Ascoli-Arzelà Theorem¹

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Summary. In this article we formalize the Ascoli-Arzelà theorem [5], [6], [8] in Mizar [1], [2]. First, we gave definitions of equicontinuousness and equiboundedness of a set of continuous functions [12], [7], [3], [9]. Next, we formalized the Ascoli-Arzelà theorem using those definitions, and proved this theorem.

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1. EQUICONTINUOUSNESS AND EQUIBOUNDEDNESS OF CONTINUOUS FUNCTIONS

From now on S , T denote real normed spaces and F denotes a subset of (the carrier of T)^(the carrier of S).

Let X be a non empty metric space and Y be a subset of X . The functor $\bar{Y}$ yielding a subset of X is defined by

(Def. 1) there exists a subset Z of X_{top} such that $Z = Y$ and $it = \bar{Z}$.

Now we state the proposition:

- (1) Let us consider a real normed space X , a subset Y of X , and a subset Z of $\text{MetricSpaceNorm } X$. If $Y = Z$, then $\bar{Y} = \bar{Z}$.

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Let X be a non empty metric space and H be a non empty subset of X . Observe that $\overline{H}$ is non empty.

Now we state the propositions:

- (2) Let us consider a topological space S , and a finite sequence F of elements of 2^α . Suppose for every natural number i such that $i \in \text{Seg len } F$ holds F_i is compact. Then $\bigcup \text{rng } F$ is compact, where α is the carrier of S .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of $2^{(\text{the carrier of } S)}$ such that $\text{len } F = \$_1$ and for every natural number i such that $i \in \text{Seg len } F$ holds F_i is compact holds $\bigcup \text{rng } F$ is compact. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

- (3) Let us consider a non empty topological space S , a normed linear topological space T , a function f from S into T , and a point x of S . Then f is continuous at x if and only if for every real number e such that $0 < e$ there exists a subset H of S such that H is open and $x \in H$ and for every point y of S such that $y \in H$ holds $\|f(x) - f(y)\| < e$.

PROOF: For every subset G of T such that G is open and $f(x) \in G$ there exists a subset H of S such that H is open and $x \in H$ and $f^\circ H \subseteq G$. $\square$

- (4) Let us consider a non empty metric space S , a non empty, compact topological space V , a normed linear topological space T , and a function f from V into T . Suppose $V = S_{\text{top}}$. Then f is continuous if and only if for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f_{/x_1} - f_{/x_2}\| < e$.

PROOF: For every point x of V , f is continuous at x . $\square$

Let S be a non empty metric space, T be a real normed space, and F be a subset of $(\text{the carrier of } T)^{(\text{the carrier of } S)}$. We say that F is equibounded if and only if

- (Def. 2) there exists a real number K such that for every function f from the carrier of S into the carrier of T such that $f \in F$ for every element x of S , $\|f(x)\| \leq K$.

Let x_0 be a point of S . We say that F is equicontinuous at x_0 if and only if

- (Def. 3) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, x_0) < d$ holds $\|f(x) - f(x_0)\| < e$.

We say that F is equicontinuous if and only if

- (Def. 4) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier

of T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$.

Now we state the proposition:

- (5) Let us consider a non empty metric space S , a real normed space T , and a subset F of $(\text{the carrier of } T)^\alpha$. Suppose S_{top} is compact. Then F is equicontinuous if and only if for every point x of S , F is equicontinuous at x , where α is the carrier of S .

PROOF: Define $\mathcal{P}[\text{element of } S, \text{real number}] \equiv 0 < \$_2$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, \$_1) < \$_2$ holds $\|f(x) - f(\$_1)\| < \frac{e}{2}$. For every element x_0 of the carrier of S , there exists an element d of $\mathbb{R}$ such that $\mathcal{P}[x_0, d]$.

Consider D being a function from the carrier of S into $\mathbb{R}$ such that for every element x_0 of the carrier of S , $\mathcal{P}[x_0, D(x_0)]$. Set $C_1 =$ the set of all $\text{Ball}(x_0, \frac{D(x_0)}{2})$ where x_0 is an element of S . $C_1 \subseteq 2^\alpha$, where α is the carrier of S_{top} . For every subset P of S_{top} such that $P \in C_1$ holds P is open. The carrier of $S_{\text{top}} \subseteq \bigcup C_1$. Consider G being a family of subsets of S_{top} such that $G \subseteq C_1$ and G is cover of $\Omega_{S_{\text{top}}}$ and finite. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point x_0 of S such that $\$2 = x_0$ and $\$1 = \text{Ball}(x_0, \frac{D(x_0)}{2})$. For every object Z such that $Z \in G$ there exists an object x_0 such that $x_0 \in$ the carrier of S and $\mathcal{Q}[Z, x_0]$.

Consider H being a function from G into the carrier of S such that for every object Z such that $Z \in G$ holds $\mathcal{Q}[Z, H(Z)]$. For every object Z such that $Z \in G$ holds $Z = \text{Ball}(H/Z, \frac{D(H(Z))}{2})$. Reconsider $D_0 = D^\circ(\text{rng } H)$ as a finite subset of $\mathbb{R}$. $G \neq \emptyset$. Consider x_3 being an object such that $x_3 \in G$. Consider x_3 being an object such that $x_3 \in \text{rng } H$. Set $d_0 = \inf D_0$. Consider x_3 being an object such that $x_3 \in \text{dom } D$ and $x_3 \in \text{rng } H$ and $d_0 = D(x_3)$. For every function f from S into T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

2. ASCOLI-ARZELÀ THEOREM

From now on S, Z denote real normed spaces, T denotes a real Banach space, and F denotes a subset of $(\text{the carrier of } T)^{(\text{the carrier of } S)}$.

Now we state the proposition:

- (6) Let us consider a real normed space Z . Then Z is complete if and only if $\text{MetricSpaceNorm } Z$ is complete.

PROOF: For every sequence s of Z such that s is Cauchy sequence by norm holds s is convergent by [10, (8)], [4, (5)]. $\square$

Let us consider a real normed space Z and a non empty subset H of $\text{MetricSpaceNorm } Z$. Now we state the propositions:

- (7) If Z is complete, then $\text{MetricSpaceNorm } Z \upharpoonright \overline{H}$ is complete.

PROOF: Reconsider $F = H$ as a non empty subset of Z . $\overline{F} = \overline{H}$. Set $N = \text{MetricSpaceNorm } Z \upharpoonright \overline{H}$. For every sequence S_2 of N such that S_2 is Cauchy holds S_2 is convergent. $\square$

- (8) $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded if and only if $\text{MetricSpaceNorm } Z \upharpoonright \overline{H}$ is totally bounded.

PROOF: Reconsider $F = H$ as a non empty subset of Z . Consider D being a subset of $(\text{MetricSpaceNorm } Z)_{\text{top}}$ such that $D = H$ and $\overline{H} = \overline{D}$. $\overline{F} = \overline{H}$. $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded. $\square$

- (9) Let us consider a real normed space Z , a non empty subset F of Z , and a non empty subset H of $\text{MetricSpaceNorm } Z$. Suppose Z is complete and $H = F$ and $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded. Then

- (i) $\overline{H}$ is sequentially compact, and
- (ii) $\text{MetricSpaceNorm } Z \upharpoonright \overline{H}$ is compact, and
- (iii) $\overline{F}$ is compact.

The theorem is a consequence of (1), (7), and (8).

- (10) Let us consider a real normed space Z , a non empty subset F of Z , a non empty subset H of $\text{MetricSpaceNorm } Z$, and a subset T of $\text{TopSpaceNorm } Z$. Suppose Z is complete and $H = F$ and $H = T$. Then

- (i) $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded iff $\overline{H}$ is sequentially compact, and
- (ii) $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded iff $\text{MetricSpaceNorm } Z \upharpoonright \overline{H}$ is compact, and
- (iii) $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded iff $\overline{F}$ is compact, and
- (iv) $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded iff $\overline{T}$ is compact.

The theorem is a consequence of (1), (7), and (8).

- (11) Let us consider a non empty, compact topological space S , and a normed linear topological space T . Suppose T is complete. Let us consider a non empty subset H of $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T)$.

Then $\overline{H}$ is sequentially compact if and only if $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T) \upharpoonright H$ is totally bounded. The theorem is a consequence of (7) and (8).

- (12) Let us consider a non empty, compact topological space S , and a normed linear topological space T . Suppose T is complete. Let us consider a non

empty subset F of the $\mathbb{R}$ -norm space of continuous functions of S and T , and a non empty subset H of $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T)$. Suppose $H = F$. Then $\overline{F}$ is compact if and only if $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T) \upharpoonright H$ is totally bounded. The theorem is a consequence of (1) and (11).

Let us consider a non empty metric space M , a non empty, compact topological space S , a normed linear topological space T , a subset G of $(\text{the carrier of } T)^{(\text{the carrier of } M)}$, and a non empty subset H of $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T)$. Now we state the propositions:

- (13) Suppose $S = M_{\text{top}}$ and T is complete. Then suppose $G = H$ and $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T) \upharpoonright H$ is totally bounded. Then G is equibounded and equicontinuous.

PROOF: Set $Z = \text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T$. Set $M_1 = \text{MetricSpaceNorm } Z \upharpoonright H$. Consider L being a family of subsets of M_1 such that L is finite and the carrier of $M_1 = \bigcup L$ and for every subset C of M_1 such that $C \in L$ there exists an element w of M_1 such that $C = \text{Ball}(w, 1)$.

Define $\mathcal{Q}[\text{object}, \text{object}] \equiv \text{there exists a point } w \text{ of } M_1 \text{ such that } \$2 = w \text{ and } \$1 = \text{Ball}(w, 1)$. For every object D such that $D \in L$ there exists an object w such that $w \in \text{the carrier of } M_1$ and $\mathcal{Q}[D, w]$. Consider U being a function from L into the carrier of M_1 such that for every object D such that $D \in L$ holds $\mathcal{Q}[D, U(D)]$. For every object D such that $D \in L$ holds $D = \text{Ball}(U/D, 1)$. Set $N_1 = \text{the norm of } Z$. Reconsider $N_2 = N_1 \circ (\text{rng } U)$ as a finite subset of $\mathbb{R}$. Consider x_3 being an object such that $x_3 \in L$. Consider x_3 being an object such that $x_3 \in \text{rng } U$. Set $d_0 = \sup N_2$. Set $K = d_0 + 1$.

For every function f from the carrier of M into the carrier of T such that $f \in G$ for every element x of M , $\|f(x)\| \leq K$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in G$ for every points x_1, x_2 of M such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

- (14) Suppose $S = M_{\text{top}}$ and T is complete. Then suppose $G = H$ and $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T) \upharpoonright H$ is totally bounded. Then

- (i) for every point x of S and for every non empty subset H_2 of $\text{MetricSpaceNorm } T$ such that $H_2 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in H\}$ holds $\text{MetricSpaceNorm } T \upharpoonright H_2$ is totally bounded, and

(ii) G is equicontinuous.

PROOF: For every point x of S and for every non empty subset H_2 of $\text{MetricSpaceNorm } T$ such that $H_2 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in H\}$ holds $\text{MetricSpaceNorm } T \upharpoonright H_2$ is totally bounded. $\square$

(15) Let us consider a normed linear topological space T , and a real normed space R . Suppose $R =$ the normed structure of T and the topology of $T =$ the topology of $\text{TopSpaceNorm } R$. Then

(i) the distance by norm of $R =$ the distance by norm of T , and

(ii) $\text{MetricSpaceNorm } R = \text{MetricSpaceNorm } T$, and

(iii) $\text{TopSpaceNorm } T = \text{TopSpaceNorm } R$.

PROOF: For every points x, y of R , (the distance by norm of T)(x, y) = $\|x - y\|$ by [11, (19)]. $\square$

Let us consider a non empty metric space M , a non empty, compact topological space S , a normed linear topological space T , a subset G of (the carrier of T)^(the carrier of M), and a non empty subset H of MetricSpaceNorm (the $\mathbb{R}$ -norm space of continuous functions of S and T). Now we state the propositions:

(16) Suppose $S = M_{\text{top}}$ and T is complete and $G = H$. Then MetricSpaceNorm (the $\mathbb{R}$ -norm space of continuous functions of S and T) $\upharpoonright H$ is totally bounded if and only if G is equicontinuous and for every point x of S and for every non empty subset H_2 of $\text{MetricSpaceNorm } T$ such that $H_2 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in H\}$ holds $\text{MetricSpaceNorm } T \upharpoonright \overline{H_2}$ is compact.

PROOF: Set $Z =$ the $\mathbb{R}$ -norm space of continuous functions of S and T . Set $M_1 = \text{MetricSpaceNorm } Z \upharpoonright H$. For every real number e such that $e > 0$ there exists a family L of subsets of M_1 such that L is finite and the carrier of $M_1 = \bigcup L$ and for every subset C of M_1 such that $C \in L$ there exists an element w of M_1 such that $C = \text{Ball}(w, e)$. $\square$

(17) Suppose $S = M_{\text{top}}$ and T is complete and $G = H$. Then $\overline{H}$ is sequentially compact if and only if G is equicontinuous and for every point x of S and for every non empty subset H_2 of $\text{MetricSpaceNorm } T$ such that $H_2 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in H\}$ holds $\text{MetricSpaceNorm } T \upharpoonright \overline{H_2}$ is compact. The theorem is a consequence of (11) and (16).

Let us consider a non empty metric space M , a non empty, compact topological space S , a normed linear topological space T , a non empty subset F of the $\mathbb{R}$ -norm space of continuous functions of S and T , and a subset G of (the carrier of T)^(the carrier of M). Now we state the propositions:

- (18) Suppose $S = M_{\text{top}}$ and T is complete and $G = F$. Then $\overline{F}$ is compact if and only if G is equicontinuous and for every point x of S and for every non empty subset F_1 of $\text{MetricSpaceNorm } T$ such that $F_1 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in F\}$ holds $\text{MetricSpaceNorm } T \upharpoonright \overline{F_1}$ is compact. The theorem is a consequence of (12) and (16).
- (19) Suppose $S = M_{\text{top}}$ and T is complete and $G = F$. Then $\overline{F}$ is compact if and only if for every point x of M , G is equicontinuous at x and for every point x of S and for every non empty subset F_1 of $\text{MetricSpaceNorm } T$ such that $F_1 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in F\}$ holds $\text{MetricSpaceNorm } T \upharpoonright \overline{F_1}$ is compact. The theorem is a consequence of (18) and (5).
- (20) Let us consider a normed linear topological space T . Then T is compact if and only if $\text{TopSpaceNorm } T$ is compact. The theorem is a consequence of (15).
- (21) Let us consider a normed linear topological space T , and a set X . Then X is a compact subset of T if and only if X is a compact subset of $\text{TopSpaceNorm } T$. The theorem is a consequence of (15).
- (22) Let us consider a normed linear topological space T . If T is compact, then T is complete. The theorem is a consequence of (20) and (6).

Let us observe that every normed linear topological space which is compact is also complete.

Now we state the proposition:

- (23) Let us consider a non empty metric space M , a non empty, compact topological space S , a normed linear topological space T , a compact subset U of T , a non empty subset F of the $\mathbb{R}$ -norm space of continuous functions of S and T , and a subset G of $(\text{the carrier of } T)^\alpha$. Suppose $S = M_{\text{top}}$ and T is complete and $G = F$ and for every function f such that $f \in F$ holds $\text{rng } f \subseteq U$. Then

- (i) if $\overline{F}$ is compact, then G is equibounded and equicontinuous, and
- (ii) if G is equicontinuous, then $\overline{F}$ is compact,

where α is the carrier of M .

PROOF: Reconsider $H = F$ as a non empty subset of $\text{MetricSpaceNorm}(\text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T)$. Set $Z = \text{the } \mathbb{R}\text{-norm space of continuous functions of } S \text{ and } T$. $\text{MetricSpaceNorm } Z \upharpoonright H$ is totally bounded iff $\overline{F}$ is compact. For every point x of S and for every non empty subset F_1 of $\text{MetricSpaceNorm } T$ such that $F_1 = \{f(x), \text{ where } f \text{ is a function from } S \text{ into } T : f \in F\}$ holds $\text{MetricSpaceNorm } T \upharpoonright \overline{F_1}$ is compact. $\square$

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