Algebraic Extensions

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Summary. In this article we further develop field theory in Mizar [1], [2], [3] towards splitting fields. We deal with algebraic extensions [4], [5]: a field extension E of a field F is algebraic, if every element of E is algebraic over F . We prove amongst others that finite extensions are algebraic and that field extensions generated by a finite set of algebraic elements are finite. From this immediately follows that field extensions generated by roots of a polynomial over F are both finite and algebraic. We also define the field of algebraic elements of E over F and show that this field is an intermediate field of $E|F$.

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1. PRELIMINARIES

Let L_1, L_2 be double loop structures. We say that $L_1 \approx L_2$ if and only if
(Def. 1) the double loop structure of $L_1 =$ the double loop structure of L_2 .

One can verify that the predicate is reflexive and symmetric.

Now we state the propositions:

- (1) Let us consider rings R, S . Then $R \approx S$ if and only if there exists a function f from R into S such that $f = \text{id}_R$ and f is isomorphism.
- (2) Let us consider strict rings R, S . Then $R \approx S$ if and only if $R = S$.

Let F_1, F_2 be fields. Let us note that $F_1 \approx F_2$ if and only if the condition
(Def. 2) is satisfied.

(Def. 2) F_1 is a subfield of F_2 and F_2 is a subfield of F_1 .

Now we state the proposition:

- (3) Let us consider a field F , an extension E of F , and a subset T of E . Then $\text{FAdj}(F, T) \approx F$ if and only if T is a subset of F .

Let us consider a field F and extensions E_1, E_2 of F . Now we state the propositions:

- (4) If $E_1 \approx E_2$, then $\text{VecSp}(E_1, F) = \text{VecSp}(E_2, F)$.
 (5) If $E_1 \approx E_2$, then $\deg(E_1, F) = \deg(E_2, F)$. The theorem is a consequence of (4).

Let F be a field and E be an extension of F . Note that there exists an extension of F which is E -homomorphic and there exists an extension of F which is E -monomorphic and there exists an extension of F which is E -isomorphic.

Let R be a ring and a, b be elements of R . One can check that the functor $\{a, b\}$ yields a subset of R . Let F be a field, V be a vector space over F , and a, b be elements of V . Note that the functor $\{a\}$ yields a subset of V . Let a, b be elements of V . Let us observe that the functor $\{a, b\}$ yields a subset of V . Let us note that every basis of V is linearly independent.

Now we state the proposition:

- (6) Let us consider a field F , a vector space V over F , and a subset X of V . Then X is linearly independent if and only if for every linear combinations l_1, l_2 of X such that $\sum l_1 = \sum l_2$ holds $l_1 = l_2$.

Let F be a field and E be an extension of F . Observe that every basis of $\text{VecSp}(E, F)$ is non empty and $\deg(E, F)$ is non zero.

Let E be an F -finite extension of F . Observe that every basis of $\text{VecSp}(E, F)$ is finite. Let us consider a field F and an extension E of F . Now we state the propositions:

- (7) $\deg(E, F) = 1$ if and only if the carrier of $E =$ the carrier of F .
 (8) $\deg(E, F) = 1$ if and only if $E \approx F$. The theorem is a consequence of (7).
 (9) $\deg(E, F) = 1$ if and only if $\{1_E\}$ is a basis of $\text{VecSp}(E, F)$. The theorem is a consequence of (7).

Let F be a field and E be an extension of F . One can check that there exists a subset of $\text{VecSp}(E, F)$ which is non empty, finite, and linearly independent.

Now we state the proposition:

- (10) Let us consider a field F , an extension E of F , and subsets T_1, T_2 of E . Suppose $T_1 \subseteq T_2$. Then $\text{FAdj}(F, T_1)$ is a subfield of $\text{FAdj}(F, T_2)$.

Let F be a field and p be a polynomial over F . The functor $\text{Coeff}(p)$ yielding a subset of F is defined by the term

(Def. 3) $\{p(i), \text{ where } i \text{ is an element of } \mathbb{N} : p(i) \neq 0_F\}$.

Let us note that $\text{Coeff}(p)$ is finite. Now we state the propositions:

- (11) Let us consider a field F , an extension E of F , and a polynomial p over E . Suppose $\text{Coeff}(p) \subseteq \text{the carrier of } F$. Then p is a polynomial over F .
- (12) Let us consider a field F , an extension E of F , and a non zero polynomial p over E . Suppose $\text{Coeff}(p) \subseteq \text{the carrier of } F$. Then p is a non zero polynomial over F . The theorem is a consequence of (11).
- (13) Let us consider a ring R , a ring extension S of R , an element p of the carrier of $\text{PolyRing}(R)$, and an element q of the carrier of $\text{PolyRing}(S)$. If $p = q$, then $\text{Roots}(S, p) = \text{Roots}(q)$.

Let R be an integral domain and p be a non zero element of the carrier of $\text{PolyRing}(R)$. Note that $\text{Roots}(p)$ is finite. Let S be a domain ring extension of R . One can check that $\text{Roots}(S, p)$ is finite. Let F be a field and E be an extension of F . Let us observe that there exists an extension of E which is F -extending. Let E be an F -finite extension of F . Note that there exists an F -extending extension of E which is F -finite and there exists an F -extending extension of E which is E -finite. Now we state the propositions:

- (14) Let us consider a field F , an element p of the carrier of $\text{PolyRing}(F)$, an extension E of F , an E -extending extension U of F , an element a of E , and an element b of U . If $a = b$, then $\text{ExtEval}(p, a) = \text{ExtEval}(p, b)$.
- (15) Let us consider a field F , an element p of the carrier of $\text{PolyRing}(F)$, an extension E of F , and an element q of the carrier of $\text{PolyRing}(E)$. Suppose $q = p$. Let us consider an E -extending extension U of F , and an element a of U . Then $\text{ExtEval}(q, a) = \text{ExtEval}(p, a)$.

Let R be a ring, S be a ring extension of R , and a be an element of R . The functor ${}^{\textcircled{a}}(a, S)$ yielding an element of S is defined by the term

(Def. 4) a .

Let a be an element of S . We say that a is R -membered if and only if

(Def. 5) $a \in \text{the carrier of } R$.

One can verify that there exists an element of S which is R -membered.

Let a be an element of S . Assume a is R -membered. The functor ${}^{\textcircled{a}}(R, a)$ yielding an element of R is defined by the term

(Def. 6) a .

Let a be an R -membered element of S . Let us observe that ${}^{\textcircled{a}}(R, a)$ reduces to a . Let F be a field and E be an extension of F . One can check that there exists an element of E which is non zero and F -algebraic.

Let a be an element of F . One can check that ${}^{\textcircled{a}}(a, E)$ is F -algebraic.

Let K be an E -extending extension of F and a be an F -algebraic element of E . Note that ${}^{\textcircled{a}}(a, K)$ is F -algebraic.

2. MORE ON FINITE EXTENSIONS

Now we state the propositions:

- (16) Let us consider a field F , an extension E of F , and an E -extending extension K of F . Then every linear combination of $\text{VecSp}(K, F)$ is a linear combination of $\text{VecSp}(K, E)$.
- (17) Let us consider a field F , an extension E of F , an E -extending extension K of F , a subset B_E of $\text{VecSp}(K, E)$, and a subset B_F of $\text{VecSp}(K, F)$. Suppose $B_F \subseteq B_E$. Then every linear combination of B_F is a linear combination of B_E . The theorem is a consequence of (16).
- (18) Let us consider a field F , an extension E of F , an E -extending extension K of F , a finite subset B_E of $\text{VecSp}(K, E)$, a finite subset B_F of $\text{VecSp}(K, F)$, a linear combination l_1 of B_F , and a linear combination l_2 of B_E . If $l_1 = l_2$ and $B_F \subseteq B_E$, then $\sum l_1 = \sum l_2$.

PROOF: by induction on $\text{card}(\text{the support of } l_1)$.

Let F be a field, E be an extension of F , K be an F -extending extension of E , B_E be a subset of $\text{VecSp}(E, F)$, and B_K be a subset of $\text{VecSp}(K, E)$. The functor $\text{Base}(B_E, B_K)$ yielding a subset of $\text{VecSp}((K \text{ qua extension of } F), F)$ is defined by the term

(Def. 7) $\{a \cdot b, \text{ where } a, b \text{ are elements of } K : a \in B_E \text{ and } b \in B_K\}$.

Let B_E be a non empty subset of $\text{VecSp}(E, F)$ and B_K be a non empty subset of $\text{VecSp}(K, E)$. One can verify that $\text{Base}(B_E, B_K)$ is non empty.

Now we state the propositions:

- (19) Let us consider a field F , an extension E of F , an F -extending extension K of E , a linearly independent subset B_E of $\text{VecSp}(E, F)$, a linearly independent subset B_K of $\text{VecSp}(K, E)$, and elements a_1, a_2, b_1, b_2 of K . Suppose $a_1, a_2 \in B_E$ and $b_1, b_2 \in B_K$. If $a_1 \cdot b_1 = a_2 \cdot b_2$, then $a_1 = a_2$ and $b_1 = b_2$.
- (20) Let us consider a field F , an extension E of F , an F -extending extension K of E , a non empty, linearly independent subset B_E of $\text{VecSp}(E, F)$, and a non empty, linearly independent subset B_K of $\text{VecSp}(K, E)$. Then $\overline{\text{Base}(B_E, B_K)} = \overline{B_E} \times \overline{B_K}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist elements a, b of K such that $a \in B_E$ and $b \in B_K$ and $\$1 = a \cdot b$ and $\$2 = \langle a, b \rangle$. Consider f being a function from $\text{Base}(B_E, B_K)$ into $B_E \times B_K$ such that for every object

x such that $x \in \text{Base}(B_E, B_K)$ holds $\mathcal{P}[x, f(x)]$. $\text{rng } f = B_E \times B_K$. f is one-to-one. $\square$

- (21) Let us consider a field F , an extension E of F , an F -extending extension K of E , a non empty, finite, linearly independent subset B_E of $\text{VecSp}(E, F)$, and a non empty, finite, linearly independent subset B_K of $\text{VecSp}(K, E)$. Then $\overline{\text{Base}(B_E, B_K)} = \overline{B_E} \cdot \overline{B_K}$. The theorem is a consequence of (20).

Let F be a field, E be an extension of F , K be an F -extending extension of E , B_E be a non empty, finite, linearly independent subset of $\text{VecSp}(E, F)$, and B_K be a non empty, finite, linearly independent subset of $\text{VecSp}(K, E)$. Observe that $\text{Base}(B_E, B_K)$ is finite.

Let B_K be a non empty, linearly independent subset of $\text{VecSp}(K, E)$, l be a linear combination of $\text{Base}(B_E, B_K)$, and b be an element of K . The functor $\text{down}(l, b)$ yielding a linear combination of B_E is defined by

- (Def. 8) for every element a of K such that $a \in B_E$ and $b \in B_K$ holds $it(a) = l(a \cdot b)$ and for every element a of E such that $a \notin B_E$ or $b \notin B_K$ holds $it(a) = 0_F$.

Let B_K be a non empty, finite, linearly independent subset of $\text{VecSp}(K, E)$. The functor $\text{down } l$ yielding a linear combination of B_K is defined by

- (Def. 9) for every element b of K such that $b \in B_K$ holds $it(b) = \sum \text{down}(l, b)$.

Let E be an F -finite extension of F , B_E be a basis of $\text{VecSp}(E, F)$, and l_1 be a linear combination of B_K . The functor $\text{lift}(l_1, B_E)$ yielding a linear combination of $\text{Base}(B_E, B_K)$ is defined by

- (Def. 10) for every element b of K such that $b \in B_K$ there exists a linear combination l_2 of B_E such that $\sum l_2 = l_1(b)$ and for every element a of K such that $a \in B_E$ and $a \cdot b \in \text{Base}(B_E, B_K)$ holds $it(a \cdot b) = l_2(a)$.

Now we state the propositions:

- (22) Let us consider a field F , an F -finite extension E of F , an E -finite, F -extending extension K of E , a basis B_E of $\text{VecSp}(E, F)$, a basis B_K of $\text{VecSp}(K, E)$, and a linear combination l of $\text{Base}(B_E, B_K)$. Then $\text{lift}(\text{down } l, B_E) = l$. The theorem is a consequence of (6).
- (23) Let us consider a field F , an F -finite extension E of F , an E -finite, F -extending extension K of E , a basis B_E of $\text{VecSp}(E, F)$, a basis B_K of $\text{VecSp}(K, E)$, and a linear combination l of B_K . Then $\text{down lift}(l, B_E) = l$.
- (24) Let us consider a field F , an extension E of F , an F -extending extension K of E , a non empty, finite, linearly independent subset B_E of $\text{VecSp}(E, F)$, a non empty, finite, linearly independent subset B_K of $\text{VecSp}(K, E)$, and linear combinations l, l_1, l_2 of $\text{Base}(B_E, B_K)$. Suppo-

se $l = l_1 + l_2$. Let us consider an element b of K . Then $\text{down}(l, b) = \text{down}(l_1, b) + \text{down}(l_2, b)$.

- (25) Let us consider a field F , an extension E of F , an F -extending extension K of E , a non empty, finite, linearly independent subset B_E of $\text{VecSp}(E, F)$, a non empty, finite, linearly independent subset B_K of $\text{VecSp}(K, E)$, and linear combinations l, l_1, l_2 of $\text{Base}(B_E, B_K)$. If $l = l_1 + l_2$, then $\text{down } l = \text{down } l_1 + \text{down } l_2$. The theorem is a consequence of (24).

Let us consider a field F , an F -finite extension E of F , an E -finite, F -extending extension K of E , a basis B_E of $\text{VecSp}(E, F)$, a basis B_K of $\text{VecSp}(K, E)$, and a linear combination l of $\text{Base}(B_E, B_K)$. Now we state the propositions:

- (26) $\sum l = \sum \text{down } l$.

PROOF: by induction on $\text{card}(\text{the support of } l)$.

- (27) If $\sum l = 0_{\text{VecSp}((K \text{ qua extension of } F), F)}$, then the support of $l = \emptyset$. The theorem is a consequence of (26).

Let us consider a field F , an F -finite extension E of F , an E -finite, F -extending extension K of E , a basis B_E of $\text{VecSp}(E, F)$, and a basis B_K of $\text{VecSp}(K, E)$. Now we state the propositions:

- (28) $\text{Lin}(\text{Base}(B_E, B_K)) = \text{the vector space structure of } \text{VecSp}((K \text{ qua extension of } F), F)$. The theorem is a consequence of (23) and (26).
- (29) $\text{Base}(B_E, B_K)$ is a basis of $\text{VecSp}((K \text{ qua extension of } F), F)$. The theorem is a consequence of (27) and (28).
- (30) Let us consider a field F , an F -finite extension E of F , and an E -finite, F -extending extension K of E . Then $\deg(K, F) = (\deg(K, E)) \cdot (\deg(E, F))$. The theorem is a consequence of (29) and (21).
- (31) Let us consider a field F , an extension E of F , and an E -extending extension K of F . Suppose K is F -finite. Then
- (i) E is F -finite, and
 - (ii) $\deg(E, F) \leq \deg(K, F)$, and
 - (iii) K is E -finite, and
 - (iv) $\deg(K, E) \leq \deg(K, F)$.

PROOF: Set $B_F = \text{the basis of } \text{VecSp}(K, F)$. Reconsider $B_E = B_F$ as a finite subset of $\text{VecSp}(K, E)$. $\text{Lin}(B_E) = \text{VecSp}(K, E)$. Consider I being a subset of $\text{VecSp}(K, E)$ such that $I \subseteq B_E$ and I is linearly independent and $\text{Lin}(I) = \text{VecSp}(K, E)$. $\square$

Let F be a field and E be an F -finite extension of F . One can check that every E -finite, F -extending extension of E is F -finite.

3. ALGEBRAIC EXTENSIONS

Let F be a field and E be an extension of F . We say that E is F -algebraic if and only if

(Def. 11) every element of E is F -algebraic.

One can verify that every extension of F which is F -finite is also F -algebraic.

Let E be an F -algebraic extension of F . Note that every element of E is F -algebraic. Now we state the propositions:

(32) Let us consider a field F , and an extension E of F . Then E is F -algebraic if and only if for every element a of E , $\text{FAdj}(F, \{a\})$ is F -finite.

(33) Let us consider a field F , an extension E of F , and an element a of E . Then a is F -algebraic if and only if there exists an F -finite extension B of F such that E is B -extending and $a \in B$.

Let F be a field, E be an extension of F , and T be a subset of E . We say that T is F -algebraic if and only if

(Def. 12) for every element a of E such that $a \in T$ holds a is F -algebraic.

One can verify that there exists a subset of E which is finite and F -algebraic.

Now we state the propositions:

(34) Let us consider a field F , an extension E of F , an element b of E , a subset T of E , an extension E_1 of $\text{FAdj}(F, T)$, and an element b_1 of E_1 . Suppose $E_1 = E$ and $b_1 = b$. Then $\text{FAdj}(F, \{b\} \cup T) = \text{FAdj}(\text{FAdj}(F, T), \{b_1\})$.

PROOF: $\{b\} \cup T \subseteq$ the carrier of $\text{FAdj}(\text{FAdj}(F, T), \{b_1\})$ by [6, (35), (36)]. $\text{FAdj}(F, T)$ is a subfield of $\text{FAdj}(F, \{b\} \cup T)$. $\square$

(35) Let us consider a field F , an extension E of F , an element b of E , a subset T of E , an extension E_1 of $\text{FAdj}(F, \{b\})$, and a subset T_1 of E_1 . Suppose $E_1 = E$ and $T_1 = T$. Then $\text{FAdj}(F, \{b\} \cup T) = \text{FAdj}(\text{FAdj}(F, \{b\}), T_1)$.

PROOF: $\{b\} \cup T \subseteq$ the carrier of $\text{FAdj}(\text{FAdj}(F, \{b\}), T_1)$ by [6, (35), (36)]. $\text{FAdj}(F, \{b\})$ is a subfield of $\text{FAdj}(F, \{b\} \cup T)$. $\square$

Let F be a field, E be an extension of F , and T be a finite, F -algebraic subset of E . One can verify that $\text{FAdj}(F, T)$ is F -finite.

Now we state the propositions:

(36) Let us consider a field F , an extension E of F , and an F -algebraic element a of E . Then $E \approx \text{FAdj}(F, \{a\})$ if and only if $\deg \text{MinPoly}(a, F) = \deg(E, F)$. The theorem is a consequence of (5), (31), (30), and (8).

(37) Let us consider a field F , and an extension E of F . Then E is F -finite if and only if there exists a finite, F -algebraic subset T of E such that $E \approx \text{FAdj}(F, T)$.

PROOF: by induction on $\deg(E, F)$.

Let F be a field, E be an extension of F , and p be a non zero element of the carrier of $\text{PolyRing}(F)$. Note that $\text{Roots}(E, p)$ is F -algebraic.

Now we state the proposition:

- (38) Let us consider a field F , an extension E of F , and a non zero element p of the carrier of $\text{PolyRing}(F)$. Then $\text{FAdj}(F, \text{Roots}(E, p))$ is F -algebraic.

Let us consider a field F , an extension E of F , and an E -extending extension K of F . Now we state the propositions:

- (39) If K is E -algebraic and E is F -algebraic, then K is F -algebraic. The theorem is a consequence of (12), (15), and (33).
 (40) If K is F -algebraic, then K is E -algebraic and E is F -algebraic. The theorem is a consequence of (15).

4. THE FIELD OF ALGEBRAIC ELEMENTS

Let F be a field, E be an extension of F , and a, b be F -algebraic elements of E . Observe that $\text{FAdj}(F, \{a, b\})$ is F -finite and 0_E is F -algebraic and 1_E is F -algebraic.

Let a, b be F -algebraic elements of E . One can verify that $a + b$ is F -algebraic and $a - b$ is F -algebraic and $a \cdot b$ is F -algebraic.

Let a be an F -algebraic element of E . Let us note that $-a$ is F -algebraic.

Let a be a non zero, F -algebraic element of E . Let us observe that a^{-1} is F -algebraic.

The functor $\text{Alg-Elem}(E)$ yielding a subset of E is defined by the term

(Def. 13) the set of all a where a is an F -algebraic element of E .

The functor $\text{Field-Alg-Elem}(E)$ yielding a strict double loop structure is defined by

(Def. 14) the carrier of $it = \text{Alg-Elem}(E)$ and the addition of $it =$ (the addition of E) $\upharpoonright$ (the carrier of it) and the multiplication of $it =$ (the multiplication of E) $\upharpoonright$ (the carrier of it) and the one of $it = 1_E$ and the zero of $it = 0_E$.

We introduce the notation $\text{F-Alg}(E)$ as a synonym of $\text{Field-Alg-Elem}(E)$.

Observe that $\text{F-Alg}(E)$ is non degenerated and $\text{F-Alg}(E)$ is Abelian, add-associative, right zeroed, and right complementable and $\text{F-Alg}(E)$ is commutative, associative, well unital, distributive, and almost left invertible and $\text{F-Alg}(E)$ is F -extending and $\text{F-Alg}(E)$ is F -algebraic. Now we state the propositions:

- (41) Let us consider a field F , and an extension E of F . Then $\text{F-Alg}(E)$ is an extension of F .
 (42) Let us consider a field F , and an extension E of F . Then E is an extension of $\text{F-Alg}(E)$.

- (43) Let us consider a field F , an extension E of F , and an extension K of E . Then $\mathbf{F}\text{-Alg}(K)$ is an extension of $\mathbf{F}\text{-Alg}(E)$.
- (44) Let us consider a field F , and an F -algebraic extension E of F . Then $\mathbf{F}\text{-Alg}(E) \approx E$.

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