Functional Space Consisted by Continuous Functions on Topological Space

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Summary. In this article, using the Mizar system [1], [2], first we give a definition of a functional space which is constructed from all continuous functions defined on a compact topological space [5]. We prove that this functional space is a Banach space [3]. Next, we give a definition of a function space which is constructed from all continuous functions with bounded support. We also prove that this function space is a normed space.

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1. REAL VECTOR SPACE OF CONTINUOUS FUNCTIONS

From now on S denotes a non empty topological space, T denotes a linear topological space, and X denotes a non empty subset of the carrier of S .

Now we state the propositions:

- (1) Let us consider a non empty topological space X , a non empty linear topological space S , functions f, g from X into S , and a point x of X . Suppose f is continuous at x and g is continuous at x . Then $f + g$ is continuous at x .

PROOF: For every neighbourhood G of $(f + g)(x)$, there exists a neighbourhood H of x such that $(f + g)^\circ H \subseteq G$. $\square$

- (2) Let us consider a non empty topological space X , a non empty linear topological space S , a function f from X into S , a point x of X , and a real number a . If f is continuous at x , then $a \cdot f$ is continuous at x .

PROOF: For every neighbourhood G of $(a \cdot f)(x)$, there exists a neighbourhood H of x such that $(a \cdot f)^\circ H \subseteq G$. $\square$

- (3) Let us consider a non empty topological space X , a non empty linear topological space S , and functions f, g from X into S . If f is continuous and g is continuous, then $f + g$ is continuous.

PROOF: For every point x of X , $f + g$ is continuous at x . $\square$

- (4) Let us consider a non empty topological space X , a non empty linear topological space S , a function f from X into S , and a real number a . If f is continuous, then $a \cdot f$ is continuous. The theorem is a consequence of (2).

Let S be a non empty topological space and T be a non empty linear topological space. The continuous functions of S and T yielding a subset of $\text{RealVectSpace}((\text{the carrier of } S), T)$ is defined by the term

(Def. 1) $\{f, \text{ where } f \text{ is a function from the carrier of } S \text{ into the carrier of } T : f \text{ is continuous}\}$.

Let us observe that the continuous functions of S and T is non empty and functional.

Let us consider a non empty topological space S and a non empty linear topological space T . Now we state the propositions:

- (5) The continuous functions of S and T is linearly closed.

PROOF: Set $W =$ the continuous functions of S and T . For every vectors v, u of $\text{RealVectSpace}((\text{the carrier of } S), T)$ such that $v, u \in$ the continuous functions of S and T holds $v + u \in$ the continuous functions of S and T . For every real number a and for every vector v of $\text{RealVectSpace}((\text{the carrier of } S), T)$ such that $v \in W$ holds $a \cdot v \in W$. $\square$

- (6) $\langle \text{the continuous functions of } S \text{ and } T, \text{Zero}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Add}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Mult}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)) \rangle$ is a subspace of $\text{RealVectSpace}((\text{the carrier of } S), T)$.

Let S be a non empty topological space and T be a non empty linear topological space.

One can verify that $\langle \text{the continuous functions of } S \text{ and } T, \text{Zero}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Add}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Mult}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)) \rangle$ is Abelian, add-

associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The $\mathbb{R}$ -vector space of continuous functions of S and T yielding a strict real linear space is defined by the term

(Def. 2) $\langle$ the continuous functions of S and T , Zero(the continuous functions of S and T , RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T , RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T , RealVectSpace((the carrier of S), T)) $\rangle$.

Observe that the $\mathbb{R}$ -vector space of continuous functions of S and T is constituted functions. Let f be a vector of the $\mathbb{R}$ -vector space of continuous functions of S and T and v be an element of S . Let us note that the functor $f(v)$ yields a vector of T . Now we state the propositions:

- (7) Let us consider a non empty topological space S , a non empty linear topological space T , and vectors f, g, h of the $\mathbb{R}$ -vector space of continuous functions of S and T . Then $h = f + g$ if and only if for every element x of S , $h(x) = f(x) + g(x)$. The theorem is a consequence of (5).
- (8) Let us consider a non empty topological space S , a non empty linear topological space T , vectors f, h of the $\mathbb{R}$ -vector space of continuous functions of S and T , and a real number a . Then $h = a \cdot f$ if and only if for every element x of S , $h(x) = a \cdot f(x)$. The theorem is a consequence of (5).
- (9) Let us consider a non empty topological space S , and a non empty linear topological space T . Then $0_\alpha = (\text{the carrier of } S) \mapsto 0_T$, where α is the $\mathbb{R}$ -vector space of continuous functions of S and T . The theorem is a consequence of (5).

Let S be a non empty topological space and T be a non empty linear topological space. Let us note that the carrier of the $\mathbb{R}$ -vector space of continuous functions of S and T is functional.

2. REAL VECTOR SPACE OF CONTINUOUS FUNCTIONS (NORM SPACE VERSION)

In the sequel S, T denote real normed spaces and X denotes a non empty subset of the carrier of S .

Now we state the proposition:

- (10) Let us consider a point x of T . Then $(\text{the carrier of } S) \mapsto x$ is continuous on the carrier of S .

Let S, T be real normed spaces. The continuous functions of S and T yielding a subset of RealVectSpace((the carrier of S), T) is defined by the term

(Def. 3) $\{f, \text{ where } f \text{ is a function from the carrier of } S \text{ into the carrier of } T : f \text{ is continuous on the carrier of } S\}$.

One can check that the continuous functions of S and T is non empty and functional.

Let us consider real normed spaces S, T . Now we state the propositions:

(11) The continuous functions of S and T is linearly closed.

PROOF: Set $W =$ the continuous functions of S and T . For every vectors v, u of $\text{RealVectSpace}((\text{the carrier of } S), T)$ such that $v, u \in$ the continuous functions of S and T holds $v+u \in$ the continuous functions of S and T . For every real number a and for every vector v of $\text{RealVectSpace}((\text{the carrier of } S), T)$ such that $v \in W$ holds $a \cdot v \in W$ by [4, (27)]. $\square$

(12) $\langle \text{the continuous functions of } S \text{ and } T, \text{Zero}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Add}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Mult}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)) \rangle$ is a subspace of $\text{RealVectSpace}((\text{the carrier of } S), T)$.

Let S, T be real normed spaces. Observe that $\langle \text{the continuous functions of } S \text{ and } T, \text{Zero}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Add}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Mult}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)) \rangle$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The $\mathbb{R}$ -vector space of continuous functions of S and T yielding a strict real linear space is defined by the term

(Def. 4) $\langle \text{the continuous functions of } S \text{ and } T, \text{Zero}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Add}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Mult}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)) \rangle$.

Note that the $\mathbb{R}$ -vector space of continuous functions of S and T is constituted functions.

Let f be a vector of the $\mathbb{R}$ -vector space of continuous functions of S and T and v be an element of S . One can check that the functor $f(v)$ yields a vector of T . Now we state the propositions:

(13) Let us consider real normed spaces S, T , and vectors f, g, h of the $\mathbb{R}$ -vector space of continuous functions of S and T . Then $h = f + g$ if and only if for every element x of S , $h(x) = f(x) + g(x)$. The theorem is a consequence of (11).

(14) Let us consider real normed spaces S, T , vectors f, h of the $\mathbb{R}$ -vector space of continuous functions of S and T , and a real number a . Then

$h = a \cdot f$ if and only if for every element x of S , $h(x) = a \cdot f(x)$. The theorem is a consequence of (11).

Let us consider real normed spaces S, T . Now we state the propositions:

- (15) The $\mathbb{R}$ -vector space of continuous functions of S and T is a subspace of $\text{RealVectSpace}((\text{the carrier of } S), T)$.
- (16) $0_\alpha = (\text{the carrier of } S) \mapsto 0_T$, where α is the $\mathbb{R}$ -vector space of continuous functions of S and T . The theorem is a consequence of (11).

Let S, T be real normed spaces and f be an object. Assume $f \in$ the continuous functions of S and T . The functor $\text{PartFuncs}(f, S, T)$ yielding a function from S into T is defined by

(Def. 5) $it = f$ and it is continuous on the carrier of S .

3. NORMED TOPOLOGICAL LINEAR SPACE

We consider normed real linear topological structures which extend real linear topological structures and normed structures and are systems

$$\langle \text{a carrier, a zero, an addition, an external multiplication,} \\ \text{a topology, a norm} \rangle$$

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $\mathbb{R} \times (\text{the carrier})$ into the carrier, the topology is a family of subsets of the carrier, the norm is a function from the carrier into $\mathbb{R}$.

Let X be a non empty set, O be an element of X , F be a binary operation on X , G be a function from $\mathbb{R} \times X$ into X , T be a family of subsets of X , and N be a function from X into $\mathbb{R}$. Observe that $\langle X, O, F, G, T, N \rangle$ is non empty and there exists a normed real linear topological structure which is strict and non empty.

Let X be a non empty normed real linear topological structure. We say that X is normed structure if and only if

- (Def. 6) there exists a real normed space R such that $R =$ the normed structure of X and the topology of $X =$ the topology of $\text{TopSpaceNorm } R$.

One can verify that there exists a non empty normed real linear topological structure which is strict, add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, real normed space-like, normed structure, and T_2 .

A normed linear topological space is a strict, add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, real normed space-like, normed structure, T_2 , non empty normed real linear topological structure. Now we state the propositions:

- (17) Every normed linear topological space is a linear topological space.
- (18) Every normed linear topological space is a real normed space.
- (19) Let us consider a normed linear topological space X , and a real normed space R . Suppose R = the normed structure of X . Let us consider points x, y of X , points x_1, y_1 of R , and a real number a . Suppose $x_1 = x$ and $y_1 = y$. Then
 - (i) $x + y = x_1 + y_1$, and
 - (ii) $a \cdot x = a \cdot x_1$, and
 - (iii) $x - y = x_1 - y_1$, and
 - (iv) $\|x\| = \|x_1\|$.

Let us consider a normed linear topological space X , a sequence S of X , and a point x of X . Now we state the propositions:

- (20) S is convergent to x if and only if for every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leq n$ holds $\|S(n) - x\| < r$. The theorem is a consequence of (19).
- (21) S is convergent and $x = \lim S$ if and only if for every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leq n$ holds $\|S(n) - x\| < r$. The theorem is a consequence of (20).
- (22) Let us consider a normed linear topological space X , and a sequence S of X . Suppose S is convergent. Let us consider a real number r . Suppose $0 < r$. Then there exists a natural number m such that for every natural number n such that $m \leq n$ holds $\|S(n) - \lim S\| < r$. The theorem is a consequence of (20).
- (23) Let us consider a normed linear topological space X , and a subset V of X . Then V is open if and only if for every point x of X such that $x \in V$ there exists a real number r such that $r > 0$ and $\{y, \text{ where } y \text{ is a point of } X : \|x - y\| < r\} \subseteq V$. The theorem is a consequence of (19).

Let us consider a normed linear topological space X , a point x of X , a real number r , and a subset V of X . Now we state the propositions:

- (24) If $V = \{y, \text{ where } y \text{ is a point of } X : \|x - y\| < r\}$, then V is open. The theorem is a consequence of (19).
- (25) Suppose $V = \{y, \text{ where } y \text{ is a point of } X : \|x - y\| \leq r\}$. Then V is closed. The theorem is a consequence of (19).

Now we state the propositions:

- (26) Let us consider a normed linear topological space X , a real normed space R , a sequence t of X , and a sequence s of R . Suppose $R =$ the normed structure of X and $t = s$ and t is convergent. Then

- (i) s is convergent, and
- (ii) $\lim s = \lim t$.

The theorem is a consequence of (22) and (19).

- (27) Let us consider a normed linear topological space X , a real normed space R , a sequence s of X , and a sequence t of R . Suppose $R =$ the normed structure of X and $s = t$. Then s is convergent if and only if t is convergent. The theorem is a consequence of (26), (19), and (21).
- (28) Let us consider a normed linear topological space X , and a subset V of X . Then V is closed if and only if for every sequence s_1 of X such that $\text{rng } s_1 \subseteq V$ and s_1 is convergent holds $\lim s_1 \in V$. The theorem is a consequence of (26) and (27).
- (29) Let us consider a normed linear topological space X , a real normed space R , a subset V of X , and a subset W of R . Suppose $R =$ the normed structure of X and the topology of $X =$ the topology of $\text{TopSpaceNorm } R$ and $V = W$. Then V is closed if and only if W is closed. The theorem is a consequence of (27), (26), and (28).
- (30) Let us consider a normed linear topological space X , a subset V of X , and a point x of X . Then V is a neighbourhood of x if and only if there exists a real number r such that $r > 0$ and $\{y, \text{ where } y \text{ is a point of } X : \|y - x\| < r\} \subseteq V$. The theorem is a consequence of (23) and (24).
- (31) Let us consider a normed linear topological space X , and a subset V of X . Then V is compact if and only if for every sequence s_1 of X such that $\text{rng } s_1 \subseteq V$ there exists a sequence s_2 of X such that s_2 is subsequence of s_1 and convergent and $\lim s_2 \in V$. The theorem is a consequence of (27) and (26).
- (32) Let us consider a normed linear topological space X , a real normed space R , a subset V of X , and a subset W of R . Suppose $R =$ the normed structure of X and the topology of $X =$ the topology of $\text{TopSpaceNorm } R$ and $V = W$. Then V is compact if and only if W is compact. The theorem is a consequence of (31), (26), and (27).

4. REAL NORM SPACE OF CONTINUOUS FUNCTIONS

Now we state the propositions:

- (33) Let us consider sets X , X_1 , a real normed space S , and a partial function f from S to $\mathbb{R}$. Suppose f is continuous on X and $X_1 \subseteq X$. Then f is continuous on X_1 .

PROOF: $f|_{X_1}$ is continuous in r . $\square$

- (34) Let us consider a non empty, compact topological space S , a normed linear topological space T , and a set x . Suppose $x \in$ the continuous functions of S and T . Then $x \in \text{BdFuncs}((\text{the carrier of } S), T)$.
- (35) Let us consider a non empty, compact topological space S , and a normed linear topological space T . Then the $\mathbb{R}$ -vector space of continuous functions of S and T is a subspace of the set of bounded real sequences from the carrier of S into T . The theorem is a consequence of (34) and (5).

Let S be a non empty, compact topological space and T be a normed linear topological space. The continuous functions norm of S and T yielding a function from the continuous functions of S and T into $\mathbb{R}$ is defined by the term

- (Def. 7) $\text{BdFuncsNorm}((\text{the carrier of } S), T) \upharpoonright (\text{the continuous functions of } S \text{ and } T)$.

The $\mathbb{R}$ -norm space of continuous functions of S and T yielding a strict normed structure is defined by the term

- (Def. 8) $\langle \text{the continuous functions of } S \text{ and } T, \text{Zero}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Add}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{Mult}(\text{the continuous functions of } S \text{ and } T, \text{RealVectSpace}((\text{the carrier of } S), T)), \text{the continuous functions norm of } S \text{ and } T \rangle$.

One can check that the $\mathbb{R}$ -norm space of continuous functions of S and T is non empty.

Now we state the propositions:

- (36) Let us consider a non empty, compact topological space S , a normed linear topological space T , a point x of the $\mathbb{R}$ -norm space of continuous functions of S and T , and a point y of the real normed space of bounded functions from the carrier of S into T . If $x = y$, then $\|x\| = \|y\|$.
- (37) Let us consider a non empty, compact topological space S , a normed linear topological space T , a point f of the $\mathbb{R}$ -norm space of continuous functions of S and T , and a function g from S into T . Suppose $f = g$. Let us consider a point t of S . Then $\|g(t)\| \leq \|f\|$. The theorem is a consequence of (34).

- (38) Let us consider a non empty, compact topological space S , a normed linear topological space T , points x_1, x_2 of the $\mathbb{R}$ -norm space of continuous functions of S and T , and points y_1, y_2 of the real normed space of bounded functions from the carrier of S into T . If $x_1 = y_1$ and $x_2 = y_2$, then $x_1 + x_2 = y_1 + y_2$. The theorem is a consequence of (5).
- (39) Let us consider a non empty, compact topological space S , a normed linear topological space T , a real number a , a point x of the $\mathbb{R}$ -norm space of continuous functions of S and T , and a point y of the real normed space of bounded functions from the carrier of S into T . If $x = y$, then $a \cdot x = a \cdot y$. The theorem is a consequence of (5).

Let S be a non empty, compact topological space and T be a normed linear topological space. One can verify that the $\mathbb{R}$ -norm space of continuous functions of S and T is non empty, right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

Let us consider a non empty, compact topological space S and a normed linear topological space T . Now we state the propositions:

- (40) (The carrier of S) $\mapsto 0_T = 0_\alpha$, where α is the $\mathbb{R}$ -norm space of continuous functions of S and T . The theorem is a consequence of (9).
- (41) $0_\alpha = 0_\beta$, where α is the $\mathbb{R}$ -norm space of continuous functions of S and T and β is the real normed space of bounded functions from the carrier of S into T . The theorem is a consequence of (40).

Let us consider a non empty, compact topological space S , a normed linear topological space T , and a point F of the $\mathbb{R}$ -norm space of continuous functions of S and T . Now we state the propositions:

- (42) $0 \leq \|F\|$. The theorem is a consequence of (34).
- (43) If $F = 0_\alpha$, then $0 = \|F\|$, where α is the $\mathbb{R}$ -norm space of continuous functions of S and T . The theorem is a consequence of (34) and (40).
- (44) Let us consider a non empty, compact topological space S , a normed linear topological space T , points F, G, H of the $\mathbb{R}$ -norm space of continuous functions of S and T , and functions f, g, h from S into T . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of S , $h(x) = f(x) + g(x)$. The theorem is a consequence of (7).
- (45) Let us consider a real number a , a non empty, compact topological space S , a normed linear topological space T , points F, G of the $\mathbb{R}$ -norm space of continuous functions of S and T , and functions f, g from S into T . Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of S , $g(x) = a \cdot f(x)$. The theorem is a consequence of (8).

- (46) Let us consider a real number a , a non empty, compact topological space

S , a normed linear topological space T , and points F, G of the $\mathbb{R}$ -norm space of continuous functions of S and T . Then

- (i) $\|F\| = 0$ iff $F = 0_\alpha$, and
- (ii) $\|a \cdot F\| = |a| \cdot \|F\|$, and
- (iii) $\|F + G\| \leq \|F\| + \|G\|$,

where α is the $\mathbb{R}$ -norm space of continuous functions of S and T . The theorem is a consequence of (34), (38), (36), (41), and (39).

Let S be a non empty, compact topological space and T be a normed linear topological space. Let us observe that the $\mathbb{R}$ -norm space of continuous functions of S and T is reflexive, discernible, and real normed space-like.

Now we state the propositions:

- (47) Let us consider a non empty, compact topological space S , a normed linear topological space T , points x_1, x_2 of the $\mathbb{R}$ -norm space of continuous functions of S and T , and points y_1, y_2 of the real normed space of bounded functions from the carrier of S into T . If $x_1 = y_1$ and $x_2 = y_2$, then $x_1 - x_2 = y_1 - y_2$. The theorem is a consequence of (39) and (38).
- (48) Let us consider a non empty, compact topological space S , a normed linear topological space T , points F, G, H of the $\mathbb{R}$ -norm space of continuous functions of S and T , and functions f, g, h from S into T . Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element x of S , $h(x) = f(x) - g(x)$. The theorem is a consequence of (44).
- (49) Let us consider a non empty topological space S , a normed linear topological space T , a sequence H of partial functions from the carrier of S into the carrier of T , and a function L_1 from S into T . Suppose H is uniform-convergent on the carrier of S and for every natural number n , there exists a function H_1 from S into T such that $H_1 = H(n)$ and H_1 is continuous and $L_1 = \lim_\alpha H$. Then L_1 is continuous, where α is the carrier of S .

PROOF: For every point x of S , L_1 is continuous at x by (30), [7, (33),(11)].
□

- (50) Let us consider a non empty, compact topological space S , a normed linear topological space T , and a subset Y of the carrier of the real normed space of bounded functions from the carrier of S into T . Suppose $Y =$ the continuous functions of S and T . Then Y is closed. The theorem is a consequence of (49).
- (51) Let us consider a non empty, compact topological space S , and a normed linear topological space T . Suppose T is complete. Let us consider a sequence s_3 of the $\mathbb{R}$ -norm space of continuous functions of S and T .

If s_3 is Cauchy sequence by norm, then s_3 is convergent. The theorem is a consequence of (34), (47), (36), and (50).

- (52) Let us consider a non empty, compact topological space S , and a normed linear topological space T . Suppose T is complete. Then the $\mathbb{R}$ -norm space of continuous functions of S and T is complete. The theorem is a consequence of (51).

5. SOME PROPERTIES OF SUPPORT

Let X be a zero structure and f be a (the carrier of X)-valued function. The functor support f yielding a set is defined by

(Def. 9) for every object x , $x \in \text{it}$ iff $x \in \text{dom } f$ and $f/x \neq 0_X$.

Now we state the proposition:

- (53) Let us consider a zero structure X , and a (the carrier of X)-valued function f . Then $\text{support } f \subseteq \text{dom } f$.

Let X be a non empty topological space, T be a real linear space, and f be a function from X into T . One can verify that the functor support f yields a subset of X . Now we state the propositions:

- (54) Let us consider a non empty topological space X , a real linear space T , and functions f, g from X into T . Then $\text{support}(f + g) \subseteq \text{support } f \cup \text{support } g$.
- (55) Let us consider a non empty topological space X , a real linear space T , a function f from X into T , and a real number a . Then $\text{support}(a \cdot f) \subseteq \text{support } f$.

6. SPACE OF REAL-VALUED CONTINUOUS FUNCTIONALS WITH BOUNDED SUPPORT

Let X be a non empty topological space and T be a normed linear topological space. The functor $\text{C}_0\text{Functions}(X, T)$ yielding a non empty subset of $\text{RealVectSpace}((\text{the carrier of } X), T)$ is defined by the term

- (Def. 10) $\{f, \text{ where } f \text{ is a function from the carrier of } X \text{ into the carrier of } T : f \text{ is continuous and there exists a non empty subset } Y \text{ of } X \text{ such that } Y \text{ is compact and } \overline{\text{support } f} \subseteq Y\}$.

Now we state the propositions:

- (56) Let us consider a non empty topological space X , a normed linear topological space T , and elements v, u of $\text{RealVectSpace}((\text{the carrier of } X), T)$.

Suppose $v, u \in C_0\text{Functions}(X, T)$. Then $v + u \in C_0\text{Functions}(X, T)$. The theorem is a consequence of (5) and (54).

(57) Let us consider a non empty topological space X , a normed linear topological space T , a real number a , and an element u of $\text{RealVectSpace}((\text{the carrier of } X), T)$. Suppose $u \in C_0\text{Functions}(X, T)$. Then $a \cdot u \in C_0\text{Functions}(X, T)$. The theorem is a consequence of (5) and (55).

(58) Let us consider a non empty topological space X , and a normed linear topological space T . Then $C_0\text{Functions}(X, T)$ is linearly closed.

Let X be a non empty topological space and T be a normed linear topological space. Let us note that $C_0\text{Functions}(X, T)$ is non empty and linearly closed.

The functor $\text{RV}_{\text{SP}}C_0\text{Functions}(X, T)$ yielding a real linear space is defined by the term

(Def. 11) $\langle C_0\text{Functions}(X, T), \text{Zero}(C_0\text{Functions}(X, T), \text{RealVectSpace}((\text{the carrier of } X), T)), \text{Add}(C_0\text{Functions}(X, T), \text{RealVectSpace}((\text{the carrier of } X), T)), \text{Mult}(C_0\text{Functions}(X, T), \text{RealVectSpace}((\text{the carrier of } X), T)) \rangle$.

Now we state the propositions:

(59) Let us consider a non empty topological space X , and a normed linear topological space T . Then $\text{RV}_{\text{SP}}C_0\text{Functions}(X, T)$ is a subspace of $\text{RealVectSpace}((\text{the carrier of } X), T)$.

(60) Let us consider a non empty topological space X , a normed linear topological space T , and a set x . Suppose $x \in C_0\text{Functions}(X, T)$. Then $x \in \text{BdFuncs}((\text{the carrier of } X), T)$.

PROOF: Consider f being a function from the carrier of X into the carrier of T such that $f = x$ and f is continuous and there exists a non empty subset Y of X such that Y is compact and $\overline{\text{support } f} \subseteq Y$. Consider Y being a non empty subset of X such that Y is compact and $\overline{\text{support } f} \subseteq Y$. Consider K being a real number such that $0 \leq K$ and for every point x of X such that $x \in Y$ holds $\|f(x)\| \leq K$. For every element x of X , $\|f(x)\| \leq K$. $\square$

Let X be a non empty topological space and T be a normed linear topological space. The functor $\text{Norm}C_0\text{Functions}(X, T)$ yielding a function from $C_0\text{Functions}(X, T)$ into $\mathbb{R}$ is defined by the term

(Def. 12) $\text{BdFuncsNorm}((\text{the carrier of } X), T) \upharpoonright C_0\text{Functions}(X, T)$.

The functor $\text{NormSp}_{C_0}\text{Functions}(X, T)$ yielding a normed structure is defined by the term

(Def. 13) $\langle C_0\text{Functions}(X, T), \text{Zero}(C_0\text{Functions}(X, T), \text{RealVectSpace}((\text{the carrier of } X), T)), \text{Add}(C_0\text{Functions}(X, T), \text{RealVectSpace}((\text{the carrier of } X), T)), \text{Mult}(C_0\text{Functions}(X, T), \text{RealVectSpace}((\text{the carrier of } X), T)) \rangle$,

$\text{NormC}_0\text{Functions}(X, T)\rangle$.

Let us note that $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$ is strict and non empty.

Now we state the proposition:

- (61) Let us consider a non empty topological space X , a normed linear topological space T , and a set x . Suppose $x \in \text{C}_0\text{Functions}(X, T)$. Then $x \in$ the continuous functions of X and T .

Let us consider a non empty topological space X and a normed linear topological space T . Now we state the propositions:

- (62) $0_{\text{RVSP}_{\text{C}_0}\text{Functions}(X, T)} = X \longmapsto 0_T$.
- (63) $0_{\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)} = X \longmapsto 0_T$. The theorem is a consequence of (62).
- (64) Let us consider a non empty topological space X , a normed linear topological space T , points x_1, x_2 of $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$, and points y_1, y_2 of the real normed space of bounded functions from the carrier of X into T . If $x_1 = y_1$ and $x_2 = y_2$, then $x_1 + x_2 = y_1 + y_2$.
- (65) Let us consider a non empty topological space X , a normed linear topological space T , a real number a , a point x of $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$, and a point y of the real normed space of bounded functions from the carrier of X into T . If $x = y$, then $a \cdot x = a \cdot y$.
- (66) Let us consider a real number a , a non empty topological space X , a normed linear topological space T , and points F, G of $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$. Then
- (i) $\|F\| = 0$ iff $F = 0_{\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)}$, and
 - (ii) $\|a \cdot F\| = |a| \cdot \|F\|$, and
 - (iii) $\|F + G\| \leq \|F\| + \|G\|$.

PROOF: $\|F\| = 0$ iff $F = 0_{\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)}$. $\|a \cdot F\| = |a| \cdot \|F\|$. $\|F + G\| \leq \|F\| + \|G\|$ by (60), (64) [6, (21)]. $\square$

- (67) Let us consider a non empty topological space X , and a normed linear topological space T . Then $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$ is real normed space-like.

Let X be a non empty topological space and T be a normed linear topological space. Let us note that $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$ is reflexive, discernible, real normed space-like, vector distributive.

And let us observe that $\text{NormSp}_{\text{C}_0}\text{Functions}(X, T)$ is scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

- (68) Let us consider a non empty topological space X , and a normed linear topological space T . Then $\text{NormSp}_{C_0}\text{Functions}(X, T)$ is a real normed space.

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