Derivation of Commutative Rings and the Leibniz Formula for Power of Derivation

Yasushige Watase
Suginami-ku Matsunoki
3-21-6 Tokyo, Japan

Summary. In this article we formalize in Mizar [1], [2] a derivation of commutative rings, its definition and some properties. The details are to be referred to [5], [7]. A derivation of a ring, say D , is defined generally as a map from a commutative ring A to A -Module M with specific conditions. However we start with simpler case, namely $\text{dom } D = \text{rng } D$. This allows to define a derivation in other rings such as a polynomial ring.

A derivation is a map $D : A \longrightarrow A$ satisfying the following conditions:

- (i) $D(x + y) = Dx + Dy$,
- (ii) $D(xy) = xDy + yDx, \forall x, y \in A$.

Typical properties are formalized such as:

$$D\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n Dx_i$$

and

$$D\left(\prod_{i=1}^n x_i\right) = \sum_{i=1}^n x_1x_2 \cdots Dx_i \cdots x_n \quad (\forall x_i \in A).$$

We also formalized the Leibniz Formula for power of derivation D :

$$D^n(xy) = \sum_{i=0}^n \binom{n}{i} D^i x D^{n-i} y.$$

Lastly applying the definition to the polynomial ring of A and a derivation of polynomial ring was formalized. We mentioned a justification about compatibility of the derivation in this article to the same object that has treated as differentiations of polynomial functions [3].

MSC: 13B25 13N15 68V20

Keywords: derivation; Leibniz Formula; derivation of polynomial ring

MML identifier: RINGDER1, version: 8.1.11 5.65.1394

1. PRELIMINARIES

From now on L denotes an Abelian, left zeroed, add-associative, associative, right zeroed, right complementable, distributive, non empty double loop structure, a, b, c denote elements of L , R denotes a non degenerated commutative ring, and n, m, i, j, k denote natural numbers.

Now we state the propositions:

$$(1) \quad n \cdot a + n \cdot b = n \cdot (a + b).$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 \cdot a + \$_1 \cdot b = \$_1 \cdot (a + b)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

$$(2) \quad (n \cdot a) \cdot b = a \cdot (n \cdot b).$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\$_1 \cdot a) \cdot b = a \cdot (\$_1 \cdot b)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

$$(3) \quad n \cdot (0_L) = 0_L.$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 \cdot (0_L) = 0_L$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

$$(4) \quad 0_L \cdot n = 0_L.$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 0_L \cdot \$_1 = 0_L$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

2. DEFINITION OF DERIVATION OF RINGS AND ITS PROPERTIES

From now on D denotes a function from R into R and x, y, z denote elements of R .

Definition of derivation of rings.

Let us consider R . Let Δ be a function from R into R . We say that Δ is derivation if and only if

$$(\text{Def. 1}) \quad \text{for every elements } x, y \text{ of } R, \Delta(x + y) = \Delta(x) + \Delta(y) \text{ and } \Delta(x \cdot y) = x \cdot \Delta(y) + y \cdot \Delta(x).$$

Observe that every function from R into R which is derivation is also additive and there exists a function from R into R which is derivation.

A derivation of R is derivation function from R into R . The functor $\text{Der } R$ yielding a subset of $(\Omega_R)^{\Omega_R}$ is defined by the term

$$(\text{Def. 2}) \quad \{f, \text{ where } f \text{ is a function from } R \text{ into } R : f \text{ is derivation}\}.$$

Let us observe that $\text{Der } R$ is non empty.

From now on D denotes a derivation of R .

Now we state the propositions:

$$(5) \quad (i) \ D(1_R) = 0_R, \text{ and}$$

$$(ii) \ D(0_R) = 0_R.$$

$$(6) \ D(n \cdot x) = n \cdot D(x).$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv D(\$1 \cdot x) = \$1 \cdot D(x)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

$$(7) \ D(x^{m+1}) = (m+1) \cdot (x^m \cdot D(x)).$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv D(x^{\$1+1}) = (\$1+1) \cdot (x^{\$1} \cdot D(x))$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

$$(8) \quad (i) \ D^{n+1} = D \cdot (D^n), \text{ and}$$

$$(ii) \ \text{dom } D = \text{the carrier of } R, \text{ and}$$

$$(iii) \ \text{dom}(D^n) = \text{the carrier of } R, \text{ and}$$

$$(iv) \ D^n \text{ is a (the carrier of } R\text{)-valued function.}$$

$$(9) \ (D^{n+1})(x) = D((D^n)(x)). \text{ The theorem is a consequence of (8).}$$

$$(10) \ \text{If } z \cdot y = 1_R, \text{ then } y^2 \cdot D(x \cdot z) = y \cdot D(x) - x \cdot D(y).$$

In the sequel s denotes a finite sequence of elements of the carrier of R and h denotes a function from R into R .

Let us consider R , s , and h . One can check that the functor $h \cdot s$ yields a finite sequence of elements of the carrier of R . Now we state the proposition:

$$(11) \ \text{If } h \text{ is additive, then } h(\sum s) = \sum(h \cdot s).$$

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{for every } h \text{ and } s \text{ such that } \text{len } s = \$1 \text{ and } h \text{ is additive holds } h(\sum s) = \sum(h \cdot s)$. $\mathcal{P}[0]$ by [4, (6)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

$$(12) \ \text{FORMULA } (f_1 + f_2 + \cdots + f_n)' = f_1' + f_2' + \cdots + f_n': \\ D(\sum s) = \sum(D \cdot s).$$

Let us consider R , D , and s . The functor $\text{DProd}(D, s)$ yielding a finite sequence of elements of the carrier of R is defined by

$$(\text{Def. 3}) \ \text{len } it = \text{len } s \text{ and for every } i \text{ such that } i \in \text{dom } it \text{ holds } it(i) = \\ \prod \text{Replace}(s, i, D(s_i)).$$

Now we state the propositions:

$$(13) \ \text{If } \text{len } s = 1, \text{ then } \sum \text{DProd}(D, s) = D(\prod s).$$

- (14) Let us consider a finite sequence t of elements of the carrier of R . If $\text{len } t \geq 1$, then $\sum \text{DProd}(D, t) = D(\prod t)$.

PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv \text{for every } s \text{ such that } \text{len } s = \S_1 \text{ holds } \sum \text{DProd}(D, s) = D(\prod s)$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k , $\mathcal{P}[k]$. $\square$

3. PROOF OF THE LEIBNIZ FORMULA FOR POWER OF DERIVATIONS

The Leibniz formula for power of a derivation of a commutative ring.

Let us consider R , D , and n . Let x , y be elements of R . The functor $\text{LBZ}(D, n, x, y)$ yielding a finite sequence of elements of the carrier of R is defined by

- (Def. 4) $\text{len } it = n + 1$ and for every i such that $i \in \text{dom } it$ holds $it(i) = \binom{n}{i-1} \cdot (D^{n+1-i})(x) \cdot (D^{i-1})(y)$.

Now we state the propositions:

- (15) $\text{LBZ}(D, 0, x, y) = \langle x \cdot y \rangle$.
 (16) $\text{LBZ}(D, 1, x, y) = \langle y \cdot D(x), x \cdot D(y) \rangle$.

Let us consider R , D , and m . Let x , y be elements of R . The functor $\text{LBZ0}(D, m, x, y)$ yielding a finite sequence of elements of the carrier of R is defined by

- (Def. 5) $\text{len } it = m$ and for every i such that $i \in \text{dom } it$ holds $it(i) = \left(\binom{m}{i-1} + \binom{m}{i} \right) \cdot (D^{m+1-i})(x) \cdot (D^i)(y)$.

The functor $\text{LBZ1}(D, m, x, y)$ yielding a finite sequence of elements of the carrier of R is defined by

- (Def. 6) $\text{len } it = m$ and for every i such that $i \in \text{dom } it$ holds $it(i) = \binom{m}{i-1} \cdot (D^{m+1-i})(x) \cdot (D^i)(y)$.

The functor $\text{LBZ2}(D, m, x, y)$ yielding a finite sequence of elements of the carrier of R is defined by

- (Def. 7) $\text{len } it = m$ and for every i such that $i \in \text{dom } it$ holds $it(i) = \binom{m}{i} \cdot (D^{m+1-i})(x) \cdot (D^i)(y)$.

Now we state the propositions:

- (17) $D(\sum \text{LBZ0}(D, n, x, y)) = \sum D \cdot (\text{LBZ0}(D, n, x, y))$.
 (18) $\text{LBZ0}(D, m, x, y) = \text{LBZ1}(D, m, x, y) + \text{LBZ2}(D, m, x, y)$.

PROOF: Set $p = \text{LBZ1}(D, m, x, y)$. Set $q = \text{LBZ2}(D, m, x, y)$. Set $r = \text{LBZ0}(D, m, x, y)$. For every k such that $1 \leq k \leq \text{len}(p + q)$ holds $(p + q)(k) = r(k)$. $\square$

- (19) $\sum \text{LBZ0}(D, n, x, y) = \sum \text{LBZ1}(D, n, x, y) + \sum \text{LBZ2}(D, n, x, y)$. The theorem is a consequence of (18).
- (20) $D \cdot (\text{LBZ0}(D, n, x, y)) = (\text{LBZ2}(D, n+1, x, y))_{\uparrow n+1} + (\text{LBZ1}(D, n+1, x, y))_{\uparrow 1}$.
 PROOF: Set $p = \text{LBZ2}(D, n+1, x, y)$. Set $q = \text{LBZ1}(D, n+1, x, y)$. Set $r = \text{LBZ0}(D, n, x, y)$. Reconsider $p_1 = p_{\uparrow n+1}$ as a finite sequence of elements of the carrier of R . Reconsider $q_1 = q_{\uparrow 1}$ as a finite sequence of elements of the carrier of R . For every i such that $1 \leq i \leq \text{len } D \cdot r$ holds $(D \cdot r)(i) = (p_1 + q_1)(i)$. $\square$
- (21) $\sum D \cdot (\text{LBZ0}(D, n, x, y)) = -(\text{LBZ1}(D, n+1, x, y))_{/1} + \sum \text{LBZ0}(D, n+1, x, y) - (\text{LBZ2}(D, n+1, x, y))_{/n+1}$. The theorem is a consequence of (20) and (19).
- (22) $\text{LBZ}(D, n+1, x, y) = (\langle (D^{n+1})(x) \cdot y \rangle \wedge \text{LBZ0}(D, n, x, y)) \wedge \langle x \cdot (D^{n+1})(y) \rangle$.
 PROOF: Set $p = \text{LBZ}(D, n+1, x, y)$. Set $q = \text{LBZ0}(D, n, x, y)$. Set $r = (\langle (D^{n+1})(x) \cdot y \rangle \wedge q) \wedge \langle x \cdot (D^{n+1})(y) \rangle$. For every k such that $1 \leq k \leq \text{len } p$ holds $p(k) = r(k)$. $\square$
- (23) $\sum (\langle (D^{n+1})(x) \cdot y \rangle \wedge \text{LBZ0}(D, n, x, y)) \wedge \langle x \cdot (D^{n+1})(y) \rangle = (D^{n+1})(x) \cdot y + \sum \text{LBZ0}(D, n, x, y) + x \cdot (D^{n+1})(y)$.
- (24) $D(\sum \text{LBZ}(D, n+1, x, y)) = \sum \text{LBZ}(D, n+2, x, y)$. The theorem is a consequence of (9), (21), (11), (22), and (23).
- (25) THE LEIBNIZ FORMULA FOR POWER OF DERIVATION:
 $(D^n)(x \cdot y) = \sum \text{LBZ}(D, n, x, y)$. The theorem is a consequence of (16), (9), (24), and (15).

4. EXAMPLE OF DERIVATION OF POLYNOMIAL RING OVER A COMMUTATIVE RING

Let us consider R . Let f be a function from $\text{PolyRing}(R)$ into $\text{PolyRing}(R)$ and p be an element of the carrier of $\text{PolyRing}(R)$. Observe that the functor $f(p)$ yields an element of the carrier of $\text{PolyRing}(R)$. Let R be a ring. The functor $\text{Der1}(R)$ yielding a function from $\text{PolyRing}(R)$ into $\text{PolyRing}(R)$ is defined by (Def. 8) for every element f of the carrier of $\text{PolyRing}(R)$ and for every natural number i , $it(f)(i) = (i+1) \cdot f(i+1)$.

Let us consider R . One can verify that $\text{Der1}(R)$ is additive.

In the sequel R denotes an integral domain and f, g denote elements of the carrier of $\text{PolyRing}(R)$.

Now we state the proposition:

- (26) Let us consider an element f of the carrier of $\text{PolyRing}(R)$, and a polynomial f_1 over R . Suppose $f = f_1$ and f_1 is constant. Then $(\text{Der1}(R))(f) = \mathbf{0}.R$.

PROOF: For every element i of $\mathbb{N}$, $(\text{Der1}(R))(f)(i) = (\mathbf{0}.R)(i)$. $\square$

In the sequel a denotes an element of R . Now we state the propositions:

- (27) Let us consider a natural number i , and an element p of the carrier of $\text{PolyRing}(R)$. Then $((a \upharpoonright R) * p)(i) = a \cdot p(i)$.
- (28) Let us consider elements f, g of the carrier of $\text{PolyRing}(R)$, and an element a of R . Suppose $f = a \upharpoonright R$. Then $(\text{Der1}(R))(f \cdot g) = (a \upharpoonright R) * (\text{Der1}(R))(g)$.
 PROOF: For every natural number n , $(\text{Der1}(R))(f \cdot g)(n) = ((a \upharpoonright R) * (\text{Der1}(R))(g))(n)$. $\square$

Let us consider an element f of the carrier of $\text{PolyRing}(R)$ and an element a of R . Now we state the propositions:

- (29) If $f = \text{anpoly}(a, 0)$, then $(\text{Der1}(R))(f) = \mathbf{0}.R$.
 PROOF: For every element n of $\mathbb{N}$, $(\text{Der1}(R))(f)(n) = (\mathbf{0}.R)(n)$. $\square$
- (30) If $f = \text{anpoly}(a, 1)$, then $(\text{Der1}(R))(f) = \text{anpoly}(a, 0)$.
 PROOF: For every element n of $\mathbb{N}$, $(\text{Der1}(R))(f)(n) = (\text{anpoly}(a, 0))(n)$.
 $\square$
- (31) Let us consider polynomials p, q over R . Suppose $p = \text{anpoly}(1_R, 1)$. Let us consider an element i of $\mathbb{N}$. Then

- (i) $(p * q)(i + 1) = q(i)$, and
 (ii) $(p * q)(0) = 0_R$.

PROOF: For every element i of $\mathbb{N}$, $(p * q)(i + 1) = q(i)$. Consider F_1 being a finite sequence of elements of the carrier of R such that $\text{len } F_1 = 0 + 1$ and $(p * q)(0) = \sum F_1$ and for every element k of $\mathbb{N}$ such that $k \in \text{dom } F_1$ holds $F_1(k) = p(k -' 1) \cdot q(0 + 1 -' k)$. $\square$

- (32) Let us consider elements f, g of the carrier of $\text{PolyRing}(R)$. Suppose $f = \text{anpoly}(1_R, 1)$. Then $(\text{Der1}(R))(f \cdot g) = (\text{Der1}(R))(f) \cdot g + f \cdot (\text{Der1}(R))(g)$.
 PROOF: Reconsider $d_1 = (\text{Der1}(R))(f)$, $d_2 = (\text{Der1}(R))(g)$ as a polynomial over R . Reconsider $f_1 = f$, $g_1 = g$ as a polynomial over R . For every element i of $\mathbb{N}$, $(\text{Der1}(R))(f \cdot g)(i) = (d_1 * g_1 + f_1 * d_2)(i)$. $\square$
- (33) Let us consider constant elements f, g of the carrier of $\text{PolyRing}(R)$. Then $(\text{Der1}(R))(f \cdot g) = (\text{Der1}(R))(f) \cdot g + f \cdot (\text{Der1}(R))(g)$. The theorem is a consequence of (29).
- (34) Let us consider elements f, g of the carrier of $\text{PolyRing}(R)$. Suppose f is constant. Then $(\text{Der1}(R))(f \cdot g) = (\text{Der1}(R))(f) \cdot g + f \cdot (\text{Der1}(R))(g)$. The theorem is a consequence of (29) and (28).
- (35) Let us consider elements x, y of the carrier of $\text{PolyRing}(R)$. Suppose x is not constant. Then $(\text{Der1}(R))(x \cdot y) = (\text{Der1}(R))(x) \cdot y + x \cdot (\text{Der1}(R))(y)$.
 PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every elements f, g of the carrier of $\text{PolyRing}(R)$ for every elements f_0, g_0 of the carrier of $\text{PolyRing}(R)$ such

that $f_0 = f$ and $g_0 = g$ and $\deg f_0 - 1 = \$_1$ holds $(\text{Der1}(R))(f_0 \cdot g_0) = (\text{Der1}(R))(f_0) \cdot g_0 + f_0 \cdot (\text{Der1}(R))(g_0)$. For every natural number k such that for every natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$ by [8, (4)]. For every natural number n , $\mathcal{P}[n]$. $\square$

(36) $(\text{Der1}(R))(f \cdot g) = (\text{Der1}(R))(f) \cdot g + f \cdot (\text{Der1}(R))(g)$. The theorem is a consequence of (35) and (34).

Let us consider R . Let us observe that $\text{Der1}(R)$ is derivation.

Now we state the propositions:

(37) Let us consider an element x of $\text{PolyRing}(R)$, and a polynomial f over R . If $x = f$, then for every natural number n , $x^n = f^n$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv x^{\$1} = f^{\$1}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [6, (19)]. For every natural number n , $\mathcal{P}[n]$. $\square$

(38) Let us consider an element x of $\text{PolyRing}(R)$. Suppose $x = \text{anpoly}(1_R, 1)$. Then there exists an element y of $\text{PolyRing}(R)$ such that

- (i) $y = \text{anpoly}(1_R, n)$, and
- (ii) $(\text{Der1}(R))(x^{n+1}) = (n+1) \cdot y$.

The theorem is a consequence of (30), (37), and (7).

From now on p denotes a polynomial over $\mathbb{R}_F$.

Let us consider p . The functor p' yielding a sequence of $\mathbb{R}_F$ is defined by

(Def. 9) for every natural number n , $it(n) = p(n+1) \cdot (n+1)$.

Now we state the proposition:

(39) Let us consider an element p_0 of $\text{PolyRing}(\mathbb{R}_F)$, and a polynomial p over $\mathbb{R}_F$. If $p_0 = p$, then $p' = (\text{Der1}(\mathbb{R}_F))(p_0)$.

PROOF: For every n , $(p')(n) = (\text{Der1}(\mathbb{R}_F))(p_0)(n)$. $\square$

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Accepted March 30, 2021