**ANALYSIS OF HOUSEHOLD INCOME IN POLAND
BASED ON THE ZENGA DISTRIBUTION AND SELECTED
INCOME INEQUALITY MEASURE**

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Abstract

Research background: A lot of research has been directed at describing empirical distributions by using a theoretical model. In the literature there are proposals for various types of mathematical functions. In 2010 Zenga proposed a new three-parameter model for economic size distribution which possesses interesting statistical properties which can be used to model income, wealth and financial variables.

Purpose: The aim of this paper is to apply the Zenga model to income distributions in Poland by voivodeship.

Research methodology: The basis for the calculations presented in the paper has been based on the individual data coming from a random sample obtained within a Household Budget Survey conducted by the central Statistical Office in the year 2014. The parameters estimates of the Zenga distribution were obtained by means of the D'Addario's invariants methods, mainly with the Pietra index.

Results: The results of the conducted approximations, presented in the paper confirmed the good consistency of the Zenga distribution with the empirical income distribution in Poland, both in total and for households.

Novelty: The study contributed to the application of a new three-parameter income distribution model to describe income distributions in Poland.

Keywords: income distribution, Zenga distribution, the Gini income inequality index, the Zenga income inequality index

JEL classification: C1, C10, C13, C15

Introduction

The analysis of the income distribution has a long history. Its evolution has been considered essential to explain not only the causes but also the potential consequences of inequality and poverty. For a majority of countries we can observe similar, characteristic shapes of the income distribution. For many countries income distributions are single-modal with right-sided asymmetry. A lot of research has also been directed at describing empirical distributions by using a theoretical model. This approach can be useful for many reasons. Knowledge of the model of income distribution, which is a simple approximation of empirical distribution and knowledge of tendency of its parameters development may be used to predict the behavior of a particular variable in a following period of time. Nevertheless, the approximation of the empirical wage and income distributions by means of theoretical curves can smooth the irregularities coming from the method of data collecting. While choosing a statistical model it is important both, to find out a theoretical distribution function that would characterize empirical frequency distribution and to choose suitable methods to calculate the parameters of the model. In the literature there are proposals for various types of mathematical functions. Especially the three-parameter distributions have represented a good approximation of income distribution for many countries. Very high accuracy with empirical distribution is characteristic for the Dagum and Singh-Maddala models. These models are the three-parameter density functions with relatively simple analytical forms.

Various theoretical distributions for example Pareto, logarithmic-normal, and gamma were used to approximate the income distributions of a Polish population. In the literature, many papers perform this problem for various families of distributions (see Wiśniewski, 1934; Vielrose, 1960; Lange, 1967; Kordos, 1968; Kordos, 1973; Kot, 1999; Kot 2000). The analysis of the income and wage distribution in Poland confirmed that the Dagum and Singh-Maddala distribution are particularly well fitted to empirical data. This observation is widely documented in the literature (see Brzeziński, 2013; Jędrzejczak, 1993; Jędrzejczak, 2006; Łukasiewicz, Orłowski, 2004; Ostasiewicz, 2013; Salamaga, 2016).

In 2010, Zenga proposed a new three-parameter model for distribution by size. The model is a Beta-mixture defined for non-negative distributions that has positive skewness and a Paretian right-tail. Its parameters separately control the location and inequality. In this model μ is the scale parameter and is equal to the expect value, α and θ are shape parameters that the inequality depends on. Studies performed in various countries show that the Zenga distribution exhibits high conformance to the empirical distributions of incomes. This distribution has not

been used so far to analyze income distributions in Poland. The aim of this paper is to apply Zenga distribution for Polish Voivodeships.

1. Zenga Distribution

The three-parameter model was introduced by Zenga (Zenga, 2010; Zenga, Pasquazzi, Poliscichio, 2011; Zenga, Pasquazzi, Zenga, 2012). The probability density function $f(x; \mu; \alpha; \theta)$, ($\mu > 0; \alpha > 0; \theta > 0$) for non-negative variables takes the form:

$$f(x; \mu; \alpha; \theta) = \int_0^1 v(x; \mu; k) g(k; \alpha; \theta) dk$$

$$= \begin{cases} \frac{1}{2\mu B(\alpha; \theta)} \left(\frac{x}{\mu}\right)^{-1.5} \int_0^{\frac{x}{\mu}} k^{\alpha-0.5} (1-k)^{\theta-2} dk, & \text{for } 0 < x < \mu \\ \frac{1}{2\mu B(\alpha; \theta)} \left(\frac{\mu}{x}\right)^{1.5} \int_0^{\frac{\mu}{x}} k^{\alpha-0.5} (1-k)^{\theta-2} dk, & \text{for } x > \mu \end{cases} \quad (1)$$

The density $f(x; \mu; \alpha; \theta)$ has been obtained as a mixture of Poliscichio's (2008) following truncated Pareto density

$$v(x; \mu; k) = \begin{cases} \frac{\sqrt{\mu}}{2} k^{0.5} (1-k)^{-1} x^{-1.5}, & \mu k \leq x \leq \frac{\mu}{k}; \quad \mu > 0, \quad 0 < k < 1 \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

with a fixed $\mu > 0$ and all the values of k in the interval $(0; 1)$. The density on the parameter k is given by the beta density and has the following form

$$g(k; \alpha; \theta) = \begin{cases} \frac{k^{\alpha-1} (1-k)^{\theta-1}}{\beta(\alpha; \theta)}, & 0 < k < 1; \quad \theta > 0, \quad \alpha > 0 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

where $B(\alpha; \theta)$ is the beta function.

The parameter α is an inverse inequality indicator and θ is a direct inequality indicator. In particular the bigger value of the parameter α the less unequal the distribution and the bigger the value, the more unequal the distribution (Porro, 2015). The expected value $E(X)$ is always equal to the parameter μ .

The moment of order r for the Zenga distribution is defined as follows:

$$E(X^r) = \frac{\mu^r}{2r-1} \frac{1}{B(\alpha; \theta)} \{B(\alpha-r+1; \theta-1) - B(\alpha+r; \theta-1)\}, \text{ where } r \in \mathbb{N}, r < \alpha+1, \theta > 1$$

$$E(X^r) = \frac{\mu^r}{2r-1} \frac{1}{B(\alpha; \theta)} \sum_{i=1}^{2r-1} B(\alpha-r+i; \theta), \text{ where } r \in \mathbb{N}, r < \alpha+1, \theta > 0$$
(4)

In the case $\theta > 0$, the distribution function $F(x; \mu; \alpha; \theta)$ is described by the equation

$$F(x; \mu; \alpha; \theta) = \begin{cases} \frac{1}{B(\alpha, \theta)} \sum_{i=1}^{\infty} \left\{ IB\left(\frac{x}{\mu}; \alpha+i-1; \theta\right) - \left(\frac{\mu}{x}\right)^{0.5} IB\left(\frac{x}{\mu}; \alpha+i-0.5; \theta\right) \right\} & \text{if } 0 < x \leq \mu \\ 1 - \frac{1}{B(\alpha, \theta)} \sum_{i=1}^{\infty} \left\{ \left(\frac{\mu}{x}\right)^{0.5} IB\left(\frac{\mu}{x}; \alpha+i-0.5; \theta\right) - IB\left(\frac{\mu}{x}; \alpha+i; \theta\right) \right\} & \text{if } \mu < x \end{cases}$$
(5)

where:

$$IB(x; \alpha; \theta) = \int_0^x t^{\alpha-1} (1-t)^{\theta-1} dt, \quad 0 < x < 1$$
(6)

is the incomplete beta function.

Figure 1. The density function $f(x; 3; 3; \theta)$ of Zenga distribution for $\mu = 3$ and $\alpha = 3$

Source: authors calculations.

Figure 2. The density function $f(x; 3; 0.5; \theta)$ of Zenga Distribution for $\mu = 3$ and $\alpha = 0.5$

Source: author's calculations.

Figure 3. The density function $f(x; 3; \alpha; 7)$ of Zenga Distribution for $\mu = 3$ and $\theta = 7$

Source: author's calculations.

Figure 4. The density function $f(x; 3; \alpha; 0.5)$ of Zenga distribution for $\mu = 3$ and $\theta = 0.5$

Source: author's calculations.

It is easy to see in graphs 1–4 how many behaviours it can have, changing the values of the parameters. The new density allows for a wider variety of shapes than the traditional three-parameter models of income distributions as for example the Dagum and Singh-Maddala distribution. The distribution depends on three parameters: in particular μ is a scale parameter, α in an inverse inequality indicator and it controls the tails of the distribution, while θ is a direct inequality indicator and it controls the distribution around the expected value μ (Arcagni, Porro, 2013).

2. Measures of income inequality

The most popular measures of income inequality are the Gini coefficient and Zenga coefficient. The Gini coefficient is half the relative mean difference and is usually defined while being based on the Lorenz curve. The Zenga coefficient is defined and based on the Zenga curve.

The graphs of the Lorenz curve for Zenga distribution can be obtained by putting in the abscissa $F(x)$ and inordinate axis $H(x)$, for a sufficient number of values of x . It is worth noting here that these measures are invariant to scale transformation, so it is enough to consider the case $f(x; 1; \alpha; \theta)$. We know that:

$$F(x; 1; \alpha; \theta) = \begin{cases} F_1(x; 1; \alpha; \theta) = \frac{1}{B(\alpha; \theta)} \sum_{i=1}^{\infty} \{IB(x; \alpha + i - 1; \theta) - x^{-0.5} IB(x; \alpha + i - 0.5; \theta)\}, & \text{if } 0 < x \leq 1 \\ F_2(x; 1; \alpha; \theta) = 1 - \frac{1}{B(\alpha; \theta)} \sum_{i=1}^{\infty} \left\{ x^{-0.5} IB\left(\frac{1}{x}; \alpha + i - 0.5; \theta\right) - IB\left(\frac{1}{x}; \alpha + i; \theta\right) \right\}, & \text{if } 1 < x \end{cases} \quad (7)$$

The first incomplete moment of the Zenga density $f(x; \mu; \alpha; \theta)$ is given by

$$H(x; 1; \alpha; \theta) = \int_0^x t f(t; 1; \alpha; \theta) dt \quad (8)$$

By the elementary transformations we get:

$$H(x; 1; \alpha; \theta) = \begin{cases} H_1(x; 1; \alpha; \theta) = \frac{1}{B(\alpha; \theta)} \sum_{i=1}^{\infty} [x^{0.5} IB(x; \alpha + i - 0.5; \theta) - IB(x; \alpha + i; \theta)], & \text{if } 0 < x \leq 1 \\ H_2(x; 1; \alpha; \theta) = 1 - \frac{1}{B(\alpha; \theta)} \sum_{i=1}^{\infty} \left[IB\left(\frac{1}{x}; \alpha + i - 1; \theta\right) + x^{0.5} IB\left(\frac{1}{x}; \alpha + i - 0.5; \theta\right) \right], & \text{if } 1 < x \end{cases} \quad (9)$$

Zenga point inequality (Zenga, 2007a) has a form

$$A(x) = \frac{F(x; 1; \alpha; \theta) - H(x; 1; \alpha; \theta)}{F(x; 1; \alpha; \theta)[1 - H(x; 1; \alpha; \theta)]} \quad (10)$$

The graphs of Zenga's point inequality can be obtained putting in abscissa $F(x; 1; \alpha; \theta)$ and $A(x)$ on the ordinate axis for a sufficient number of values of x .

The Gini inequality index G can be obtained by numerical integration. Similarly, the Zenga inequality measure I can be obtained by numerical integration of the Zenga point inequality curve for the entire income range. See Zenga (2007a, 2007b) for further details about the Zenga inequality measure.

3. Results

The basis for the calculations presented in the paper has been based on the individual data coming from the random sample obtained within the Household Budget Survey conducted by the central Statistical Office in the year 2014. The data obtained from the Household Budget Survey (HBS) allow for an analysis of living conditions in Poland, being the basic source of information on the revenues and expenditure of the population.

To find a degree of adjustment of a theoretical distribution to the empirical one we have calculated the goodness of fit measures: the Mortara index A_1 , the quadratic K. Pearson index A_2 and the modified quadratic index A'_2 which are described by the following formulas:

$$A_1 = \frac{1}{n} \sum_{j=1}^s |n_j - \hat{n}_j| \quad (11)$$

$$A_2 = \sqrt{\frac{1}{n} \sum_{j=1}^s \frac{(n_j - \hat{n}_j)^2}{\hat{n}_j}} \quad (12)$$

$$A'_2 = \sqrt{\frac{1}{n} \sum_{j=1}^s \frac{(n_j - \hat{n}_j)^2}{n_j}} \quad (13)$$

Where n_j and $\hat{n}_j$ are respectively the observed and estimated frequencies of the j -th interval. The coefficient of distributions similarity W_p for the empirical data arranged into a grouped frequency distribution with s -class intervals can be calculated by the formula:

$$W_p = \sum_{i=1}^s \min(w_i ; w'_i) \quad (14)$$

where w_i and w'_i represent empirical and theoretical frequencies, respectively.

The results of the approximation of the empirical income distributions in Poland by means of the Zenga model, together with the goodness of the fit measures A_1, A_2, A'_2 and W_p are presented in Table 1 and in Figure 5. The approximating methods for the Zenga distribution have been analysed in the paper (Zenga, 2010). Table 1 contains the parameter estimates obtained by means of D'Addario's invariants method, mainly with the Pietra index, which gave the best fitting. In 2014 the randomly selected sample covered 37,148 households, i.e. approximately 0.3% of the total number of households in Poland. The households have been classified according to sixteen voivodeships. These are: dolnośląskie, kujawsko-pomorskie, lubelskie, lubuskie, łódzkie, małopolskie, mazowieckie, opolskie, podkarpackie, podlaskie, pomorskie, śląskie, świętokrzyskie, warmińsko-mazurskie, wielkopolskie, and zachodnio-pomorskie.

Table 1. Results of the estimation methods and goodness of fit measures for income distributions in Poland from a Household Budget Survey in 2014

Voivodeship	Estimated values of parameters			Indexes of goodness of fit			w_p
	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\theta}$	A_1	A_2	A'_2	
Dolnośląskie	3844.423	3.5282	4.8954	0.1059	0.1291	0.1366	0.9468
Kujawsko-pomorskie	3571.206	3.1567	5.0032	0.0812	0.0945	0.0960	0.9594
Lubelskie	3642.403	2.9708	4.7894	0.0622	0.1008	0.1056	0.9688
Lubuskie	3889.118	3.6807	4.3216	0.1338	0.1691	0.1616	0.9329
Łódzkie	3779.297	2.4118	3.4963	0.1071	0.1071	0.1378	0.9465
Małopolskie	3752.958	3.2573	4.0968	0.0841	0.1069	0.1209	0.9579
Mazowieckie	4602.032	2.1478	3.3247	0.0665	0.0889	0.0876	0.9667
Opolskie	3724.747	3.3297	4.4413	0.1217	0.2038	0.1674	0.9358
Podkarpackie	3519.052	2.3493	3.0077	0.1046	0.1501	0.1669	0.9476
Podlaskie	3485.184	3.4404	5.6692	0.1028	0.1251	0.1353	0.9486
Pomorskie	3917.293	2.9737	3.7733	0.0797	0.1252	0.1064	0.9601
Śląskie	3772.442	2.4485	2.6784	0.1393	0.1832	0.2112	0.9300
Świętokrzyskie	3613.180	2.5637	3.7145	0.1024	0.1484	0.1629	0.9487
Warmińsko-mazurskie	3447.347	2.7436	3.8567	0.1131	0.1429	0.1609	0.9434
Wielkopolskie	3999.477	2.6198	3.4849	0.0895	0.1192	0.1201	0.9552
Zachodnio-pomorskie	3678.149	3.0759	3.7393	0.1035	0.1342	0.1328	0.9481
Total	3755.330	2.7943	3.9244	0.0599	0.0815	0.0052	0.9700

Source: author's calculations based on the HBS data.

Voivodeship: dolnośląskie

Voivodeship: kujawsko-pomorskie

Voivodeship: lubelskie

Voivodeship: lubuskie

Voivodeship: łódzkie

Voivodeship: małopolskie

Voivodeship: mazowieckie

Voivodeship: opolskie

Voivodeship: podkarpackie

Voivodeship: podlaskie

Voivodeship: pomorskie

Voivodeship: śląskie

Voivodeship: świętokrzyskie

Voivodeship: warmińsko-mazurskie

Voivodeship: wielkopolskie

Voivodeship: zachodnio-pomorskie

Figure 5. Zenga density function $f(x; \mu; \alpha; \theta)$ fitted to empirical 2014 Household income distribution by voivodeship

Source: author's calculations.

Through the goodness of fit indexes and the presented graphs it can be observed that the Zenga distribution well describes the empirical distribution of incomes both in total and voivodeships in Polish households. A very good agreement is for the following regions: lubelskie, mazowieckie, pomorskie, kujawsko-pomorskie, małopolskie and wielkopolskie. For these regions we can observe small differences of the goodness of fit indexes and high values of the coefficient of distributions similarity W_p . Note that, parameters estimated for the śląskie and lubuskie regions do not fit well. There are also wide differences between the A_1 index and quadratic indexes. The households of the following voivodeships: mazowieckie, wielkopolskie and pomorskie have the highest average income. Furthermore the households of the following voivodeships: warmińsko-mazurskie and podlaskie have the lowest average income.

The Zenga distribution was applied to the estimation of point and synthetic inequality measures. Table 2 describes the results of income inequality decomposition by Polish HBS. The values of the Gini and Zenga coefficients were obtained by means of numerical integration based on the Lorenz and Zenga curves.

Figures from 6 to 11 report on the graphs of the Lorenz curve $L(p)$ and Zenga curve $I(p)$, where $p = F(x)$ for Polish voivodeships.

Table 2. Estimation results for income distributions in Poland from a Household Budget Survey in 2014

Voivodeship	Gini coefficient G	Zenga coefficient Z
Dolnośląskie	0.3252	0.6609
Kujawsko-pomorskie	0.3543	0.6921
Lubelskie	0.3602	0.6982
Lubuskie	0.2940	0.6240
Łódzkie	0.3509	0.6893
Małopolskie	0.3108	0.6449
Mazowieckie	0.3697	0.7068
Opolskie	0.3204	0.6558
Podkarpackie	0.3297	0.6676
Podlaskie	0.3579	0.6956
Pomorskie	0.3164	0.6517
Śląskie	0.2999	0.6339
Świętokrzyskie	0.3452	0.6859
Warmińsko-mazurskie	0.3386	0.6764
Wielkopolskie	0.3308	0.6684
Zachodnio-pomorskie	0.3137	0.6412
Total	0.3426	0.6783

Source: author's calculations based on the HBS data.

The formulas presented in section 3 can be applied to the data provided by the Household Budget Survey in 2014. Figures 6–11 present the most known inequality curves. The considered curves are the Lorenz curve $L(p)$ and the Zenga curve $I(p)$. Figures 1–3 show the behavior of the empirical Lorenz curves $L(p)$ for the following voivodeships: mazowieckie, wielkopolskie, warmińsko-mazurskie and the national Total. Moreover, figures 4–6 show the behavior of the empirical Zenga curves $I(p)$ for selected voivodeships. The values of the coefficients in Table 2 corresponding to the areas below the curves are included.

Figure 6. Lorenz curves $L(p)$ for the mazowieckie and wielkopolskie voivodeships

Source: author's calculations.

Figure 7. Lorenz curves $L(p)$ for the mazowieckie and warmińsko-mazurskie voivodeships

Source: author's calculations.

Figure 8. Lorenz curves $L(p)$ for the Total and mazowieckie voivodeship

Source: author's calculations.

Figure 9. Zenga curves $I(p)$ for the mazowieckie, wielkopolskie and pomorskie voivodeship

Source: author's calculations.

Figure 10. Zenga curves $I(p)$ for the mazowieckie and warmińsko-mazurskie voivodeships

Source: author's calculations.

Figure 11. Zenga curves $I(p)$ for the mazowieckie, wielkopolskie, and pomorskie warmińsko-mazurskie voivodeships and Total

Source: author's calculations.

The results of the calculations concerning the level of distribution inequality, revealed substantial differences between voivodeships in Poland. The Gini and Zenga coefficients were calculated on the basis of the Zenga distribution. The highest values, in the Gini and Zenga coefficients were observed for the mazowieckie voivodeship ($G_{\text{maz}} = 0.3697$, $Z_{\text{maz}} = 0.7068$). The overall income inequality in Poland in 2014, measured by means of the Gini and Zenga coefficients and estimated on the basis of the Polish HBS, were $G_{\text{total}} = 0.3426$, $Z_{\text{total}} = 0.6783$. In terms of mean income levels, the largest discrepancy was observed between the mazowieckie and warmińsko-mazurskie voivodeships, while the highest difference in inequality levels we observed between the mazowieckie and lubuskie voivodeships ($G_{\text{maz}} = 0.3426$, $Z_{\text{maz}} = 0.6783$; $G_{\text{lubus}} = 0.2940$, $Z_{\text{lubus}} = 0.6240$). The comparison between the $L(p)$ and $I(p)$ curves highlights the important results. In Figure 9 we can discover that the inequality curves $I(p)$ for wielkopolskie and pomorskie lies below the curve for mazowieckie for the entire income range. It is worth mentioning that, income inequality patterns for warmińsko-mazurskie, wielkopolskie and the national Total are very similar. Furthermore, we can observe that the inequality for mazowieckie is always beneath the national Total. In figures 1–3 we can observe that the inequality curves $L(p)$ for wielkopolskie, warmińsko-mazurskie and Total lies above mazowieckie. It is easy to see that the considerable coherence of the results summarized in (Table 2) and (Figures 6–11) confirm the adequacy of both these tools to measure the inequality degree for polish HBS.

Conclusions

In this work we use a new three-parameter model of income distribution which seems to provide a better fit to the experimental data. The Zenga density function can take on various shapes and this attribute allows a good fitting also for small incomes. This feature of the Zenga density function allows for a better fitting to experimental data than other probability density functions, which have been proposed as income distribution models.

The application of the Zenga model is better than the existing approaches to income analysis because the Zenga distribution parameters can be interpreted in economic terms. The μ parameter is interpreted as an arithmetic mean to individual income or an arithmetic household income mean, and other parameters α and θ can be interpreted in terms of income inequality.

The results of the conducted approximations, presented in the paper confirmed the good consistency of the Zenga distribution with the empirical income distribution in Poland, both in total and for voivodeships and households. The results of the calculations concerning the level of distribution inequality, revealed differences between the Polish voivodeships. Results concerning the distributions of incomes of Polish households for the year 2014 are an extension of our previous paper (Jędrzejczak, Trzcińska, 2018).

The Zenga distribution meets the criteria set for the theoretical income distribution, and the transparent economic interpretation of the Zenga density parameters is an additional argument for using this model to describe a Polish Household Budget Survey.

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