Calculus for the intermediate point associated with a mean value theorem of the integral calculus ¹

Emilia-Loredana Pop, Dorel Duca, Augusta Rațiu

Abstract

If $f, g : [a, b] \rightarrow \mathbb{R}$ are two continuous functions, then there exists a point $c \in (a, b)$ such that

$$\int_a^c f(x) dx + (c - a)g(c) = \int_c^b g(x) dx + (b - c)f(c).$$

In this paper, we study the approaching of the point c towards a , when b approaches a .

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1 Introduction and Preliminaries

Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g : [a, b] \rightarrow \mathbb{R}$ two continuous functions. By Rolle's theorem (see, for example [1], [2]), applied to the function $h : [a, b] \rightarrow \mathbb{R}$ defined by

$$h(x) = (b - x) \int_a^x f(t) dt + (x - a) \int_x^b g(t) dt, \text{ for all } x \in [a, b],$$

there exists a point $c \in (a, b)$ such that

$$\int_a^c f(x) dx + (c - a)g(c) = \int_c^b g(x) dx + (b - c)f(c).$$

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It follows that, for each $x \in (a, b]$, there exists a point $c_x \in (a, x)$ such that

$$(1) \quad \int_a^{c_x} f(t) dt + (c_x - a)g(c_x) = \int_{c_x}^x g(t) dt + (x - c_x)f(c_x).$$

If for each $x \in (a, b]$ we choose one $c_x \in (a, x)$ such that (1) holds, we can define the function $c : (a, b] \rightarrow [a, b]$ by

$$(2) \quad c(x) = c_x, \text{ for all } x \in (a, b],$$

with the property

$$(3) \quad \int_a^{c(x)} f(t) dt + (c(x) - a)g(c(x)) = \int_{c(x)}^x g(t) dt + (x - c(x))f(c(x)),$$

for all $x \in (a, b]$.

So, we have that

Theorem 1 *Let $a, b \in \mathbb{R}$ with $a < b$. If the functions $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous, then there exists a function $c : (a, b] \rightarrow [a, b]$ such that (3) holds.*

When $x \in (a, b]$ tends to a , because $|c(x) - a| \leq |x - a|$, we have

$$\lim_{x \rightarrow a} c(x) = a.$$

Then the function $\bar{c} : [a, b] \rightarrow [a, b]$ defined by

$$\bar{c}(x) = \begin{cases} c(x), & \text{if } x \in (a, b] \\ a, & \text{if } x = a. \end{cases}$$

is continuous at $x = a$.

For $x \in (a, b]$, we have

$$\frac{\bar{c}(x) - \bar{c}(a)}{x - a} = \frac{c(x) - a}{x - a}.$$

If we denote by

$$\theta(x) = \frac{c(x) - a}{x - a},$$

then

$$\theta(x) \in (0, 1]$$

and

$$c(x) = a + (x - a)\theta(x)$$

and hence

$$(4) \quad \begin{aligned} & \int_a^{a+(x-a)\theta(x)} f(t) dt + (x - a)\theta(x)g(a + (x - a)\theta(x)) = \\ & = \int_{a+(x-a)\theta(x)}^x g(t) dt + (x - a)(1 - \theta(x))f(a + (x - a)\theta(x)), \\ & \text{for all } x \in (a, b]. \end{aligned}$$

Hence, the following result is true.

Theorem 2 Let $a, b \in \mathbb{R}$ with $a < b$. If the functions $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous, then there exists a function $\theta : (a, b] \rightarrow (0, 1]$ such that (4) holds.

If F is a primitive of the function f and G is a primitive of the function g , then, from the Leibniz-Newton Theorem we have

$$\int_a^{a+(x-a)\theta(x)} f(t)dt = F(a + (x-a)\theta(x)) - F(a),$$

$$\int_{a+(x-a)\theta(x)}^x g(t)dt = G(x) - G(a + (x-a)\theta(x))$$

and the equality (4) can be written as

$$\begin{aligned} & F(a + (x-a)\theta(x)) - F(a) + g(a + (x-a)\theta(x))\theta(x)(x-a) = \\ (5) \quad & = G(x) - G(a + (x-a)\theta(x)) + f(a + (x-a)\theta(x))(1-\theta(x))(x-a), \\ & \text{for all } x \in (a, b]. \end{aligned}$$

The purpose of this paper is to establish under which circumstances the function $\bar{c}$ is differentiable at the point $x = a$ and to compute its derivatives.

The idea was presented for different approaches, by using other hypothesis and obtaining corresponding results for other interesting problems, like the ones presented in [3], [4], [5], [6], [7], [8].

2 Main results

Theorem 3 Let $a, b \in \mathbb{R}$ with $a < b$ and let f, g be two differentiable functions on $[a, b]$. If $f(a) + g(a) \neq 0$, then:

1° The function θ has the property that

$$\lim_{x \rightarrow a} \theta(x) = \frac{1}{2}.$$

2° The function $\bar{c}$ is differentiable at $x = a$ and

$$\bar{c}'(a) = \frac{1}{2}.$$

Proof. We consider relation (5) and we divide it by $(x-a)$ and get

$$\begin{aligned} & \frac{F(a+(x-a)\theta(x)) - F(a)}{x-a} + g(a + (x-a)\theta(x))\theta(x) = \\ & = \frac{G(x) - G(a+(x-a)\theta(x))}{x-a} + f(a + (x-a)\theta(x))(1-\theta(x)), \\ & \text{for all } x \in (a, b]. \end{aligned}$$

Next, we adjust the relation with $\theta(x)$ and manage $G(a)$, such that

$$(6) \quad \begin{aligned} & \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)}\theta(x) + g(a+(x-a)\theta(x))\theta(x) = \frac{G(x)-G(a)}{x-a} + \\ & + \frac{G(a)-G(a+(x-a)\theta(x))}{(x-a)\theta(x)}\theta(x) + f(a+(x-a)\theta(x))(1-\theta(x)), \\ & \text{for all } x \in (a, b]. \end{aligned}$$

When we pass to $\lim_{x \rightarrow a}$, we get

$$F'(a)\lim_{x \rightarrow a}\theta(x) + g(a)\lim_{x \rightarrow a}\theta(x) = G'(a) - G'(a)\lim_{x \rightarrow a}\theta(x) + f(a)(1 - \lim_{x \rightarrow a}\theta(x)).$$

But $F' = f$ and $G' = g$. Consequently, we have

$$f(a)\lim_{x \rightarrow a}\theta(x) + g(a)\lim_{x \rightarrow a}\theta(x) = g(a) - g(a)\lim_{x \rightarrow a}\theta(x) + f(a)(1 - \lim_{x \rightarrow a}\theta(x)).$$

But $f(a) + g(a) \neq 0$ and hence $2\lim_{x \rightarrow a}\theta(x) = 1$ and so,

$$\lim_{x \rightarrow a}\theta(x) = \frac{1}{2} \text{ and } \bar{c}'(a) = \frac{1}{2}.$$

Theorem 4 Let $a, b \in \mathbb{R}$ with $a < b$ and let f, g be two times differentiable functions on $[a, b]$. If $f(a) + g(a) \neq 0$, then

1° The function $\bar{\theta} : [a, b] \rightarrow [0, 1]$ defined by

$$\bar{\theta}(x) = \begin{cases} \theta(x), & \text{if } x \in (a, b] \\ \frac{1}{2}, & \text{if } x = a \end{cases}$$

is differentiable at $x = a$ and

$$\bar{\theta}'(a) = \frac{f'(a) + g'(a)}{8[f(a) + g(a)]}.$$

2° The function $\bar{c}$ is two times differentiable at $x = a$ and

$$\bar{c}''(a) = \frac{f'(a) + g'(a)}{4[f(a) + g(a)]}.$$

Proof. First, we rewrite, relation (6) in the following form:

$$(7) \quad \begin{aligned} & \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)}\theta(x) + \frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)}\theta(x) + g(a+(x-a)\theta(x))\theta(x) + \\ & + f(a+(x-a)\theta(x))\theta(x) = \frac{G(x)-G(a)}{x-a} + f(a+(x-a)\theta(x)), \\ & \text{for all } x \in (a, b]. \end{aligned}$$

We construct $\frac{\bar{\theta}(x)-\bar{\theta}(a)}{x-a} = \frac{\theta(x)-\frac{1}{2}}{x-a}$. From relation (7) we have

$$\begin{aligned}
 & \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)} \left(\theta(x) - \frac{1}{2} \right) + \frac{1}{2} \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)} + \\
 & + \frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)} \left(\theta(x) - \frac{1}{2} \right) + \frac{1}{2} \frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)} + \\
 (8) \quad & + \left(\theta(x) - \frac{1}{2} \right) g(a + (x-a)\theta(x)) + \frac{1}{2} g(a + (x-a)\theta(x)) + \\
 & + \left(\theta(x) - \frac{1}{2} \right) f(a + (x-a)\theta(x)) + \frac{1}{2} f(a + (x-a)\theta(x)) = \\
 & = \frac{G(x)-G(a)}{x-a} + f(a + (x-a)\theta(x)), \text{ for all } x \in (a, b].
 \end{aligned}$$

We divide this relation by $(x-a)$ and we get

$$\begin{aligned}
 & \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)} \frac{\theta(x)-\frac{1}{2}}{x-a} + \frac{1}{2} \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)^2\theta(x)} + \\
 & + \frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)} \frac{\theta(x)-\frac{1}{2}}{x-a} + \frac{1}{2} \frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)^2\theta(x)} + \\
 (9) \quad & + \frac{\theta(x)-\frac{1}{2}}{x-a} g(a + (x-a)\theta(x)) + \frac{1}{2(x-a)} g(a + (x-a)\theta(x)) + \\
 & + \frac{\theta(x)-\frac{1}{2}}{x-a} f(a + (x-a)\theta(x)) + \frac{1}{2(x-a)} f(a + (x-a)\theta(x)) = \\
 & = \frac{G(x)-G(a)}{(x-a)^2} + \frac{f(a+(x-a)\theta(x))}{x-a}, \\
 & \text{for all } x \in (a, b].
 \end{aligned}$$

We pass to $\lim_{x \rightarrow a}$ in relation (9) and we obtain

$$(10) \quad \bar{\theta}'(a) [f(a) + g(a)] = I,$$

where

$$\begin{aligned}
 (11) \quad I = \lim_{x \rightarrow a} & \left[\frac{G(x)-G(a)}{(x-a)^2} - \frac{1}{2} \frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)^2\theta(x)} - \frac{1}{2} \frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)^2\theta(x)} + \right. \\
 & \left. + \frac{f(a+(x-a)\theta(x))}{2(x-a)} - \frac{g(a+(x-a)\theta(x))}{2(x-a)} \right]
 \end{aligned}$$

We separate this limit in 2 parts $I = I_1 + I_2$ and we have:

$$\begin{aligned}
 (12) \quad I_1 &= \frac{-1}{2} \lim_{x \rightarrow a} \left[\frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)^2\theta(x)} - \frac{f(a+(x-a)\theta(x))}{x-a} \right] = \\
 &= \frac{-1}{2} \lim_{x \rightarrow a} \left[\frac{\frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)} - f(a+(x-a)\theta(x))}{(x-a)} \right] =
 \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{2} \lim_{x \rightarrow a} \left[\frac{\frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)} - F'(a)}{(x-a)} - \frac{f(a+(x-a)\theta(x)) - f(a)}{x-a} \right] = \\
&= \frac{-1}{2} \lim_{x \rightarrow a} \theta(x) \left[\frac{\frac{F(a+(x-a)\theta(x))-F(a)}{(x-a)\theta(x)} - F'(a)}{(x-a)\theta(x)} - \frac{f(a+(x-a)\theta(x)) - f(a)}{(x-a)\theta(x)} \right] \\
&= \frac{-1}{4} \left[\frac{F''(a)}{2} - f'(a) \right] = \frac{f'(a)}{8}.
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \lim_{x \rightarrow a} \left[\frac{G(x) - G(a)}{(x-a)^2} - \frac{G(a+(x-a)\theta(x)) - G(a)}{2(x-a)^2\theta(x)} - \frac{g(a+(x-a)\theta(x))}{2(x-a)} \right] = \\
&= \frac{-1}{2} \lim_{x \rightarrow a} \left[\frac{\frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)} + g(a+(x-a)\theta(x)) - 2\frac{G(x)-G(a)}{x-a}}{(x-a)} \right] = \\
&= \frac{-1}{2} \lim_{x \rightarrow a} \left[\frac{\frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)} - G'(a)}{(x-a)} + \frac{G'(a) + g(a+(x-a)\theta(x)) - 2\frac{G(x)-G(a)}{x-a}}{x-a} \right] = \\
&= \frac{-1}{2} \lim_{x \rightarrow a} \frac{\frac{G(a+(x-a)\theta(x))-G(a)}{(x-a)\theta(x)} - G'(a)}{\theta(x)(x-a)} \theta(x) + \\
&\quad + \lim_{x \rightarrow a} \frac{\frac{G(x)-G(a)}{x-a} - G'(a)}{x-a} - \frac{1}{2} \lim_{x \rightarrow a} \frac{g(a+(x-a)\theta(x)) - g(a)}{\theta(x)(x-a)} \theta(x) = \\
&= -\frac{G''(a)}{4} \theta(a) + \frac{G''(a)}{2} - \frac{g'(a)}{2} \theta(a) = \frac{g'(a)}{8}.
\end{aligned}$$

From

$$I = I_1 + I_2,$$

we obtain

$$\bar{\theta}'(a) [f(a) + g(a)] = \frac{f'(a) + g'(a)}{8}.$$

Hence

$$\bar{\theta}'(a) = \frac{f'(a) + g'(a)}{8[f(a) + g(a)]}$$

and

$$\bar{c}''(a) = \frac{f'(a) + g'(a)}{4[f(a) + g(a)]}.$$

3 Conclusions and further challenges

In this paper we introduced conditions for the functions f and g such that the intermediate point of functions $\bar{c}$ and $\bar{\theta}$ to be derivable at the point a and we provided the derivative $\bar{\theta}'(a)$, $\bar{c}'(a)$ and $\bar{c}''(a)$.

In future we want to see in which conditions the functions $\bar{c}$ and $\bar{\theta}$ are derivable of order n at the point a and to calculate the corresponding derivatives.

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Emilia-Loredana Pop

Babeș-Bolyai University

Faculty of Mathematics and Computer Science

Department of Computer Science

No. 1 Mihail Kogalniceanu Street, RO-400084 Cluj-Napoca, Romania

e-mail: pop_emilia_loredana@yahoo.com

Dorel Duca

Babeș-Bolyai University

Faculty of Mathematics and Computer Science

Department of Mathematics

No. 1 Mihail Kogalniceanu Street, RO-400084 Cluj-Napoca, Romania

e-mail: dorelduca@yahoo.com

Augusta Rațiu

Lucian Blaga University of Sibiu

Faculty of Science

Department of Mathematics and Computer Science

No. 5-7 Dr. I. Rațiu Street, RO-550012 Sibiu, Romania

e-mail: augu2003@yahoo.com