A refinement of Grüss inequality for the complex integral¹

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Abstract

Assume that f and g are continuous on γ , $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the complex Čebyšev functional is defined by

$$\mathcal{D}_\gamma(f, g) := \frac{1}{w - u} \int_\gamma f(z) g(z) dz - \frac{1}{w - u} \int_\gamma f(z) dz \frac{1}{w - u} \int_\gamma g(z) dz.$$

In this paper we establish some Grüss type inequalities for $\mathcal{D}_\gamma(f, g)$ under some complex boundedness conditions for the functions f and g .

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1 Introduction

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the Čebyšev functional defined by

$$C(f, g) := \frac{1}{b - a} \int_a^b f(t) g(t) dt - \frac{1}{b - a} \int_a^b f(t) dt \frac{1}{b - a} \int_a^b g(t) dt.$$

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In 1934, G. Grüss [17] showed that

$$(1) \quad |C(f, g)| \leq \frac{1}{4} (M - m)(N - n),$$

provided m, M, n, N are real numbers with the property that

$$(2) \quad -\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b].$$

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller one.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$(3) \quad |C(f, g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_{\infty} \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}}, \end{cases}$$

where $p > 1, 1/p + 1/q = 1$.

For $\gamma = 0$, we get from the first inequality in (3)

$$(4) \quad |C(f, g)| \leq \|g\|_{\infty} \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a, b]$, then $\|g - \frac{m+M}{2}\|_{\infty} \leq \frac{1}{2}(M - m)$ and by the first inequality in (3) we can deduce the following result obtained by Cheng and Sun [9]

$$(5) \quad |C(f, g)| \leq \frac{1}{2} (M - m) \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.$$

The constant $\frac{1}{2}$ is best in (5) as shown by Cerone and Dragomir in [7].

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ . Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z) dz = \int_{\gamma_{u,w}} f(z) dz := \int_a^b f(z(t)) z'(t) dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z) dz := \int_{\gamma_{u,v}} f(z) dz + \int_{\gamma_{v,w}} f(z) dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z) |dz| := \int_a^b f(z(t)) |z'(t)| dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)| dt.$$

Let f and g be holomorphic in G , and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$(6) \quad \int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz.$$

We recall also the *triangle inequality* for the complex integral, namely

$$(7) \quad \left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma, \infty} \ell(\gamma)$$

where $\|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the p -norm with $p \geq 1$ by

$$\|f\|_{\gamma, p} := \left(\int_{\gamma} |f(z)|^p |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma, 1} := \int_{\gamma} |f(z)| |dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma, 1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma, p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $f : \gamma \rightarrow \mathbb{C}$ a continuous function on γ . Define the quantity:

$$(8) \quad \begin{aligned} \mathcal{P}_{\gamma}(f, \bar{f}) &= \frac{1}{\ell(\gamma)} \int_{\gamma} |f(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 \\ &= \frac{1}{\ell(\gamma)} \int_{\gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 |dv| \geq 0. \end{aligned}$$

If f and g are continuous on γ , we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_{\gamma}(f, g) := \frac{1}{w-u} \int_{\gamma} f(z) g(z) dz - \frac{1}{w-u} \int_{\gamma} f(z) dz \frac{1}{w-u} \int_{\gamma} g(z) dz.$$

In this paper we establish some Grüss type inequalities for $\mathcal{D}_{\gamma}(f, g)$ under some complex boundedness conditions for the functions f and g .

2 Some Preliminary Results

We have the following equalities:

Lemma 1 *Assume that the path γ is piecewise smooth and $f : \gamma \rightarrow \mathbb{C}$ is continuous on γ . Then for $\phi, \Phi \in \mathbb{C}$ with $\phi \neq \Phi$ we have*

$$(9) \quad \mathcal{P}_\gamma(f, \bar{f}) = \operatorname{Re} \left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| - \bar{\phi} \right) \right] \\ - \frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re} \left[(\Phi - f(z)) (\overline{f(z)} - \bar{\phi}) \right] |dz|$$

and, equivalently,

$$(10) \quad \mathcal{P}_\gamma(f, \bar{f}) = \frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \\ - \frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re} \left[(\Phi - f(z)) (\overline{f(z)} - \bar{\phi}) \right] |dz|.$$

Proof. We have

$$I_1 := \operatorname{Re} \left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| - \bar{\phi} \right) \right] \\ = \operatorname{Re} \left[\frac{\Phi}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| + \frac{\bar{\phi}}{\ell(\gamma)} \int_\gamma f(z) |dz| - \Phi \bar{\phi} - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right|^2 \right] \\ = \operatorname{Re} \left(\frac{\Phi}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| \right) + \operatorname{Re} \left(\frac{\bar{\phi}}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \\ - \operatorname{Re}(\Phi \bar{\phi}) - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right|^2$$

and

$$I_2 := \frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re} \left[(\Phi - f(z)) (\overline{f(z)} - \bar{\phi}) \right] |dz| \\ = \frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re} \left[\Phi \overline{f(z)} + \bar{\phi} f(z) - \Phi \bar{\phi} - |f(z)|^2 \right] |dz| \\ = \frac{1}{\ell(\gamma)} \int_\gamma \left[\operatorname{Re}(\Phi \overline{f(z)}) + \operatorname{Re}(\bar{\phi} f(z)) - \operatorname{Re}(\Phi \bar{\phi}) - |f(z)|^2 \right] |dz| \\ = \operatorname{Re} \left(\frac{\Phi}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| \right) + \operatorname{Re} \left(\frac{\bar{\phi}}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \\ - \operatorname{Re}(\Phi \bar{\phi}) - \frac{1}{\ell(\gamma)} \int_\gamma |f(z)|^2 |dz|.$$

Therefore

$$I_1 - I_2 = \frac{1}{\ell(\gamma)} \int_{\gamma} |f(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2$$

proving the identity (9).

We have the equality for complex numbers

$$\operatorname{Re} [(\Phi - w)(\bar{w} - \bar{\phi})] = \frac{1}{4} |\Phi - \phi|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2$$

for $w \in \mathbb{C}$, then by taking

$$w = \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz|$$

in this equality, we get

$$\begin{aligned} \operatorname{Re} \left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_{\gamma} \overline{f(z)} |dz| - \bar{\phi} \right) \right] \\ = \frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \end{aligned}$$

and by (9) we obtain the desired result (10).

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. Now, for $\phi, \Phi \in \mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_{\gamma}(\phi, \Phi) := \left\{ f : \gamma \rightarrow \mathbb{C} \mid \operatorname{Re} [(\Phi - f(z))(\overline{f(z)} - \bar{\phi})] \geq 0 \text{ for each } z \in \gamma \right\}$$

and

$$\bar{\Delta}_{\gamma}(\phi, \Phi) := \left\{ f : \gamma \rightarrow \mathbb{C} \mid \left| f(z) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 2 *For any $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{\gamma}(\phi, \Phi)$ and $\bar{\Delta}_{\gamma}(\phi, \Phi)$ are nonempty, convex and closed sets and*

$$(11) \quad \bar{U}_{\gamma}(\phi, \Phi) = \bar{\Delta}_{\gamma}(\phi, \Phi).$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re} [(\Phi - w)(\bar{w} - \bar{\phi})] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re} [(\Phi - w)(\bar{w} - \bar{\phi})]$$

that holds for any $w \in \mathbb{C}$.

The equality (11) is thus a simple consequence of this fact.

On making use of the complex numbers field properties we can also state that:

Corollary 3 *For any $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$(12) \quad \bar{U}_\gamma(\phi, \Phi) = \{f : \gamma \rightarrow \mathbb{C} \mid (\operatorname{Re} \Phi - \operatorname{Re} f(z))(\operatorname{Re} f(z) - \operatorname{Re} \phi) + (\operatorname{Im} \Phi - \operatorname{Im} f(z))(\operatorname{Im} f(z) - \operatorname{Im} \phi) \geq 0 \text{ for each } z \in \gamma\}.$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$(13) \quad \bar{S}_\gamma(\phi, \Phi) := \{f : \gamma \rightarrow \mathbb{C} \mid \operatorname{Re}(\Phi) \geq \operatorname{Re} f(z) \geq \operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(z) \geq \operatorname{Im}(\phi) \text{ for each } z \in \gamma\}.$$

One can easily observe that $\bar{S}_\gamma(\phi, \Phi)$ is closed, convex and

$$(14) \quad \emptyset \neq \bar{S}_\gamma(\phi, \Phi) \subseteq \bar{U}_\gamma(\phi, \Phi).$$

Theorem 4 *Assume that the path γ is piecewise smooth and $f : \gamma \rightarrow \mathbb{C}$ is continuous on γ and there exist $\phi, \Phi \in \mathbb{C}$ with $\phi \neq \Phi$ such that $f \in \bar{\Delta}_\gamma(\phi, \Phi)$. Then*

$$(15) \quad \mathcal{P}_\gamma(f, \bar{f}) \leq \operatorname{Re} \left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| - \bar{\phi} \right) \right] \leq \frac{1}{4} |\Phi - \phi|^2$$

or, equivalently,

$$(16) \quad \mathcal{P}_\gamma(f, \bar{f}) \leq \frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \leq \frac{1}{4} |\Phi - \phi|^2.$$

Proof. Since $f \in \bar{\Delta}_\gamma(\phi, \Phi)$, hence

$$\operatorname{Re} [(\Phi - f(z))(\overline{f(z)} - \bar{\phi})] \geq 0 \text{ for each } z \in \gamma.$$

Therefore

$$\frac{1}{\ell(\gamma)} \int_\gamma \operatorname{Re} [(\Phi - f(z))(\overline{f(z)} - \bar{\phi})] |dz| \geq 0$$

and by Lemma 1 we deduce the desired result.

We have:

Lemma 5 Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ , then for all $\lambda \in \mathbb{C}$ we have

$$(17) \quad \mathcal{P}_\gamma(f, \bar{f}) = \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) (\overline{f(v)} - \lambda) |dv|.$$

Proof. We observe that

$$\begin{aligned} & \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) (\overline{f(v)} - \lambda) |dv| \\ &= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \frac{1}{\ell(\gamma)} \int_\gamma \overline{f(v)} |dv| \\ & \quad - \lambda \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) |dv| \\ &= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \frac{1}{\ell(\gamma)} \left(\int_\gamma \overline{f(v)} |dv| \right) \\ &= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2 |dv| - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right|^2 \end{aligned}$$

for any $\lambda \in \mathbb{C}$, which proves (17).

We have:

Lemma 6 Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ and there exists $c \in \mathbb{C}$ and $\rho > 0$ such that

$$(18) \quad f \in \overline{D}(c, \rho) := \{z \in \mathbb{C} \mid |z - c| \leq \rho\},$$

then

$$(19) \quad 0 \leq \mathcal{P}_\gamma(f, \bar{f}) \leq \rho \frac{1}{\ell(\gamma)} \left| \int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right| |dv|$$

and

$$(20) \quad \begin{aligned} 0 \leq \mathcal{P}_\gamma^2(f, \bar{f}) &\leq \rho^2 \left(\frac{1}{\ell(\gamma)} \left| \int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right| |dv| \right)^2 \\ &\leq \rho^2 \mathcal{P}_\gamma(f, \bar{f}). \end{aligned}$$

Proof. For the equality (17) for $\lambda = \bar{c}$ we have

$$\begin{aligned} \mathcal{P}_\gamma(f, \bar{f}) &= \left| \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) (\overline{f(v)} - \bar{c}) |dv| \right| \\ &\leq \frac{1}{\ell(\gamma)} \left| \int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right| |\overline{f(v)} - \bar{c}| |dv| \\ &= \frac{1}{\ell(\gamma)} \left| \int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right| |f(v) - c| |dv| \\ &\leq \rho \frac{1}{\ell(\gamma)} \left| \int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right| |dv|, \end{aligned}$$

which proves (19).

Using Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}
 (21) \quad & \frac{1}{\ell(\gamma)} \left| \int_{\gamma} f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| |dv| \\
 & \leq \left(\frac{1}{\ell(\gamma)} \int_{\gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 |dv| \right)^{1/2} \\
 & = \left(\frac{1}{\ell(\gamma)} \int_{\gamma} |f(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right|^2 \right)^{1/2},
 \end{aligned}$$

where for the last equality we used (8).

From (19) and (21) we have

$$0 \leq \mathcal{P}_{\gamma}(f, \bar{f}) \leq \rho \frac{1}{\ell(\gamma)} \left| \int_{\gamma} f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| |dv| \leq \rho [\mathcal{P}_{\gamma}(f, \bar{f})]^{1/2},$$

which implies (20).

Corollary 7 *Assume that the path γ is piecewise smooth and $f : \gamma \rightarrow \mathbb{C}$ is continuous on γ and there exist $\phi, \Phi \in \mathbb{C}$ with $\phi \neq \Phi$ such that $f \in \bar{\Delta}_{\gamma}(\phi, \Phi)$. Then*

$$(22) \quad 0 \leq \mathcal{P}_{\gamma}(f, \bar{f}) \leq \frac{1}{2} |\Phi - \phi| \frac{1}{\ell(\gamma)} \left| \int_{\gamma} f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| |dv|$$

and

$$\begin{aligned}
 (23) \quad 0 \leq \mathcal{P}_{\gamma}^2(f, \bar{f}) & \leq \frac{1}{4} |\Phi - \phi|^2 \left(\frac{1}{\ell(\gamma)} \int_{\gamma} \left| f(v) - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right| |dv| \right)^2 \\
 & \leq \frac{1}{4} |\Phi - \phi|^2 \operatorname{Re} \left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_{\gamma} \overline{f(z)} |dz| - \bar{\phi} \right) \right] \\
 & = \frac{1}{4} |\Phi - \phi|^2 \left[\frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \right] \\
 & \leq \frac{1}{16} |\Phi - \phi|^4.
 \end{aligned}$$

3 Refinements of Grüss' Inequality

We start with the following identity of interest:

Lemma 8 Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ , then

$$\begin{aligned}
 (24) \quad \mathcal{D}_\gamma(f, g) &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right) dz \\
 &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dz \right) dw \\
 &= \frac{1}{2(w-u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w)) dz dw.
 \end{aligned}$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw$ exists and

$$\begin{aligned}
 I(z) &:= \int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \\
 &= \int_\gamma (f(z)g(z) + f(w)g(w) - g(z)f(w) - f(z)g(w)) dw \\
 &= f(z)g(z) \int_\gamma dw + \int_\gamma f(w)g(w) dw - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw \\
 &= (w-u)f(z)g(z) + \int_\gamma f(w)g(w) dw - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw.
 \end{aligned}$$

The function $I(z)$ is also continuous on γ , then the integral $\int_\gamma I(z) dz$ exists and

$$\begin{aligned}
 \int_\gamma I(z) dz &= \int_\gamma \left[(w-u)f(z)g(z) + \int_\gamma f(w)g(w) dw \right. \\
 &\quad \left. - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw \right] dz \\
 &= (w-u) \int_\gamma f(z)g(z) dz + (w-u) \int_\gamma f(w)g(w) dw \\
 &\quad - \int_\gamma f(w) dw \int_\gamma g(z) dz - \int_\gamma g(w) dw \int_\gamma f(z) dz \\
 &= 2(w-u) \int_\gamma f(z)g(z) dz - 2 \int_\gamma f(z) dz \int_\gamma g(z) dz = 2(w-u)^2 \mathcal{P}_\gamma(f, g),
 \end{aligned}$$

which proves the first equality in (24).

The rest follows in a similar manner and we omit the details.

We have:

Lemma 9 Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ , then

$$(25) \quad |\mathcal{D}_\gamma(f, g)| \leq \epsilon^2(\gamma) [\mathcal{P}_\gamma(f, \bar{f})]^{1/2} [\mathcal{P}_\gamma(g, \bar{g})]^{1/2},$$

where $\epsilon(\gamma) := \frac{\ell(\gamma)}{|w-u|}$ can be interpreted as the deviation of the path γ from the segment joining the points u and w in $\mathbb{C}$.

Proof. Taking the modulus in the first equality in (24), we get

$$\begin{aligned} |\mathcal{D}_\gamma(f, g)| &= \frac{1}{2|w-u|^2} \left| \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right) dz \right| \\ &\leq \frac{1}{2|w-u|^2} \int_\gamma \left| \int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right| |dz| =: A \end{aligned}$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned} \left| \int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right| \\ \leq \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2}, \end{aligned}$$

which implies that

$$\begin{aligned} A &\leq \frac{1}{2|w-u|^2} \int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2} |dz| \\ &=: B. \end{aligned}$$

By the Cauchy-Bunyakovsky-Schwarz integral inequality, we also have

$$\begin{aligned} &\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2} |dz| \\ &\leq \left(\int_\gamma \left[\left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \right]^2 |dz| \right)^{1/2} \\ &\quad \times \left(\int_\gamma \left[\left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2} \right]^2 |dz| \right)^{1/2} \\ &= \left(\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right) |dz| \right)^{1/2} \left(\int_\gamma \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right) |dz| \right)^{1/2}, \end{aligned}$$

which implies that

$$\begin{aligned} (26) \quad B &\leq \frac{1}{2|w-u|^2} \left(\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right) |dz| \right)^{1/2} \\ &\quad \times \left(\int_\gamma \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right) |dz| \right)^{1/2}. \end{aligned}$$

Now, observe that

$$\begin{aligned}
 (27) \quad & \int_{\gamma} \left(\int_{\gamma} |f(z) - f(w)|^2 |dw| \right) |dz| \\
 &= \int_{\gamma} \left(\int_{\gamma} \left(|f(z)|^2 - 2 \operatorname{Re} \left(f(z) \overline{f(w)} \right) + |f(w)|^2 \right) |dw| \right) |dz| \\
 &= \int_{\gamma} \left(\ell(\gamma) |f(z)|^2 - 2 \operatorname{Re} \left(f(z) \int_{\gamma} \overline{f(w)} |dw| \right) + \int_{\gamma} |f(w)|^2 |dw| \right) |dz| \\
 &= \ell(\gamma) \int_{\gamma} |f(z)|^2 |dz| - 2 \operatorname{Re} \left(\int_{\gamma} f(z) |dz| \int_{\gamma} \overline{f(w)} |dw| \right) + \ell(\gamma) \int_{\gamma} |f(w)|^2 |dw| \\
 &= 2\ell(\gamma) \int_{\gamma} |f(z)|^2 |dz| - 2 \operatorname{Re} \left(\int_{\gamma} f(z) |dz| \overline{\left(\int_{\gamma} f(w) |dw| \right)} \right) \\
 &= 2 \left[\ell(\gamma) \int_{\gamma} |f(z)|^2 |dz| - \left| \int_{\gamma} f(z) |dz| \right|^2 \right] = 2\ell^2(\gamma) \mathcal{P}_{\gamma}(f, \bar{f})
 \end{aligned}$$

and, similarly

$$(28) \quad \int_{\gamma} \left(\int_{\gamma} |g(z) - g(w)|^2 |dw| \right) |dz| = 2\ell^2(\gamma) \mathcal{P}_{\gamma}(g, \bar{g}).$$

Making use of (27) and (28), we get

$$\begin{aligned}
 B &\leq \frac{1}{2|w-u|^2} [2\ell^2(\gamma) \mathcal{P}_{\gamma}(f, \bar{f})]^{1/2} [2\ell^2(\gamma) \mathcal{P}_{\gamma}(g, \bar{g})]^{1/2} \\
 &= \frac{\ell^2(\gamma)}{|w-u|^2} [\mathcal{P}_{\gamma}(f, \bar{f})]^{1/2} [\mathcal{P}_{\gamma}(g, \bar{g})]^{1/2},
 \end{aligned}$$

which proves the desired result (25).

Remark 1 For $g = f$ we have

$$(29) \quad \mathcal{D}_{\gamma}(f, f) = \frac{1}{w-u} \int_{\gamma} f^2(z) dz - \left(\frac{1}{w-u} \int_{\gamma} f(z) dz \right)^2$$

and by (25) we get

$$(30) \quad |\mathcal{D}_{\gamma}(f, f)| \leq \epsilon^2(\gamma) \mathcal{P}_{\gamma}(f, \bar{f}).$$

For $g = \bar{f}$ we have

$$(31) \quad \mathcal{D}_{\gamma}(f, \bar{f}) = \frac{1}{w-u} \int_{\gamma} |f(z)|^2 dz - \frac{1}{w-u} \int_{\gamma} f(z) dz \frac{1}{w-u} \int_{\gamma} \overline{f(z)} dz$$

and by (25) we get

$$(32) \quad |\mathcal{D}_{\gamma}(f, \bar{f})| \leq \epsilon^2(\gamma) \mathcal{P}_{\gamma}(f, \bar{f}).$$

Theorem 10 Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ and there exists $\phi, \Phi, \psi, \Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_\gamma(\phi, \Phi)$ and $g \in \bar{\Delta}_\gamma(\psi, \Psi)$ then

$$\begin{aligned}
 (33) \quad |\mathcal{D}_\gamma(f, g)| &\leq \epsilon^2(\gamma) \left(\frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \right)^{1/2} \\
 &\quad \times \left(\frac{1}{4} |\Psi - \psi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma g(z) |dz| - \frac{\psi + \Psi}{2} \right|^2 \right)^{1/2} \\
 &\leq \epsilon^2(\gamma) \left[\frac{1}{4} |\Phi - \phi| |\Psi - \psi| \right. \\
 &\quad \left. - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right| \left| \frac{1}{\ell(\gamma)} \int_\gamma g(z) |dz| - \frac{\psi + \Psi}{2} \right| \right] \\
 &\leq \frac{1}{4} \epsilon^2(\gamma) |\Phi - \phi| |\Psi - \psi|.
 \end{aligned}$$

Proof. The first inequality in (33) follows by Corollary 7 and Lemma 9.

Using the elementary inequality

$$(m^2 - n^2)^{1/2} (p^2 - q^2)^{1/2} \leq mp - nq$$

that holds for $m \geq n \geq 0$ and $p \geq q \geq 0$, for the choices

$$m = \frac{1}{2} |\Phi - \phi|, \quad n = \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|$$

and

$$p = \frac{1}{2} |\Psi - \psi|, \quad q = \left| \frac{1}{\ell(\gamma)} \int_\gamma g(z) |dz| - \frac{\psi + \Psi}{2} \right|$$

we get the second inequality in (33).

The last part is obvious.

Remark 2 If there is information on the boundedness of only one function, namely $f \in \bar{\Delta}_\gamma(\phi, \Phi)$, where $g : \gamma \rightarrow \mathbb{C}$ is continuous, then we have the "premature" Grüss inequality

$$\begin{aligned}
 (34) \quad |\mathcal{D}_\gamma(f, g)| &\leq \epsilon^2(\gamma) \left(\frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \right)^{1/2} \\
 &\quad \times [\mathcal{P}_\gamma(g, \bar{g})]^{1/2}.
 \end{aligned}$$

4 Applications for Trapezoid Inequality

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f holomorphic in G , an open domain and suppose $\gamma \subset G$. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. Using the integration by parts formula (6) twice we have

$$\int_{\gamma_{u,v}} (z - v) f'(z) dz = (v - u) f(u) - \int_{\gamma_{u,v}} f(z) dz$$

and

$$\int_{\gamma_{v,w}} (z - v) f'(z) dz = (w - v) f(w) - \int_{\gamma_{v,w}} f(z) dz,$$

for any $v \in \gamma$.

If we add these two equalities, we get the *generalized trapezoid equality*

$$\begin{aligned} (35) \quad & (v - u) f(u) + (w - v) f(w) - \int_{\gamma} f(z) dz \\ &= \int_{\gamma_{u,v}} (z - v) f'(z) dz + \int_{\gamma_{v,w}} (z - v) f'(z) dz = \int_{\gamma} (z - v) f'(z) dz \end{aligned}$$

with the above assumptions for u, v and w on γ .

If we take $v = \frac{u+w}{2}$, then we get the *trapezoid equality*

$$(36) \quad (w - u) \frac{f(u) + f(w)}{2} - \int_{\gamma} f(z) dz = \int_{\gamma} \left(z - \frac{u+w}{2} \right) f'(z) dz,$$

which also can be written as

$$\frac{f(u) + f(w)}{2} - \frac{1}{w - u} \int_{\gamma} f(z) dz = \mathcal{D}_{\gamma}(f', h)$$

where $h(z) = z - \frac{u+w}{2}$, $z \in \mathbb{C}$, since $\int_{\gamma} \left(z - \frac{u+w}{2} \right) dz = 0$.

If $f' \in \bar{\Delta}_{\gamma}(\theta, \Theta)$ for some $\theta, \Theta \in \mathbb{C}$ with $\theta \neq \Theta$, then by (34) we get the following trapezoid inequality of interest:

$$\begin{aligned} & \left| \frac{f(u) + f(w)}{2} - \frac{1}{w - u} \int_{\gamma} f(z) dz \right| = \mathcal{D}_{\gamma}(f', h) \\ & \leq \epsilon^2(\gamma) \left(\frac{1}{4} |\Theta - \theta|^2 - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f'(z) |dz| - \frac{\theta + \Theta}{2} \right|^2 \right)^{1/2} [\mathcal{P}_{\gamma}(h, \bar{h})]^{1/2} \\ & \leq \frac{1}{2} |\Theta - \theta| \epsilon^2(\gamma) [\mathcal{P}_{\gamma}(h, \bar{h})]^{1/2}, \end{aligned}$$

where

$$\mathcal{P}_{\gamma}(h, \bar{h}) = \frac{1}{\ell(\gamma)} \int_{\gamma} \left| z - \frac{u+w}{2} \right|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} \left(z - \frac{u+w}{2} \right) |dz| \right|^2.$$

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_{\gamma} f(z) dz$ as $\int_u^w f(z) dz$. We then have

$$\epsilon(\gamma) = \frac{\ell(\gamma)}{|w - u|} = \frac{|w - u|}{|w - u|} = 1,$$

$$\frac{1}{\ell([u, w])} \int_u^w f'(z) |dz| = \frac{|w - u|}{|w - u|} \int_0^1 f'((1 - t)u + tw) dt = \frac{f(w) - f(u)}{w - u}$$

and

$$\begin{aligned} \mathcal{P}_{[u, w]}(h, \bar{h}) &= \frac{|w - u|}{|w - u|} \int_0^1 \left| (1 - t)u + tw - \frac{u + w}{2} \right|^2 dt \\ &\quad - \left| \frac{|w - u|}{|w - u|} \int_0^1 \left((1 - t)u + tw - \frac{u + w}{2} \right) dt \right|^2 \\ &= |w - u|^2 \int_0^1 \left(t - \frac{1}{2} \right)^2 dt - |w - u|^2 \left| \int_0^1 \left(t - \frac{1}{2} \right) dt \right|^2 = \frac{1}{12} |w - u|^2. \end{aligned}$$

Based on the calculations outlined above, we can state the following result as well:

Proposition 11 *Assume that f is holomorphic in G , an open domain and suppose $[u, w] \subset G$. If $f' \in \bar{\Delta}_{[u, w]}(\theta, \Theta)$ for some $\theta, \Theta \in \mathbb{C}$ with $\theta \neq \Theta$, then*

$$\begin{aligned} (37) \quad &\left| \frac{f(u) + f(w)}{2} - \frac{1}{w - u} \int_u^w f(z) dz \right| \\ &\leq \frac{\sqrt{3}}{6} |w - u| \left(\frac{1}{4} |\Theta - \theta|^2 - \left| \frac{f(w) - f(u)}{w - u} - \frac{\theta + \Theta}{2} \right|^2 \right)^{1/2} \\ &\leq \frac{\sqrt{3}}{12} |w - u| |\Theta - \theta|. \end{aligned}$$

Since

$$\begin{aligned} \operatorname{Re} \left[\left(\Phi - \frac{f(w) - f(u)}{w - u} \right) \left(\overline{\frac{f(w) - f(u)}{w - u}} - \bar{\phi} \right) \right] \\ = \frac{1}{4} |\Phi - \phi|^2 - \left| \frac{f(w) - f(u)}{w - u} - \frac{\phi + \Phi}{2} \right|^2, \end{aligned}$$

then the inequality (37) can also be written as

$$\begin{aligned} (38) \quad &\left| \frac{f(u) + f(w)}{2} - \frac{1}{w - u} \int_u^w f(z) dz \right| \\ &\leq \frac{\sqrt{3}}{6} |w - u| \left(\operatorname{Re} \left[\left(\Phi - \frac{f(w) - f(u)}{w - u} \right) \left(\overline{\frac{f(w) - f(u)}{w - u}} - \bar{\phi} \right) \right] \right)^{1/2} \\ &\leq \frac{\sqrt{3}}{12} |w - u| |\Theta - \theta|. \end{aligned}$$

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