyline (26) for those equilibrium-and-efficient situations in Figure 4 whose pooled payoff is maximal

Henceforward, it is worth to look at the aggregate of 57 equilibria in Figure 4 again, in a more careful manner. Point

$$[\alpha_1 \quad \alpha_2 \quad \alpha_3] = [5.236 \quad 5.236 \quad 2.0944]$$

(easily seen as a “blob” in Figure 4 and in the bottom projection in Figure 5) corresponding to equilibrium-and-efficient situation

$$\{x_1^{**}(t), x_2^{**}(t), x_3^{**}(t)\} = \{\sin(t + 5.236), \sin(t + 5.236), \sin(t + 2.0944)\} \quad (33)$$

is very noticeable owing to, during those 32 iterations, it is “hit” four times. These are

Figure 7. Polyline (27) for those equilibrium-and-efficient situations in Figure 4 whose pooled payoff is maximal

iterations ## 2, 8, 14, 20, i. e. situation (33) is “hit” in every 6 iterations up to the $25 \times 25 \times 25$ -game (iteration #20). At iteration #26 (in the $31 \times 31 \times 31$ -game) the equilibrium-and-efficient situation slightly changes to

$$\{x_1^{**}(t), x_2^{**}(t), x_3^{**}(t)\} = \{\sin(t + 5.236), \sin(t + 5.4454), \sin(t + 2.0944)\} \quad (34)$$

where only the second player’s strategy deviates — its phase lag becomes 4 % greater compared to that in situation (33). At iteration #32 (in the $37 \times 37 \times 37$ -game) the equilibrium-and-efficient situation slightly changes once again only with the deviating second player’s strategy:

Figure 8. The pooled payoff in the 9015 efficient situations along with the pooled payoff in the 57 equilibria

Figure 9. A zoom-in on the pooled payoff in the 57 equilibria left as it is on the abscissa axis in Figure 8

$$\{x_1^{**}(t), x_2^{**}(t), x_3^{**}(t)\} = \{\sin(t + 5.236), \sin(t + 5.4105), \sin(t + 2.0944)\} \quad (35)$$

where the second player's phase lag becomes less than that in situation (34). Amazingly enough, by skipping the next 5 iterations (to follow the "every 6 iterations" extempore rule), the respective $43 \times 43 \times 43$ -game has an equilibrium-and-efficient situation

$$\{x_1^{**}(t), x_2^{**}(t), x_3^{**}(t)\} = \{\sin(t + 5.236), \sin(t + 5.3856), \sin(t + 2.0944)\} \quad (36)$$

where the second player's phase lag becomes less than that in situation (35), like being "drawn" back to situation (33).

In situation (33), the players receive payoffs

$$K_1(\sin(t + 5.236), \sin(t + 5.236), \sin(t + 2.0944)) = 7.6736, \quad (37)$$

$$K_2(\sin(t + 5.236), \sin(t + 5.236), \sin(t + 2.0944)) = 8.8943, \quad (38)$$

$$K_3(\sin(t + 5.236), \sin(t + 5.236), \sin(t + 2.0944)) = 4.5424, \quad (39)$$

and the pooled payoff is 21.1103. In situation (34), the players' payoffs are increased by 0.22 %, 0.08 %, and 1.19 %, respectively:

$$K_1(\sin(t + 5.236), \sin(t + 5.4454), \sin(t + 2.0944)) = 7.6907, \quad (40)$$

$$K_2(\sin(t + 5.236), \sin(t + 5.4454), \sin(t + 2.0944)) = 8.9015, \quad (41)$$

$$K_3(\sin(t + 5.236), \sin(t + 5.4454), \sin(t + 2.0944)) = 4.5964, \quad (42)$$

so the pooled payoff is 21.1886 being increased by 0.37 % compared to situation (33). Then, in situation (35), however, only the second player receives a 0.11 % greater payoff, whereas the first and third players lose 0.004 % and 0.307 % of their payoffs in situation (34), respectively:

$$K_1(\sin(t + 5.236), \sin(t + 5.4105), \sin(t + 2.0944)) = 7.6904, \quad (43)$$

$$K_2(\sin(t + 5.236), \sin(t + 5.4105), \sin(t + 2.0944)) = 8.9115, \quad (44)$$

$$K_3(\sin(t + 5.236), \sin(t + 5.4105), \sin(t + 2.0944)) = 4.5823, \quad (45)$$

so the pooled payoff is 21.1842 being decreased by 0.021 % compared to situation (34). Finally, in situation (36), the first and third players again lose 0.01 % and 0.192 % of their payoffs in situation (35), respectively, whereas the second player receives a 0.05 % greater payoff:

$$K_1(\sin(t+5.236), \sin(t+5.3856), \sin(t+2.0944)) = 7.6896, \quad (46)$$

$$K_2(\sin(t+5.236), \sin(t+5.3856), \sin(t+2.0944)) = 8.9159, \quad (47)$$

$$K_3(\sin(t+5.236), \sin(t+5.3856), \sin(t+2.0944)) = 4.5735, \quad (48)$$

so the pooled payoff is 21.179 being decreased by 0.025 % compared to situation (35).

Despite payoffs (37) — (39) are slightly worse than the respective payoffs (40) — (42), (43) — (45), (46) — (48), the first and third players are “attracted” anyway to their respective strategies $x_1^{**}(t) = \sin(t+5.236)$ and $x_3^{**}(t) = \sin(t+2.0944)$ used in every situation (33) — (36). So, it is obvious that situation (33) “attracts” the first and third players, as well as situations (34) — (36) do. At the same time, the second player will tend to select a phase lag closer to 5.3856, at which its payoff is close to be maximal (among the equilibrium-and-efficient situations, the maximal payoff of the second player is 8.922). Consequently, situation (36) is an acceptable-and-attractive solution for all the players.

Although it is unlikely that, in the considered example, the second player will always select strategy $x_2^{**}(t) = \sin(t+5.3856)$, the phase lag in the selected strategy will surely be between about 5.236 and 5.4454 (the latter will induce a 0.045 % greater pooled payoff in 21.1886, indeed). This implies that the acceptable-and-attractive solution is not just a situation, but rather a continuous subset (sub-interval, sub-rectangle, or sub-parallelepiped of situations associated with just the phase lags) of situations defined by its center (the most attractive situation). In this case, the center can be thought of as a situation-wanderer roaming around situations (33) — (36). To speak it with a stronger certainty, probably more iterations must be made. In general, the finite approximation may have practical issues caused by multiplicity of acceptable solutions and solution bounces (instability) [16], which subsequently may lead to new selection problems.

8. Discussion

It might have seemed that consideration of sinusoidal strategies is a theoretical idealization barely applicable in practice. However, such an idealization is nonetheless useful by two reasons. First, the sine function is a natural pattern for processes of growth-and-decrease competitive activity. Second, if even the strategy is not sinusoidal, its shape should be cyclic in order to harmonically react against external disturbance/influence [6, 12, 19, 22]. So, consideration sinusoidal strategies saves from needless complications.

The requirements with polyines (26) and (27) cannot be narrowed to using one set of polyines. Regardless of the fact that, in the considered example, the difference between the respective polyines (26) in Figure 6 and polyines (27) in Figure 7 is indeed insignificant, it is easy to see that, e. g., $\lambda_3(1) < \lambda_3(9)$ and $\eta_3(1) > \eta_3(9)$. Such a local contradiction means that regularities and interrelationships among polyines (26) are not necessarily player-wise followed by polyines (27) and vice versa. This is why it is incorrect to throw off polyines (26), nor polyines (27).

It is absolutely clear that the acceptable-and-attractive approximate solution of a CNCG can be easily implemented and practiced. In fact, the finite approximation makes the CNCG tractable itself. This justifies the applicability of the presented finite approximation method. Although it does not remove the problem of the single solution selection, it provides finite sets of possible solutions instead of continuums. Difficulties with slow convergence are probable but they can be removed by solving new, bigger, finite games to accumulate sufficient statistics of approximate solutions, whereupon to decide on which one attracts the players.

As the number of players increases, the volume of computations and decisions grows. CNCGs with four and more players may be much trickier to approximate them because then any full-scale visualization of equilibria and/or efficient situations is impossible. This is, however, only a limitation of the visualization that does not limit the presented method at all. Apart from ordinarily solving a finite game in pure strategies, the method core consists in selecting an acceptable-and-attractive situation by simultaneously controlling that polylines (26) and (27) be decreasing on average.

For theory and practice of CNCGs, the presented method is quite significant for avoiding too complicated solutions resulting from game continuities and functional spaces of pure strategies (especially for three players and more). The method allows effectively modeling growth-and-decrease competitive interaction processes, where the player can use a continuum of short-term time-varying strategies influenced/enforced by seasonality. Henceforward, by fulfilling the gradual finite approximation with the proved theorems and (21) — (27), CNCGs can be simplified dramatically by mapping them into finite games, without losing precious information. In other words, CNCGs are then just “deinstallungized” [11] by the presented finite approximation method. In particular, the “deinstallungization” can be thought of as a return from the paradigm of mixed strategies (especially, if the mixed strategy is a span of functions) to pure strategies. Thus, the model becomes more flexible and tractable, so its implementation is realized with higher fidelity. Furthermore, owing to the “deinstallungization”, the model reliability becomes incomparably stronger than that of the CNCGs.

9. Conclusion

The method of finite approximation based on sampling the sinusoidal strategy functional spaces is intended for settling CNCGs defined on products of such spaces, wherein the settlement implies reducing the solution issues to practical computability, certainty, and realizability. The sets of the players’ pure strategies (i. e., the spaces of pure strategies) are sampled uniformly so that the payoff matrices in the resulting finite game are hypercubic. The approximation procedure starts with not a great number of intervals along the player’s phase lag. Then this number is gradually increased, and new, bigger, finite games are solved until an acceptable solution of the finite game becomes sufficiently close to the same-type solutions at the preceding iterations. The closeness is expressed in terms of the respective functional spaces, in which the player’s strategies at the succeeding iterations should be not farther from each other than at the preceding iterations. Due to limited feasibility, these requirements are transformed into the relaxed conditions which allow sampling the players’ sets of pure strategies: the respective distance polylines are required

to be decreasing on average once they are smoothed (approximated) with respective polynomials of degree 2, where the parabolas must be having positive coefficients at the squared variable. The approximate solution of the initial CNCG is expected to be simultaneously acceptable and attractive for players.

The presented CNCG finite approximation has a certain promising impact on modeling growth-and-decrease competitive interaction processes, where players use short-term time-varying strategies influenced by seasonality. Besides, this is an important contribution to the fields of game theory and decision-making theory, where many questions of practical computability, certainty, and realizability are still unanswered. Thus, studying infinite games with discontinuities is an outlook for a further research.

Acknowledgments

This work was technically supported by the Faculty of Mechanical and Electrical Engineering at the Polish Naval Academy, Gdynia, Poland.

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Received 15.07.2020, Accepted 22.01.2021