A Computational Technique for Solving Singularly Perturbed Delay Partial Differential Equations

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Abstract. In this work, a matrix method based on Laguerre series to solve singularly perturbed second order delay parabolic convection-diffusion and reaction-diffusion type problems involving boundary and initial conditions is introduced. The approximate solution of the problem is obtained by truncated Laguerre series. Moreover convergence analysis is introduced and stability is explained. Besides, a test case is given and the error analysis is considered by the different norms in order to show the applicability of the method.

Keywords: Laguerre polynomials and series, singularly perturbed parabolic problem, delay partial differential equation, convergence analysis, Laguerre matrix-collocation method.

1. Introduction

Singular parabolic partial differential equations are of an extensive range of real life phenomena. Some of the applications of singularly perturbed diffusive models in physics are to describe the dynamics of convection-diffusion models and systems in chemical physics [20], [35], [21], [36], [33]. Another application of these type of models is explained as systems of thermal science and mechanics which are characterized by both temporal and spatial variables. Heat generation or loss has been considered in the physical applications of this type of differential equations [19], [37], [13], [7].

In this study, following type of singularly perturbed linear parabolic delay partial differential equations (SPDP-PDEs) is considered:

$$u_t(x, t) - \epsilon u_{xx}(x, t) + a(x, t)u(x, t) + b(x, t)u(x, t - \tau) = f(x, t), \quad (1)$$

$$(x, t) \in [a, b] \times [0, T].$$

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Then Eq. (1) is supplemented with boundary and initial conditions

$$\begin{aligned} u(x, 0) &= \phi(x), \quad x \in [0, l], \\ u_t(x, 0) &= m(x), \quad x \in [0, l], \\ u(0, t) &= h(t), \quad t \in [0, T], \end{aligned} \quad (2)$$

where ϵ is a small parameter in the neighborhood of 0, i.e. $0 < \epsilon \ll 1$, τ is also a parameter and $0 < \tau \ll 1$. Besides, $a(x, t)$, $b(x, t)$, $f(x, t)$, $\phi(x)$, $m(x)$ and $h(t)$ are continuous functions.

A considerable amount of literature exists respecting the numerical techniques for solving SPDP-PDEs. Bashier et al. showed fitted operator finite difference method [4], Ansari et al. introduced parameter-robust finite difference method [2], finite difference method has been presented by Rai et al. [30], Crank-Nicolson scheme has been used to solve SPDP-PDEs by Parthiban et al. [20].

Moreover, higher order uniformly convergent method has been presented by Salama et al. [33]. A domain decomposition Taylor Galerkin finite element approximation has been introduced by Kumar et al. [24]. Selvi and Ramanujam presented a difference scheme along with Robin type boundary condition [3]. Also, stability analysis of the heat equation with feedback has been considered by Çalışkan et al. [36].

Here, approximate solution of the problem (1)-(2) is considered in the truncated Laguerre series form

$$u(x, t) \cong u_N(x, t) = \sum_{n=0}^N \sum_{p=0}^N a_{n,p} L_{n,p}(x, t); \quad (3)$$

$$L_{n,p}(x, t) = L_n(x) L_p(t), \quad 0 \leq x \leq l, \quad 0 \leq t \leq T < \infty,$$

where $L_n(x)$ expresses the Laguerre polynomials;

$$L_n(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} x^k, \quad n \in N, \quad 0 \leq x < \infty, \quad (4)$$

and $a_{n,p}$, ($n, p = 0, 1, \dots, N$) are given as unidentified coefficients of Laguerre polynomial, and N is described as any positive integer for $N \geq 2$ [1].

2. Fundamental relations

In this section, an approximate solution in terms of Laguerre polynomials in the form (3) is presented. For our purpose, Eq. (1) is given in the form:

$$[u(x, t)] = \mathbf{L}(x) \bar{\mathbf{L}}(t) \mathbf{A}, \quad (5)$$

where

$$\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix},$$

$$\bar{\mathbf{L}}(t) = \begin{bmatrix} \mathbf{L}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{L}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{L}(t) \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,N} \end{bmatrix}^T, \quad i = 0, 1, \dots, N.$$

Then, the matrix relation is used

$$\mathbf{L}(x) = \mathbf{X}(x)\mathbf{H} \quad \text{and} \quad \bar{\mathbf{L}}(t) = \bar{\mathbf{X}}(t)\bar{\mathbf{H}}, \quad (6)$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix},$$

$$\mathbf{X}(t) = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \frac{(-1)^0}{0!} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \cdots & \frac{(-1)^0}{0!} \begin{pmatrix} N \\ 0 \end{pmatrix} \\ 0 & \frac{(-1)^1}{1!} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \cdots & \frac{(-1)^1}{1!} \begin{pmatrix} N \\ 1 \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(-1)^N}{N!} \begin{pmatrix} N \\ N \end{pmatrix} \end{bmatrix},$$

$$\bar{\mathbf{X}}(t) = \begin{bmatrix} \mathbf{X}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(t) \end{bmatrix}, \quad \bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H} & 0 & \cdots & 0 \\ 0 & \mathbf{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H} \end{bmatrix}.$$

Besides, the matrices $\mathbf{X}(x)$, $\bar{\mathbf{X}}(t)$ and their connections with the derivatives $\mathbf{X}'(x)$, $\mathbf{X}''(x)$ and $\bar{\mathbf{X}}'(t)$, $\bar{\mathbf{X}}''(t)$ can be composed as

$$\begin{aligned} \mathbf{X}'(x) &= \mathbf{X}(x)\mathbf{B}, \quad \mathbf{X}''(x) = \mathbf{X}(x)\mathbf{B}^2, \\ \bar{\mathbf{X}}'(t) &= \bar{\mathbf{X}}(t)\bar{\mathbf{B}}, \quad \text{and} \quad \bar{\mathbf{X}}''(t) = \bar{\mathbf{X}}(t)\bar{\mathbf{B}}^2, \end{aligned} \quad (7)$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}.$$

Additionally, $u(x, t)$ and its derivatives are organized regarding Eq. (5)-Eq. (7) in the matrix forms

$$\begin{aligned} [u(x, t)] &= \mathbf{L}(x)\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A}, \\ [u_x(x, t)] &= \mathbf{L}'(x)\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{B}\mathbf{H}\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{B}\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A}, \\ [u_{xx}(x, t)] &= \mathbf{L}''(x)\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{B}^2\mathbf{H}\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{B}^2\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A}, \\ [u_t(x, t)] &= \mathbf{L}(x)\bar{\mathbf{L}}'(t)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{L}}'(t)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{B}}\mathbf{H}\mathbf{A}. \end{aligned} \quad (8)$$

Also, we have

$$[u(x, t - \tau)] = \mathbf{L}(x)\bar{\mathbf{L}}(t - \tau)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t - \tau)\bar{\mathbf{H}}\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{B}}(1, -\tau)\bar{\mathbf{H}}\mathbf{A}, \quad (9)$$

where

$$\mathbf{B}(1, -\tau) = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} (-\tau)^0 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} (-\tau)^1 & \begin{pmatrix} 2 \\ 0 \end{pmatrix} (-\tau)^2 & \cdots & \begin{pmatrix} N \\ 0 \end{pmatrix} (-\tau)^N \\ 0 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-\tau)^0 & \begin{pmatrix} 2 \\ 1 \end{pmatrix} (-\tau)^1 & \cdots & \begin{pmatrix} N \\ 1 \end{pmatrix} (-\tau)^{N-1} \\ 0 & 0 & \begin{pmatrix} 2 \\ 2 \end{pmatrix} (-\tau)^2 & \cdots & \begin{pmatrix} N \\ 2 \end{pmatrix} (-\tau)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \begin{pmatrix} N \\ N \end{pmatrix} (-\tau)^0 \end{bmatrix}.$$

Using the relations Eq. (8) and Eq. (9), then the matrix form of Eq. (1) is obtained as

$$\begin{aligned} \{ \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{B}}\bar{\mathbf{H}} - \epsilon\mathbf{X}(x)\mathbf{B}^2\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}} - a(x, t)\mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}} \\ + b(x, t)\mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(t)\mathbf{B}(1, -\tau)\bar{\mathbf{H}} \} \mathbf{A} = [f(x, t)]. \end{aligned} \quad (10)$$

Similarly, the matrix equations for the boundary and initial conditions Eq. (2) are organized by using Eq. (8):

$$\begin{aligned} [u(x, 0)] &= \mathbf{L}(x)\bar{\mathbf{L}}(0)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{L}}(0)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(0)\bar{\mathbf{H}}\mathbf{A} = [\phi(x)] = \lambda, \\ [u_t(x, 0)] &= \mathbf{L}(x)\bar{\mathbf{L}}'(0)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{L}}'(0)\mathbf{A} = \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(0)\bar{\mathbf{B}}\mathbf{H}\mathbf{A} = [m(x)] = \mu, \\ [u(0, t)] &= \mathbf{L}(0)\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(0)\mathbf{H}\bar{\mathbf{L}}(t)\mathbf{A} = \mathbf{X}(0)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A} = [h(t)] = \gamma. \end{aligned} \quad (11)$$

Then, the modified matrix system is obtained by simplification of Eq. (11)

$$\begin{aligned} \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(0)\bar{\mathbf{H}}\mathbf{A} &= \lambda, \\ \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(0)\bar{\mathbf{B}}\mathbf{H}\mathbf{A} &= \mu, \\ \mathbf{X}(0)\mathbf{H}\bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A} &= \gamma. \end{aligned} \quad (12)$$

3. Method of solution

In this section, the numerical technique is introduced which was previously used in many studies. Here in this study, Laguerre polynomial approach which is based on finding approximate solution of the problem (1)-(2). Collocation methods have been applied on many

problems with the basic idea of collocation with piecewise polynomial functions is developed as a method for solving different types of problems [10], [11], [38]. Stability of the technique was also explained with details on specific problems [6].

The collocation points

$$x_i = \frac{l}{N}i, \quad i = 0, 1, \dots, N, \quad t_j = \frac{T}{N}j, \quad j = 0, 1, \dots, N, \quad \text{and } h = \frac{l}{N}, \quad (13)$$

are defined and where h is called as "step size" [39]. The collocation points are a number of appropriately located points which are also called in different names such as matching points, enough points and so on [31], [32]. In Fig. (1) a random grid section for the collocation points, which are defined in Eq. (13), is given.

Figure 1. A representation of the connection between arguments on a rectangular domain.

The polynomials of each argument is necessarily agreed with the SPDP-PDE at collocation points of the argument and a precise degree of the continuity together with the polynomials which is in the neighborhood through matching points. So that the collocation points are described with regard to this idea in Eq. (13). They are defined as typical collocation points and these matching points on the common sides shared by the merged subregions [34]. By increasing the number of collocation points, numerical solution approximate the exact solution. Nevertheless, this enhancing at a precise range and the scheme bear to cause divergence, singularities, etc. So that, a rectangular domain is used with typical collocation scheme in this study.

The collocation points, which are defined in (13), are put into Eq. (10), then the fundamental matrix equation is obtained as

$$\begin{aligned} &\{\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{B}}\overline{\mathbf{H}} - \epsilon\mathbf{X}(x_i)\mathbf{B}^2\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}} - a(x_i, t_j)\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}} \\ &+ b(x_i, t_j)\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\mathbf{B}(1, -\tau)\overline{\mathbf{H}}\}\mathbf{A} = [f(x_i, t_j)] = \mathbf{F}. \end{aligned} \quad (14)$$

Briefly,

$$\mathbf{W}(x_i, t_j)\mathbf{A} = \mathbf{F} \text{ or } [\mathbf{W}; \mathbf{F}]. \quad (15)$$

If the identical procedure is followed for the boundary and initial conditions Eq. (12), then

$$\begin{aligned} \mathbf{U}\mathbf{A} &= \lambda_i, & [\mathbf{U}; \lambda], \\ \mathbf{V}\mathbf{A} &= \mu_i, \text{ or } [\mathbf{V}; \mu], \\ \mathbf{Z}\mathbf{A} &= \gamma_j, & [\mathbf{Z}; \gamma]. \end{aligned} \quad (16)$$

Therefore, the solution of Eq. (1) under the conditions Eq. (2) is obtained by replacing the row matrices Eq. (16) by the last rows of the augmented matrix Eq. (15), the required augmented matrix is gained and, Eq. (15) becomes,

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{F}}]. \quad (17)$$

Laguerre coefficients can be computed by solving the augmented matrix system. Thus, the approximate solution $u_N(x, t)$, $0 \leq x \leq l$, $0 \leq t \leq T$ is found in the truncated Laguerre series Eq. (3) [14], [15].

$$u_N(x, t) = \sum_{n=0}^N \sum_{p=0}^N a_{n,p} L_{n,p}(x, t).$$

4. Error analysis

In this section, an error estimation is introduced for the numerical solution Eq. (3) of Eq. (1) in order to investigate the consistency of the method in different norms.

Definition: (Error function) The error function is described for x_i and t_j , $i, j = 0, 1, 2, \dots, N$ values in Eq. (13). It is shown as follows

$$e_i = u(x_i, t_j) - u_N(x_i, t_j), \quad (18)$$

where $u(x_i, t_j)$ and $u_N(x_i, t_j)$ are the exact and approximate solutions of the problem, respectively [12], [26], [8], [9].

Definition: (Absolute error function) The absolute error function is described for x_i and t_j , $i, j = 0, 1, 2, \dots, N$ values in Eq. (13). It is shown as follows

$$E_N = |u - (u_t)_N + \epsilon(u_{xx})_N - au_N - b(u_{-\tau})_N + f| = |e_i|, \quad (19)$$

where $E_N \leq 10^{-k}$ for k is prescribed in which decrease at each point when the truncation limit N escalates.

Definition: (Truncation error function) Truncation error occurs in numerical solutions of the problem which includes rounding error in each step of the calculation. In computational science and scientific computation it helps us for understanding convergence and stability of the present technique. In here, the truncation error is given by the notation s_i which addresses

$$s_i = (\epsilon - \tau b(x, t))(u(x_i, t_j) - 2u(x_{i+1}, t_{j+1}) + u(x_{i+2}, t_{j+2})/h^2 - u_{xx}(x_{i+1}, t_{j+1})). \quad (20)$$

Definition: (Norm) Let us define $\|\cdot\| : U \rightarrow [0, \infty]$ a map with following properties:

- (i) $\|u\| = 0 \Rightarrow u = 0$,
- (ii) $\|\alpha u\| = |\alpha| \|u\| \quad \forall \alpha \in \mathbb{R}, u \in U$,
- (iii) $\|\alpha u + v\| \leq \|u\| + \|v\|$.

The map is called as "norm" [29]. Furthermore, we consider different error norms in the followings:

$$\begin{aligned} \text{For } L_2; \|E_N\|_2(x_\alpha, t_\beta) &= \left(\sum_{i=1}^n (e_i)^2 \right)^{1/2}, \\ \text{For } L_\infty; \|E_N\|_\infty(x_\alpha, t_\beta) &= \max(e_i), \quad 0 \leq i \leq n, \\ \text{For } RMS; E_N^*(x_\alpha, t_\beta) &= \sqrt{\frac{\sum_{i=1}^n (e_i)^2}{n}}, \end{aligned} \quad (21)$$

where $e_i = u(x_i, t_j) - u_N(x_i, t_j)$ also $u(x_i, t_j)$ and $u_N(x_i, t_j)$ are the exact and approximate solutions of the problem, respectively [12], [26], [8], [9]. Besides, n is the truncation number for the error series with the relation $n \geq N$. Moreover, we define the error functions with mappings respectively.

Stability and convergence analysis

Here the stability and convergence of the numerical scheme which is already introduced in this section.

Theorem 1 *It is assumed that $(\epsilon - \tau b(x, t))$, $a(x, t) \leq M < 0$ and $b(x, t) < 0$, $\forall (x, t) \in [a, b] \times [0, T]$ is the solution to the system of equations in Eq. (17) together with the given conditions which is unique, exists and answers*

$$\|u(x, t)\|_{h, \infty} \leq M^{-1} \|f(x, t)\|_{h, \infty} + (|\phi_0| + |m_0| + |h_0|), \quad (22)$$

where M is a positive constant value with $a(x, t) \geq M > 0$ in $[a, b]$ and $\|u(x, t)\|_{h, \infty}$ is the discrete L_∞ -Norm which is defined in Eq. (21).

Then it implies that system solution of given in Eq. (17) uniformly bounded. Hence present scheme is stable (see also Proof in [23]).

Theorem 2 *Based on Theorem 1, the error function e_i , $u(x, t)$ and the solution of the problem (1)-(2) $u_N(x, t)$ satisfies the following*

$$\|e\|_{h, \infty} \leq M^{-1} \|s\|, \quad (23)$$

where s is the truncation error function which is defined in Eq. (20). Besides,

$$\begin{aligned} |s_i| &\leq \max_{x_i \leq x \leq x_{i+2}} (h/2)b|u_{xx}| + \max_{x_i \leq x \leq x_{i+2}} (h/6)b|u_{xxx}| \\ &\quad + \max_{x_i \leq x \leq x_{i+2}} (h^3/24)(2(\epsilon - \tau b) + b)|u_{xxx}|. \end{aligned}$$

Proof: Truncation error s_i is given as

$$\begin{aligned} s_i &= (\epsilon - \tau b(x, t))(u(x_i, t_j) - 2u(x_{i+1}, t_{j+1}) + u(x_{i+2}, t_{j+2}))/h^2 \\ &\quad - u_{xx}(x_{i+1}, t_{j+1}) \end{aligned}$$

$$|s_i| \leq \max_{x_i \leq x \leq x_{i+2}} (h/2)b|u_{xx}| + \max_{x_i \leq x \leq x_{i+2}} (h/6)b|u_{xxx}| \\ + \max_{x_i \leq x \leq x_{i+2}} (h^3/24)(2(\epsilon - \tau b) + b)|u_{xxxx}|.$$

Then

$$L_h(e_i) = L_h(u) - L_h(u_N) = s_i, \quad i = 0, 1, 2, \dots, N.$$

and $e_0 = 0 = e_N$ so that $\|e_{h,\infty}\| \leq M^{-1}\|\tau_{h,\infty}\|$. Thus the convergence is proven with the fixed points of ϵ . $\square$

Algorithm

In this section, the Steps of algorithm for the technique is given. A systematic approach has been taken by using our algorithm to get an approximation [16].

Step 0. Input initial data: $a(x,t), b(x,t)$ and τ . Determine the initial and boundary conditions.

Step 1. Set N where $N \in \mathbb{N}$.

Step 2. Construct the matrices $\mathbf{F}$, $\mathbf{X}(x)$, $\bar{\mathbf{X}}(t)$, $\mathbf{B}$, $\bar{\mathbf{B}}$, $\mathbf{B}(1, -\tau)$, $\mathbf{H}$, and $\bar{\mathbf{H}}$, then $\mathbf{W}$.

Step 3. Define the collocation points $x_i = \frac{l}{N}i, i = 0, 1, \dots, N; t_j = \frac{T}{N}j, j = 0, 1, \dots, N$.

Step 4. Compute $[\mathbf{W}:\mathbf{F}]$ [14].

Step 5. Compute $[\mathbf{U}:\lambda], [\mathbf{V}:\mu], [\mathbf{Z}:\gamma]$.

Step 6. Construct the augmented matrix $[\tilde{\mathbf{W}}:\tilde{\mathbf{F}}]$.

Step 7. Input: the augmented matrix arguments, forward elimination, back substitution.
Output: $\mathbf{A}$ (Solve the system by Gaussian elimination method).

Step 8. Put arguments $a_{n,p}$ in the truncated Laguerre series form.

Step 9. Output data: the approximate solution $u_N(x, t)$.

Step 10. Construct $u(x, t)$ is the exact solution of (1).

Step 11. Stop when $E_N \leq 10^{-k}$ where $k \in \mathbb{Z}^+$. Otherwise, increase N and return to Step 1.

Figure 2. Steps of algorithm.

5. Numerical experiment

In this section, we investigate a test case to show applicability of our method.

Test Case: We consider SPDP-PDE as [4]

$$u_t(x, t) - \epsilon u_{xx}(x, t) = \frac{1}{2}((2x\sqrt{\epsilon} - \epsilon) \exp(-(t + x/\sqrt{\epsilon})) - (2x\sqrt{\epsilon} + \epsilon) \exp(-(t + (1 - x)/\sqrt{\epsilon}))) - (2 + x^2)u(x, t - 1),$$

$$(x, t) \in [0, 2] \times [0, 1],$$

with the initial condition

$$u(x, t) = (2 + x^2)(\exp(-(t + x/\sqrt{\epsilon}) + \exp(-(t + (1 - x)/\sqrt{\epsilon}))),$$

$$(x, t) \in [0, 1] \times [-\tau, 0], \quad (24)$$

and with the boundary conditions

$$u(0, t) = \exp(-t) + \exp(-t - 1/\sqrt{\epsilon}), t \in [0, 2],$$

$$u(1, t) = \frac{3}{2} \exp(-t) + \exp(-t - 1/\sqrt{\epsilon}), t \in [0, 2].$$

Besides, $u(x, t) = (2 + x^2)(\exp(-(t + x/\sqrt{\epsilon}) + \exp(-(t + (1 - x)/\sqrt{\epsilon})))$ is given as the exact solution of (24). Table 1 shows that Laguerre collocation method (LCM) gives better maximum errors than Standard finite difference method (SFDM) and Fitted operator finite difference method (FOFDM) for the small enough values. We also can see the errors in different norms in Table 2. In Fig. (3), Fig. (4) and Fig. (5) exact and numerical solutions of LCM for different N values are represented.

Table 1. Comparison between maximum errors for $N = 8$ the present method (LCM) with SFDM and FOFDM [4].

$N = 8$	$\epsilon = 1$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-10}$	$\epsilon = 10^{-16}$
LCM	$1.403E - 02$	$2.193E - 02$	$6.855E - 10$	$5.106E - 15$
SFDM	$1.508E - 02$	$3.923E - 02$	$5.685E - 09$	$5.685E - 15$
FOFDM	$1.600E - 02$	$8.950E - 02$	$2.320E - 01$	$2.320E - 01$

In Table 1, we have the error comparison which is given in L_2 Norm. As we can see from this table that the approximation results are of less error with regard to smaller ϵ values. Besides, we have the specific point for both time and space values which is 0.5.

Table 2. L_2 , L_∞ and RMS errors for $N = 4$ and $t = 0.2$.

x	L_2	L_∞	RMS
1	$1.213E - 02$	$2.200E - 02$	$6.540E - 02$
2	$1.958E - 03$	$1.983E - 03$	$1.995E - 03$
3	$7.440E - 03$	$7.590E - 03$	$7.120E - 03$
4	$5.830E - 04$	$2.193E - 04$	$8.665E - 04$
5	$3.028E - 03$	$3.038E - 02$	$5.685E - 02$

Figure 3. Numerical Solution for $N = 4$.

Figure 4. Numerical Solution for $N = 6$.

Figure 5. Exact Solution of the problem (24)

6. Conclusion

In this study, we have introduced a matrix method depending on Laguerre polynomials in order to solve SPDP-PDEs numerically. Furthermore, the error analysis is given to show the accuracy of the method. The present method and its error analysis are applied on illustrative example which have been shown by figures and tables. The method has significant importance such as:

- The present method has short and concise computing procedure by writing the algorithm in Maple.
- The present method has sufficient results when N is chosen large enough [17], [18].

The method also can be extended on different type of mathematical models but some mod-

ifications are required [40], [27], [28], [22], [25], [5].

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Received 16.06.2020, Accepted 17.02.2021