Some applications of generalized Ruscheweyh derivatives involving a general fractional derivative operator to a class of analytic functions with negative coefficients II¹

Waggas Galib Atshan, S. R. Kulkarni

Abstract

In this paper, we study a class of univalent functions f as defined by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator, satisfying

$$\operatorname{Re} \left\{ \frac{z(\mathbf{J}_1^{\lambda, \mu} f(z))'}{(1 - \gamma)\mathbf{J}_1^{\lambda, \mu} f(z) + \gamma z^2(\mathbf{J}_1^{\lambda, \mu} f(z))''} \right\} > \beta.$$

A necessary and sufficient condition for a function to be in the class $A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$ is obtained. Also, our paper includes linear combination, integral operators and we introduce the subclass $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ consisting of functions with negative and fixed finitely many coefficients. We study some interesting properties of $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

2010 Mathematics Subject Classification: 30C45, 30C50.

Key words and phrases: Generalized Ruscheweyh derivatives, Fractional derivative, Integral operators, Linear combination, Radii of starlikeness, Extreme points, Weighted mean and Arithmetic mean.

¹Received 8 February, 2010

Accepted for publication (in revised form) 12 July, 2010

1 Introduction

Let Ω denote the class of functions analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $A(n)$ denote the subclass of Ω consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0, n \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (1.1)$$

where $f(z)$ is analytic and univalent in the unit disk U . Then the function $f(z) \in A(n)$ is said to be in the class $S(n, \alpha)$ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in U, 0 \leq \alpha < 1). \quad (1.2)$$

A function $f(z) \in S(n, \alpha)$ is called starlike function of order α . A function $f(z) \in A(n)$ is said to be in the class $C(n, \alpha)$ if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in U, 0 \leq \alpha < 1). \quad (1.3)$$

A function $f(z) \in C(n, \alpha)$ is called convex function of order α . It is observed that $f(z) \in C(n, \alpha)$ if and only if $zf'(z) \in S(n, \alpha) \forall n \in \{1, 2, \dots\}$ [9] (1.4)

A function $f(z) \in A(n)$ is said to be in the class $K(n, \alpha)$ if there is a convex function $g(z)$ such that

$$\operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > \alpha, \quad (\forall z \in U, 0 \leq \alpha < 1). \quad (1.5)$$

A function $f(z) \in K(n, \alpha)$ is called close-to-convex of order α .

We shall need the fractional derivative operator([18], [19]) in this paper.

Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$. The Gaussian hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1 \equiv {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (1.6)$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

Definition 1 Let $0 \leq \lambda < 1$ and $\mu, \nu \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda, \mu, \nu}$ of a function $f(z)$ is defined by:

$$J_{0,z}^{\lambda, \mu, \nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \{ z^{\lambda-\mu} \int_0^z (z-\xi)^{-\lambda} f(\xi) \cdot \\ {}_2F_1(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\xi}{z}) d\xi \} & (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n, \mu, \nu} f(z), & (n \leq \lambda < n+1, n \in \mathbb{N}) \end{cases} \quad (1.7)$$

where the function $f(z)$ is analytic in a simply - connected region of the z -plane containing the region, with the order

$$f(z) = O(|z|^\epsilon), \quad (z \rightarrow 0) \quad (1.8)$$

for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multiplicity of $(z - \xi)^{-\lambda}$ is removed by required $\log(z - \xi)$ to be real when $z - \xi > 0$.

The fractional derivative of order λ of a function $f(z)$ is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z - \xi)^\lambda} d\xi, \quad 0 \leq \lambda < 1 \quad (1.9)$$

where $f(z)$ is chosen as in (1.7), and the multiplicity of $(z - \xi)^{-\lambda}$ is removed by required $\log(z - \xi)$ to be real when $z - \xi > 0$.

By comparing (1.7) with (1.9), we find

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^\lambda f(z), \quad (0 \leq \lambda < 1). \quad (1.10)$$

In terms of gamma function, we have

$$J_{0,z}^{\lambda,\mu,\nu} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+\nu+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+\nu+k)} z^{k-\mu} \quad (1.11)$$

($0 \leq \lambda < 1, \mu, \nu \in \mathbb{R}$ and $k > \max\{0, \mu - \nu\} - 1$).

Definition 2 Let $f(z) \in A(n)$ be given by (1.1). Then the class $A_\gamma^{\lambda,\mu,\nu}(n, \beta)$ is defined by

$$A_\gamma^{\lambda,\mu,\nu}(n, \beta) = \left\{ f \in A(n) : \operatorname{Re} \left\{ \frac{z(\mathbf{J}_1^{\lambda,\mu} f(z))'}{(1-\gamma)\mathbf{J}_1^{\lambda,\mu} f(z) + \gamma z^2 (\mathbf{J}_1^{\lambda,\mu} f(z))''} \right\} > \beta, \right.$$

$$(z \in U, 0 \leq \gamma < 1, n \in \mathbb{N}; 0 \leq \beta < 1; \lambda > -1) \quad (1.12)$$

where $\mathbf{J}_1^{\lambda,\mu} f(z)$ is a generalized Ruscheweyh derivative defined as

$$\mathbf{J}_1^{\lambda,\mu} f(z) = \frac{\Gamma(\mu - \lambda + \nu - 2)}{\Gamma(\nu + 2)\Gamma(\mu + 1)} z J_{0,z}^{\lambda,\mu,\nu} (z^{\mu-1} f(z)) = z - \sum_{k=n+1}^{\infty} a_k C_1^{\lambda,\mu}(k) z^k \quad (1.13)$$

where

$$C_1^{\lambda,\mu}(k) = \frac{\Gamma(k+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu+1)}{\Gamma(k)\Gamma(k+\nu+1+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}. \quad (1.14)$$

For $\mu = \lambda = \alpha, \nu = 1$, these generalized Ruscheweyh derivatives reduce to ordinary Ruscheweyh derivatives of $f(z)$ of order α [14]:

$$D^\alpha f(z) = \frac{z}{\Gamma(\alpha+1)} D^\alpha (z^{\alpha-1} f(z)) = z - \sum_{k=n+1}^{\infty} a_k C_k(\alpha) z^k \quad (1.15)$$

where

$$C_k(\alpha) = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{(k-1)!}. \quad (1.16)$$

The class $A_\gamma^{\lambda,\mu,\nu}(n, \beta)$ contains many well-known classes of analytic functions, for example,

(i) If $\mu = \lambda = \alpha, \nu = 1, n = 1$, we get the class $A_\gamma^{\lambda,\lambda,1}(1, \beta)$ studied by Tehranchi and Kulkarni [20].

(ii) If $\mu = \lambda = 0, \nu = 1, \alpha = \beta, \gamma = 0$, we get the class of starlike functions of order $\alpha, (S(n, \alpha))$.

The same properties have been found for other classes in [2,3,4,5,6,7,8,10,16,17].

Lemma 1 .[1] Let $w = u + iv$. Then

$$\operatorname{Re} w \geq \beta \text{ if and only if } |w - (1 + \beta)| \leq |w + (1 - \beta)|.$$

2 Coefficient bounds

The following theorem gives a necessary and sufficient condition for function to be in $A_\gamma^{\lambda,\mu,\nu}(n, \beta)$.

Theorem 1 Let $f(z) \in A(n)$. Then $f(z) \in A_\gamma^{\lambda,\mu,\nu}(n, \beta)$ if and only if

$$\sum_{k=n+1}^{\infty} (\gamma\beta(1+k-k^2) + k - \beta)C_1^{\lambda,\mu}(k)a_k < 1 - \beta(1 - \gamma), \quad (2.1)$$

where $0 \leq \gamma < 1, 0 \leq \beta < 1, \lambda > -1, n \in \mathbb{N}$ and $C_1^{\lambda,\mu}(k)$ is given by (1.14).

Proof. Assume that $f \in A_\gamma^{\lambda,\mu,\nu}(n, \beta)$ so we have

$$\operatorname{Re} \left\{ \frac{z(\mathbf{J}_1^{\lambda,\mu} f(z))'}{(1 - \gamma)\mathbf{J}_1^{\lambda,\mu} f(z) + \gamma z^2(\mathbf{J}_1^{\lambda,\mu} f(z))''} \right\} > \beta$$

$$\operatorname{Re} \left\{ \frac{z - \sum_{k=n+1}^{\infty} k C_1^{\lambda,\mu}(k) a_k z^k}{(1 - \gamma) \left(z - \sum_{k=n+1}^{\infty} C_1^{\lambda,\mu}(k) a_k z^k \right) + \gamma \left(- \sum_{k=n+1}^{\infty} k(k-1) C_1^{\lambda,\mu}(k) a_k z^k \right)} \right\} > \beta.$$

Hence

$$\operatorname{Re} \left\{ \frac{1 - \sum_{k=n+1}^{\infty} k C_1^{\lambda,\mu}(k) a_k z^k}{(1 - \gamma) - \sum_{k=n+1}^{\infty} (1 - \gamma + \gamma k(k-1) C_1^{\lambda,\mu}(k) a_k z^{k-1})} \right\} > \beta.$$

or equivalently

$$\begin{aligned} & \operatorname{Re}([1 - \sum_{k=n+1}^{\infty} k C_1^{\lambda, \mu}(k) a_k z^{k-1} - \beta(1 - \gamma) \\ & + \beta \sum_{k=n+1}^{\infty} (1 - \gamma + \gamma k(k-1)) C_1^{\lambda, \mu}(k) a_k z^{k-1}] / [(1 - \gamma) \\ & - \sum_{k=n+1}^{\infty} (1 - \gamma + \gamma k(k-1)) C_1^{\lambda, \mu}(k) a_k z^{k-1}]) > 0. \end{aligned}$$

This inequality is correct for all $z \in U$. Letting $z \rightarrow 1^-$ yields

$$\operatorname{Re} \left\{ \frac{1 - \beta(1 - \gamma) - \sum_{k=n+1}^{\infty} (k - \beta(1 - \gamma + \gamma k(k-1))) C_1^{\lambda, \mu}(k) a_k}{(1 - \gamma) - \sum_{k=n+1}^{\infty} (1 - \gamma + \gamma k(k-1)) C_1^{\lambda, \mu}(k) a_k} \right\} > 0$$

and so by the mean value theorem, we have

$$\operatorname{Re} \left\{ 1 - \beta(1 - \gamma) - \sum_{k=n+1}^{\infty} (k - \beta(1 - \gamma + \gamma k^2 - \gamma k)) C_1^{\lambda, \mu}(k) a_k \right\} > 0,$$

so we have

$$\sum_{k=n+1}^{\infty} (\gamma \beta(1 + k - k^2) + k - \beta) C_1^{\lambda, \mu}(k) a_k < 1 - \beta(1 - \gamma).$$

Conversely, let (2.1) hold. We will prove that (1.12) is correct and then $f \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$. By Lemma 1 it is enough to prove that $|w - (1 + \beta)| < |w + (1 - \beta)|$ where

$$w = \frac{z(\mathbf{J}_1^{\lambda, \mu} f(z))'}{(1 - \gamma)\mathbf{J}_1^{\lambda, \mu} f(z) + \gamma z^2(\mathbf{J}_1^{\lambda, \mu} f(z))''}$$

or show that

$$\begin{aligned} T &= \frac{1}{|N(z)|} |z(\mathbf{J}_1^{\lambda, \mu} f(z))' - (1 + \beta)(1 - \gamma)\mathbf{J}_1^{\lambda, \mu} f(z) - (1 + \beta)\gamma z^2(\mathbf{J}_1^{\lambda, \mu} f(z))''| \\ &< \frac{1}{|N(z)|} |z(\mathbf{J}_1^{\lambda, \mu} f(z))' + (1 - \beta)(1 - \gamma)\mathbf{J}_1^{\lambda, \mu} f(z) + (1 - \beta)\gamma z^2(\mathbf{J}_1^{\lambda, \mu} f(z))''| \\ &= Q \end{aligned}$$

where $N(z) = (1 - \gamma)\mathbf{J}_1^{\lambda, \mu} f(z) + \gamma z^2(\mathbf{J}_1^{\lambda, \mu} f(z))''$ and it is easy to verify that $Q - T > 0$ and so the proof is complete. $\square$

Corollary 1 Let $f \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$. Then

$$a_k < \frac{1 - \beta(1 - \gamma)}{C_1^{\lambda, \mu}(k) |\gamma \beta(1 + k - k^2) + k - \beta|}, \quad k = n + 1, n + 2, \dots$$

3 Linear combination

In the following theorem, we will show that this class $A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$ is closed under linear combination.

Theorem 2 Let $f_i(z) = z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$, $i \in \{1, 2, \dots, t\}$ and $0 < C_i < 1$ such that

$$\sum_{i=1}^t C_i = 1.$$

Then the function $G(z)$ defined by $G(z) = \sum_{i=1}^t C_i f_i(z)$ is also in $A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$.

Proof. For every $i \in \{1, 2, \dots, t\}$ we obtain

$$\sum_{k=n+1}^{\infty} \frac{(\gamma\beta(1+k-k^2) + k - \beta)C_1^{\lambda, \mu}(k)}{1 - \beta(1 - \gamma)} a_{k,i} < 1.$$

Since

$$G(z) = \sum_{i=1}^t C_i f_i(z) = \sum_{i=1}^t C_i (z - \sum_{k=n+1}^{\infty} a_{k,i} z^k) = z - \sum_{k=n+1}^{\infty} (\sum_{i=1}^t C_i a_{k,i}) z^k.$$

Therefore,

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \frac{(\gamma\beta(1+k-k^2) + k - \beta)C_1^{\lambda, \mu}(k)}{1 - \beta(1 - \gamma)} [\sum_{i=1}^t C_i a_{k,i}] \\ &= \sum_{i=1}^t C_i \left[\sum_{k=n+1}^{\infty} \frac{(\gamma\beta(1+k-k^2) + k - \beta)C_1^{\lambda, \mu}(k)}{1 - \beta(1 - \gamma)} a_{k,i} \right] < \sum_{i=1}^t C_i = 1. \end{aligned}$$

Hence $G(z) \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$ and therefore the proof is complete. $\square$

4 Integral operators

In the next theorems, we obtain some results by using the integral operators for this class.

Theorem 3 Let $f(z) \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$, Komato operator [11] of f is defined by

$$y(z) = \int_0^1 \frac{(d+1)^{\theta}}{\Gamma(\theta)} \ell^d (\log \frac{1}{\ell})^{\theta-1} \frac{f(\ell z)}{\ell} d\ell, d > -1, \theta \geq 0. \quad (4.1)$$

Then $y(z) \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$.

Proof. We have $\int_0^1 \ell^d (\log \frac{1}{\ell})^{\theta-1} d\ell = \frac{\Gamma(\theta)}{(d+1)^\theta}$ and $\int_0^1 \ell^{k+d-1} (\log \frac{1}{\ell})^{\theta-1} d\ell = \frac{\Gamma(\theta)}{(d+k)^\theta}$, $k = n+1, n+2, \dots$, then we obtain

$$\begin{aligned} y(z) &= \frac{(d+1)^\theta}{\Gamma(\theta)} \left[\int_0^1 \ell^d (\log \frac{1}{\ell})^{\theta-1} z d\ell - \sum_{k=n+1}^{\infty} z^k \int_0^1 a_k \ell^{d+k-1} (\log \frac{1}{\ell})^{\theta-1} d\ell \right] \\ &= z - \sum_{k=n+1}^{\infty} \left(\frac{d+1}{d+k} \right)^\theta a_k z^k. \end{aligned}$$

Since $f \in A_\gamma^{\lambda, \mu, \nu}(n, \beta)$ and since $\left(\frac{d+1}{d+k} \right)^\theta < 1$ for $k = n+1, n+2, \dots$, then

$$\sum_{k=n+1}^{\infty} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1-\gamma)} \right] \left(\frac{d+1}{d+k} \right)^\theta a_k C_1^{\lambda, \mu}(k) < 1.$$

This completes the proof. $\square$

Theorem 4 The function $y(z)$ defined in (4.1) is starlike of order η in $|z| < r_1(\lambda, \mu, \nu, \gamma, \beta, d, \theta, \eta)$, where

$$r_1(\lambda, \mu, \nu, \gamma, \beta, d, \theta, \eta) =$$

$$\inf_k \left\{ \frac{1-\eta}{k-\eta} \left(\frac{d+k}{d+1} \right)^\theta \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1-\gamma)} \right] C_1^{\lambda, \mu}(k) \right\}^{\frac{1}{k-1}}.$$

Proof. We must show that

$$\left| \frac{zy'(z)}{y(z)} - 1 \right| < 1 - \eta.$$

But

$$\begin{aligned} \left| \frac{zy'(z)}{y(z)} - 1 \right| &= \left| \frac{z - \sum_{k=n+1}^{\infty} k \left(\frac{d+1}{d+k} \right)^\theta a_k z^k}{z - \sum_{k=n+1}^{\infty} \left(\frac{d+1}{d+k} \right)^\theta a_k z^k} - 1 \right| \\ &= \left| \frac{\sum_{k=n+1}^{\infty} (k-1) \left(\frac{d+1}{d+k} \right)^\theta a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} \left(\frac{d+1}{d+k} \right)^\theta a_k z^{k-1}} \right| \end{aligned}$$

$$\leq \frac{\sum_{k=n+1}^{\infty} (k-1) \left(\frac{d+1}{d+k}\right)^{\theta} a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} \left(\frac{d+1}{d+k}\right)^{\theta} a_k |z|^{k-1}} < 1 - \eta.$$

Then

$$\sum_{k=n+1}^{\infty} (k-1) \left(\frac{d+1}{d+k}\right)^{\theta} a_k |z|^{k-1} < 1 - \eta - (1 - \eta) \sum_{k=n+1}^{\infty} \left(\frac{d+1}{d+k}\right)^{\theta} a_k |z|^{k-1}.$$

Hence

$$\sum_{k=n+1}^{\infty} \frac{k-\eta}{1-\eta} \left(\frac{d+1}{d+k}\right)^{\theta} a_k |z|^{k-1} < 1. \quad (4.2)$$

Now in view of (2.1), therefore (4.2) holds if

$$|z|^{k-1} < \left(\frac{1-\eta}{k-\eta}\right) \left(\frac{d+k}{d+1}\right)^{\theta} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1-\gamma)} \right] C_1^{\lambda,\mu}(k).$$

This completes the proof. $\square$

Corollary 2 The function $y(z)$ defined in Theorem 3 is convex of order η in $|z| < r_2(\lambda, \mu, \nu, \gamma, \beta, d, \theta, \eta)$, where

$$r_2(\lambda, \mu, \nu, \gamma, \beta, d, \theta, \eta) = \inf_k \left\{ \frac{1-\eta}{k(k-\eta)} \left(\frac{d+k}{d+1}\right)^{\theta} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1-\gamma)} \right] C_1^{\lambda,\mu}(k) \right\}^{\frac{1}{k-1}}.$$

Theorem 5 Let $f \in A_{\gamma}^{\lambda,\mu,\nu}(n, \beta)$. Then the function $T_{\psi}(z)$ defined by $T_{\psi}(z) = (1-\psi)z + \psi \int_0^z \frac{f(\ell)}{\ell} d\ell$, $\psi \geq 0$, $z \in U$ is in $A_{\gamma}^{\lambda,\mu,\nu}(n, \beta)$ if $0 \leq \psi \leq n+1$.

Proof.

$$\begin{aligned} T_{\psi}(z) &= (1-\psi)z + \psi \int_0^z \left(\frac{\ell - \sum_{k=n+1}^{\infty} a_k \ell^k}{\ell} \right) d\ell \\ &= z - \psi z + \psi \left(\ell - \sum_{k=n+1}^{\infty} a_k \left(\frac{1}{k} \ell^k \right) \Big|_0^z \right) \\ &= z - \psi z + \psi z - \sum_{k=n+1}^{\infty} a_k \frac{\psi}{k} z^k = z - \sum_{k=n+1}^{\infty} a_k \frac{\psi}{k} z^k, \end{aligned}$$

but

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1 - \gamma)} \right] \frac{\psi a_k}{k} C_1^{\lambda, \mu}(k) \\ & \leq \sum_{k=n+1}^{\infty} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1 - \gamma)} \right] \frac{\psi a_k}{n+1} C_1^{\lambda, \mu}(k) < 1. \end{aligned}$$

This inequality completes the proof. $\square$

Remark 1 By the similar proofs which we applied for Theorem 4 and Corollary 2, we conclude that the function $T_\psi(z)$ is starlike of order η ($0 \leq \eta < 1$) in $|z| < r_1(\mu, \nu, \gamma, \beta, \psi, \eta)$, where

$$r_1(\lambda, \mu, \nu, \gamma, \beta, \psi, \eta) = \inf_k \left\{ \frac{k(1-\eta)}{\psi(k-\eta)} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1 - \gamma)} \right] C_1^{\lambda, \mu}(k) \right\}^{\frac{1}{k-1}}.$$

Also $T_\psi(z)$ is convex of order η ($0 \leq \eta < 1$) in $|z| < r_2(\lambda, \mu, \nu, \gamma, \beta, \psi, \eta)$, where

$$r_2(\lambda, \mu, \nu, \gamma, \beta, \psi, \eta) = \inf_k \left\{ \frac{(1-\eta)}{\psi(k-\eta)} \left[\frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1 - \gamma)} \right] C_1^{\lambda, \mu}(k) \right\}^{\frac{1}{k-1}}.$$

Theorem 6 (i) Let $f \in A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$. Then for every $\delta \geq 0$ the function $F_\delta(z)$ defined by $F_\delta(z) = (1 - \delta)f(z) + \delta \int_0^z \frac{f(\ell)}{\ell} d\ell$ also belongs to $A_{\gamma}^{\lambda, \mu, \nu}(n, \beta)$.
(ii) If $f \in A_0^{0,0,1}(n, \beta) = S(n, \beta)$, $\frac{f(z)}{z} \neq 0, z \in U$, also $0 \leq \delta < 1$, then $F_\delta(z)$ is close-to-convex of order δ .

Proof. (i)

$$\begin{aligned} F_\delta(z) &= (1 - \delta) \left(z - \sum_{k=n+1}^{\infty} a_k z^k \right) + \delta \int_0^z \frac{\ell - \sum_{k=n+1}^{\infty} a_k \ell^k}{\ell} d\ell \\ &= z - \delta z - (1 - \delta) \sum_{k=n+1}^{\infty} a_k z^k + \delta z - \sum_{k=n+1}^{\infty} a_k \frac{\delta}{k} z^k \\ &= z - \sum_{k=n+1}^{\infty} \left(1 - \delta + \frac{\delta}{k} \right) a_k z^k. \end{aligned}$$

By (2.1) and this fact that $1 - \delta + \frac{\delta}{k} = 1 - (\delta - \frac{\delta}{k}) < 1$ for $k \geq n+1$, we have

$$\sum_{k=n+1}^{\infty} \frac{\gamma\beta(1+k-k^2) + k - \beta}{1 - \beta(1 - \gamma)} \left(1 - \delta + \frac{\delta}{k} \right) a_k C_1^{\lambda, \mu}(k) < 1.$$

So we obtain $F_\delta(z) \in A_\gamma^{\lambda, \mu, \nu}(n, \beta)$.

(ii) Since $F'_\delta(z) = (1 - \delta)f'(z) + \delta \frac{f(z)}{z}$, so $\frac{zF'_\delta(z)}{f(z)} = \delta + (1 - \delta)\frac{zf'(z)}{f(z)}$ and

$$\operatorname{Re} \left\{ \frac{zF'_\delta(z)}{f(z)} \right\} = \delta + (1 - \delta)\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta$$

and this by (1.5) shows that $F_\delta(z)$ is close-to-convex of order δ . $\square$

Remark 2 The function $F_\delta(z)$ defined in Theorem 6 is starlike of order ϕ ($0 \leq \phi < 1$) in $|z| < r_1(\lambda, \mu, \nu, \gamma, \beta, \delta, \phi)$, where

$$r_1(\lambda, \mu, \nu, \gamma, \beta, \delta, \phi) = \inf_k \left\{ \frac{(1 - \phi)[\gamma\beta(1 + k - k^2) + k - \beta]}{(k - \phi)(1 - \delta + \frac{\delta}{k})[1 - \beta(1 - \gamma)]} C_1^{\lambda, \mu}(k) \right\}^{\frac{1}{k-1}}.$$

Also $F_\delta(z)$ is convex of order ϕ ($0 \leq \phi < 1$) in $|z| < r_2(\lambda, \mu, \nu, \gamma, \beta, \delta, \phi)$, where

$$r_2(\lambda, \mu, \nu, \gamma, \beta, \delta, \phi) = \inf_k \left\{ \frac{(1 - \phi)[\gamma\beta(1 + k - k^2) + k - \beta]}{k(k - \phi)(1 - \delta + \frac{\delta}{k})[1 - \beta(1 - \gamma)]} C_1^{\lambda, \mu}(k) \right\}^{\frac{1}{k-1}}.$$

5 Some properties of a subclass $A_{\gamma, c_m}^{\mu, \nu}(1, \beta)$

We introduce the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ the subclass of $A_\gamma^{\lambda, \mu, \nu}(1, \beta)$ where

$$A_{\gamma}^{\lambda, \mu, \nu}(1, \beta) = \left\{ f \in A(1) : \operatorname{Re} \left\{ \frac{z(\mathbf{J}_1^{\lambda, \mu} f(z))'}{(1 - \gamma)\mathbf{J}_1^{\lambda, \mu} f(z) + \gamma z^2(\mathbf{J}_1^{\lambda, \mu} f(z))''} \right\} > \beta \right\},$$

consisting of functions with negative and fixed finitely many coefficients of the form

$$f(z) = z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))C_i}{[\gamma\beta(1 + i - i^2) + i - \beta]C_1^{\lambda, \mu}(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k, \quad (5.1)$$

where $m = 2, 3, \dots$, $a_k \geq 0$ for $k = m + 1, m + 2, \dots$, $0 \leq C_i \leq 1$ for $i = 2, 3, \dots, m$ and $0 \leq \sum_{i=2}^m C_i \leq 1$.

The different cases were studied earlier by many authors e.g. [12], [13], [15].

We need the following lemma which has been proved in general case in Theorem 1.

Lemma 2 Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in A(1)$. Then $f \in A_{\gamma}^{\lambda, \mu, \nu}(1, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{\gamma\beta(1 + k - k^2) + k - \beta}{1 - \beta(1 - \gamma)} C_1^{\lambda, \mu}(k) a_k < 1.$$

The following theorem gives a necessary and sufficient condition for a function to be in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

Theorem 7 Let $f(z)$ be defined by (5.1). Then $f \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ if and only if

$$\sum_{k=m+1}^{\infty} \frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)} C_1^{\lambda, \mu}(k) a_k < 1 - \sum_{i=2}^m c_i. \quad (5.2)$$

Proof. By letting

$$a_i = \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda, \mu}(i)}$$

since $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta) \subset A_{\gamma}^{\lambda, \mu, \nu}(1, \beta)$ so $f \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ if and only if

$$\sum_{i=2}^m \frac{\gamma\beta(1+i-i^2)+i-\beta}{1-\beta(1-\gamma)} a_i C_1^{\lambda, \mu}(i) + \sum_{k=m+1}^{\infty} \frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)} a_k C_1^{\lambda, \mu}(k) < 1$$

or

$$\sum_{k=m+1}^{\infty} \frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)} a_k C_1^{\lambda, \mu}(k) < 1 - \sum_{i=2}^m c_i$$

and the proof is complete. $\square$

Corollary 3 If $f(z)$ defined by (5.1) be in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$, then for $k \geq m+1$, we have

$$a_k \leq \frac{(1-\beta(1-\gamma))(1-\sum_{i=2}^m c_i)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda, \mu}(k)}.$$

This result is sharp due to the function $f(z)$ defined by

$$f(z) = z - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda, \mu}(i)} z^i - \frac{(1-\beta(1-\gamma))(1-\sum_{i=2}^m c_i)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda, \mu}(k)} z^k.$$

Theorem 8 Let

$$f_j(z) = z - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{(\gamma\beta(1+i-i^2)+i-\beta)C_1^{\lambda, \mu}(i)} z^i - \sum_{k=m+1}^{\infty} a_{k,j} z^k \quad (5.3)$$

for $j = 1, 2, \dots, \iota$ be in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$. Then the function $F(z) = \sum_{j=1}^{\iota} \eta_j f_j(z)$ is also in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ where $\sum_{j=1}^{\iota} \eta_j = 1, 0 \leq \sum_{i=2}^m c_i \leq 1, 0 \leq c_i \leq 1$.

Proof. By Theorem 7 for every $j = 1, 2, \dots, \iota$, we have

$$\sum_{k=m+1}^{\infty} \frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)} a_{k,j} C_1^{\lambda, \mu}(k) < 1 - \sum_{i=2}^m c_i.$$

But

$$F(z) = \sum_{j=1}^{\iota} \eta_j f_j(z) = z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{(\gamma\beta(1 + i - i^2) + i - \beta)C_1^{\lambda,\mu}(i)} z^i \\ - \sum_{k=m+1}^{\infty} \left(\sum_{j=1}^{\iota} \eta_j a_{k,j} \right) z^k.$$

So

$$\sum_{k=m+1}^{\infty} \frac{\gamma\beta(1 + k - k^2) + k - \beta}{1 - \beta(1 - \gamma)} C_1^{\lambda,\mu}(k) \left(\sum_{j=1}^{\iota} \eta_j a_{k,j} \right) \\ = \sum_{j=1}^{\iota} \sum_{k=m+1}^{\infty} \left(\frac{\gamma\beta(1 + k - k^2) + k - \beta}{1 - \beta(1 - \gamma)} C_1^{\lambda,\mu}(k) a_{k,j} \right) \eta_j \\ < \sum_{j=1}^{\iota} \left(1 - \sum_{i=2}^m c_i \right) \eta_j = 1 - \sum_{i=2}^m c_i$$

and the proof is complete. $\square$

Remark 3 If $f_1(z), f_2(z)$ be in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$, then the function $H(z) = \frac{1}{2}(f_1(z) + f_2(z))$ is also in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

Remark 4 The class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ is a convex set.

In the next theorem, we will prove the arithmetic mean property.

Theorem 9 Let $f_j(z), (j = 1, 2, \dots, \iota)$ defined by (5.3) be in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$. Then the function

$$V(z) = z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{(\gamma\beta(1 + i - i^2) + i - \beta)C_1^{\lambda,\mu}(i)} z^i - \sum_{k=m+1}^{\infty} b_k z^k, (b_k \geq 0)$$

is also in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$, where $b_k = \frac{1}{\iota} \sum_{j=1}^{\iota} a_{k,j}$.

Proof. We have

$$\sum_{k=m+1}^{\infty} \frac{\gamma\beta(1 + k - k^2) + k - \beta}{1 - \beta(1 - \gamma)} b_k C_1^{\lambda,\mu}(k) \\ = \sum_{k=m+1}^{\infty} \frac{\gamma\beta(1 + k - k^2) + k - \beta}{\iota(1 - \beta(1 - \gamma))} \left(\sum_{j=1}^{\iota} a_{k,j} \right) C_1^{\lambda,\mu}(k) \\ = \frac{1}{\iota} \sum_{j=1}^{\iota} \left[\sum_{k=m+1}^{\infty} \frac{\gamma\beta(1 + k - k^2) + k - \beta}{1 - \beta(1 - \gamma)} a_{k,j} C_1^{\lambda,\mu}(k) \right]$$

(by Theorem 7))

$$< \frac{1}{\ell} \sum_{j=1}^{\ell} \left(1 - \sum_{i=2}^m c_i \right) = 1 - \sum_{i=2}^m c_i,$$

and the proof is complete. $\square$

Definition 3 Let $f(z)$ and $g(z)$ belong to $A(n)$. Then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by

$$h_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)].$$

Theorem 10 If $f(z)$ and $g(z)$ be in the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$, then the weighted mean of $f(z)$ and $g(z)$ is also in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

Proof. By using Definition 3, we obtain

$$\begin{aligned} h_j(z) &= \frac{1}{2} \left[(1-j) \left(z - \sum_{i=2}^m \frac{(1-(1-\gamma))c_i}{[\gamma\beta(1+i-i^2) + i - \beta]C_1^{\lambda, \mu}(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k \right) \right. \\ &\quad \left. + (1+j) \left(z - \sum_{i=2}^m \frac{1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2) + i - \beta]C_1^{\lambda, \mu}(i)} z^i - \sum_{k=m+1}^{\infty} b_k z^k \right) \right] \\ &= z - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2) + i - \beta]C_1^{\lambda, \mu}(i)} z^i \\ &\quad - \sum_{k=m+1}^{\infty} \frac{1}{2} [(1-j)a_k + (1+j)b_k] z^k. \end{aligned}$$

Since $f(z)$ and $g(z)$ are in the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ using Theorem 8 we have

$$\begin{aligned} &\sum_{k=m+1}^{\infty} \frac{[\gamma\beta(1+k-k^2) + k - \beta]C_1^{\lambda, \mu}(k)}{(1-\beta(1-\gamma))} \frac{1}{2} [(1-j)a_k + (1+j)b_k] \\ &= \frac{1}{2} \sum_{k=m+1}^{\infty} \frac{[\gamma\beta(1+k-k^2) + k - \beta]}{(1-\beta(1-\gamma))} (1-j)C_1^{\lambda, \mu}(k)a_k \\ &\quad + \frac{1}{2} \sum_{k=m+1}^{\infty} \frac{[\gamma\beta(1+k-k^2) + k - \beta]}{(1-\beta(1-\gamma))} (1+j)C_1^{\lambda, \mu}(k)b_k \\ &\leq \frac{1}{2}(1-j) \left(1 - \sum_{i=2}^m c_i \right) + \frac{1}{2}(1+j) \left(1 - \sum_{i=2}^m c_i \right) = 1 - \sum_{i=2}^m c_i, \end{aligned}$$

and again by Theorem 8, $h_j(z) \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$. $\square$

Now we will find the extreme points of the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ but we need the following theorem to prove.

Theorem 11 *Let*

$$f_m(z) = z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{[\gamma\beta(1 + i - i^2) + i - \beta]C_1^{\lambda,\mu}(i)} z^i \quad (5.4)$$

and for $k \geq m + 1$

$$f_k(z) = z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{[\gamma\beta(1 + i - i^2) + k - \beta]C_1^{\lambda,\mu}(i)} z^i - \frac{(1 - \beta(1 - \gamma))(1 - \sum_{i=2}^m c_i)}{[\gamma\beta(1 + k - k^2) + k - \beta]C_1^{\lambda,\mu}(k)} z^k. \quad (5.5)$$

Then the function $X(z)$ is in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ if and only if it can be expressed in the form

$$X(z) = \sum_{k=m}^{\infty} \sigma_k f_k(z)$$

where $\sigma_k \geq 0$ ($k \geq m$) and $\sum_{k=m}^{\infty} \sigma_k = 1$.

Proof. Let $X(z) = \sum_{k=m}^{\infty} \sigma_k f_k(z)$. Then

$$\begin{aligned} X(z) &= \sigma_m f_m(z) + \sum_{k=m+1}^{\infty} \sigma_k f_k(z) = \sigma_m z - \sigma_m \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{[\gamma\beta(1 + i - i^2) + i - \beta]C_1^{\lambda,\mu}(i)} z^i \\ &\quad + \sum_{k=m+1}^{\infty} \sigma_k z - \sum_{k=m+1}^{\infty} \sigma_k \left(\sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{[\gamma\beta(1 + i - i^2) + i - \beta]C_1^{\lambda,\mu}(i)} z^i \right) \\ &\quad - \sum_{k=m+1}^{\infty} \sigma_k \left(\frac{(1 - \beta(1 - \gamma))(1 - \sum_{i=2}^m c_i)}{[\gamma\beta(1 + k - k^2) + k - \beta]C_1^{\lambda,\mu}(k)} z^k \right) \\ &= \left(\sigma_m + \sum_{k=m+1}^{\infty} \sigma_k \right) z - \left(\sigma_m + \sum_{k=m+1}^{\infty} \sigma_k \right) \\ &\quad \times \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{[\gamma(1 + i - i^2) + i - \beta]C_1^{\lambda,\mu}(i)} z^i \\ &\quad - \sum_{k=m+1}^{\infty} \frac{(1 - \sum_{i=2}^m c_i)(1 - \beta(1 - \gamma))}{[\gamma\beta(1 + k - k^2) + k - \beta]C_1^{\lambda,\mu}(k)} \sigma_k z^k \\ &= z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))c_i}{[\gamma\beta(1 + i - i^2) + i - \beta]C_1^{\lambda,\mu}(i)} z^i \\ &\quad - \sum_{k=m+1}^{\infty} \frac{(1 - \sum_{i=2}^m c_i)(1 - \beta(1 - \gamma))\sigma_k}{[\gamma\beta(1 + k - k^2) + k - \beta]C_1^{\lambda,\mu}(k)} z^k. \end{aligned}$$

Finally we have

$$\begin{aligned} & \sum_{k=m+1}^{\infty} \frac{[\gamma\beta(1+k-k^2)+k-\beta](1-\sum_{i=2}^m c_i)(1-\beta(1-\gamma))\sigma_k}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)(1-\beta(1-\gamma))} C_1^{\lambda,\mu}(k) \\ &= \left(1 - \sum_{i=2}^m c_i\right) \sum_{k=m+1}^{\infty} \sigma_k = \left(1 - \sum_{i=2}^m c_i\right) (1 - \sigma_m) < 1 - \sum_{i=2}^m c_i. \end{aligned}$$

Thus $X(z) \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

Conversely, assume $X(z) \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ so

$$X(z) = z - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k.$$

By putting

$$\sigma_k = \frac{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)}{(1-\beta(1-\gamma))(1-\sum_{i=1}^m c_i)} a_k, \quad (k \geq m+1)$$

we have $\sigma_k \geq 0$ and if we set $\sigma_m = 1 - \sum_{k=m+1}^{\infty} \sigma_k$, we get

$$\begin{aligned} X(z) &= z - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} z^i - \sum_{k=m+1}^{\infty} \frac{(1-\beta(1-\gamma))(1-\sum_{i=1}^m c_i)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} \sigma_k z^k \\ &= f_m(z) - \sum_{k=m+1}^{\infty} \left(z - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} z^i - f_k(z) \right) \sigma_k \\ &= f_m(z) - \sum_{k=m+1}^{\infty} (f_m(z) - f_k(z)) \sigma_k \\ &= \left(1 - \sum_{k=m+1}^{\infty} \sigma_k\right) f_m(z) + \sum_{k=m+1}^{\infty} \sigma_k f_k(z) = \sum_{k=m}^{\infty} \sigma_k f_k(z). \quad \square \end{aligned}$$

Corollary 4 The extreme points of the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ are the functions $f_k(z)$ ($k \geq m$) defined by (5.4), (5.5).

Now, we obtain the radii of starlikeness and convexity for the elements of the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

Theorem 12 Let the function $f(z)$ defined by (5.1) be in the class $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$. Then $f(z)$ is starlike of order η ($0 \leq \eta < 1$) in $|z| < \gamma$, where γ is the largest value such that

$$\begin{aligned} & \sum_{i=2}^m \frac{c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} r^{i-1} \\ &+ \frac{1 - \sum_{i=2}^m c_i}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} r^{k-1} < \frac{1}{1-\beta(1-\gamma)}. \end{aligned}$$

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \eta. \quad (5.6)$$

Thus, we have

$$\begin{aligned} & \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \\ & \frac{\sum_{i=2}^m \frac{(i-1)(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} |z|^{i-1} + \sum_{k=m+1}^{\infty} (k-1)a_k |z|^{k-1}}{1 - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} |z|^{i-1} - \sum_{k=m+1}^{\infty} a_k |z|^{k-1}} \\ & < \frac{\sum_{i=2}^m \frac{(i-1)(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} r^{i-1} + \sum_{k=m+1}^{\infty} \frac{(k-1)(1-\beta(1-\gamma))(1-\sum_{i=2}^m c_i)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} r^{k-1}}{1 - \sum_{i=2}^m \frac{(1-\beta(1-\gamma))c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} r^{i-1} - \sum_{k=m+1}^{\infty} \frac{(1-\beta(1-\gamma))(1-\sum_{i=2}^m c_i)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} r^{k-1}}. \end{aligned}$$

Therefore (5.6) holds true if the last term of above relationship is less than $1 - \eta$ or equivalently

$$\begin{aligned} & \sum_{i=2}^m \frac{(i-\eta)(1-\beta(1-\gamma))c_i}{(1-\eta)[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} r^{i-1} \\ & + \sum_{k=m+1}^{\infty} \frac{(k-\eta)(1-\beta(1-\gamma))(1-\sum_{i=2}^m c_i)}{(1-\eta)[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} r^{k-1} < 1. \end{aligned}$$

Finally we find

$$\begin{aligned} & \sum_{i=2}^m \frac{c_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} r^{i-1} \\ & + \frac{1 - \sum_{i=2}^m c_i}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} r^{k-1} < \frac{1}{1-\beta(1-\gamma)} \end{aligned}$$

and this completes the proof. $\square$

Making use of (1.4), we obtain the following corollary.

Corollary 5 Let $f \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$. Then f is convex of order η ($0 \leq \eta < 1$) in $|z| < r$ where r is the largest value for which

$$\begin{aligned} & \sum_{i=2}^m \frac{ic_i}{[\gamma\beta(1+i-i^2)+i-\beta]C_1^{\lambda,\mu}(i)} r^{i-1} \\ & + \frac{k(1-\sum_{i=2}^m c_i)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} r^{k-1} < \frac{1}{1-\beta(1-\gamma)}. \end{aligned}$$

Theorem 13 Let $f \in A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$ and

$$d_i = \frac{(1 - \beta(1 - \gamma))c_i^2}{|\gamma\beta(1 + i - i^2) + i - \beta|C_1^{\lambda, \mu}(i)}, \quad (2 \leq i \leq m).$$

Then the function

$$h(z) = z - \sum_{i=2}^m \frac{(1 - \beta(1 - \gamma))d_i}{(\gamma\beta(1 + i - i^2) + i - \beta)C_1^{\lambda, \mu}(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k$$

is also in $A_{\gamma, c_m}^{\lambda, \mu, \nu}(1, \beta)$.

Proof. It can be verified that $|\gamma\beta(1 + i - i^2) + i - \beta|C_1^{\lambda, \mu}(i) > 1$, $i = 2, 3, \dots, m$. Therefore

$$0 \leq d_i = \frac{(1 - \beta(1 - \gamma))c_i^2}{|\gamma\beta(1 + i - i^2) + i - \beta|C_1^{\lambda, \mu}(i)} < c_i \leq 1.$$

So

$$0 \leq \sum_{i=2}^m d_i < \sum_{i=2}^m c_i \leq 1.$$

Thus

$$\begin{aligned} & \sum_{k=m+1}^{\infty} \frac{[\gamma\beta(1 + k - k^2) + k - \beta]}{(1 - \beta(1 - \gamma))(1 - \sum_{i=2}^m d_i)} C_1^{\lambda, \mu}(k) a_k \\ & < \sum_{k=m+1}^{\infty} \frac{[\gamma\beta(1 + k - k^2) + k - \beta]}{(1 - \beta(1 - \gamma))(1 - \sum_{i=2}^m c_i)} C_1^{\lambda, \mu}(k) a_k < 1 \end{aligned}$$

and this completes the proof. $\square$

References

- [1] E.S. Aqlan, *Some Problems Connected with Geometric Function Theory*, Ph.D. Thesis, Pune University, Pune, 2004.
- [2] W. G. Atshan, *Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II*, Surveys in Mathematics and its Applications, vol. 3, 2008, 67-77.
- [3] W. G. Atshan, *Application of fractional calculus operators for a new class of univalent functions with negative coefficients defined by Hohlov operator*, Mathematica Slovaca, vol.60, no.1, 2010, 75-82.
- [4] W. G. Atshan, R. H. Buti, *Fractional calculus of a class of univalent functions with negative coefficients defined by Hadamard product with Rafid-Operator*, European Journal of Pure and Applied Mathematics, vol.4, no.2, 2011, 162-173.

- [5] W. G. Atshan , S. R. Kulkarni, *Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative I*, J. Rajasthan Acad. Phy. Sci., vol.6, no.2, 2007, 129-140.
- [6] W. G. Atshan , S. R. Kulkarni, *Application of fractional derivatives and hypergeometric functions to multivalent analytic functions*, Far East Journal of Mathematical Sciences (FJMS), vol. 27, no. 2, 2007, 277-294.
- [7] W. G. Atshan, S. R. Kulkarni, *Neighborhoods and partial sums of subclass of k -uniformly convex functions and related class of k -starlike functions with negative coefficients based on integral operator*, Southeast Asian Bulletin of Mathematics, vol. 33, 2009, 623-637.
- [8] W. G. Atshan , S. R. Kulkarni, *On application of differential subordination for certain subclass of meromorphically p -valent functions with positive coefficients defined by linear operator*, Journal of Inequalities in Pure and Applied Mathematics, vol. 10, no. 2, 2009, Article 53, 11 pp.
- [9] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer - Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
- [10] S. Kanas, A. Wisniowska, *Conic regions and k -uniformly convexity II*, Folia Sci. Tech. Reso., 178, 1998, 65-78.
- [11] Y. Komato, *On analytic prolongation of a family of operators*, Mathematica (Cluj), vol. 39, no. 55, 1990, 141-145.
- [12] M. Nunokawa, *A sufficient condition for univalence and starlikeness*, Proc. Japan Acad. Ser. A Math. Sci., vol. 65, 1989, 163-164.
- [13] S. Owa, M. Nunokawa, H. M. Srivastava, *A certain class of multivalent functions*, Appl. Math. Lett., vol. 10, no. 2, 1997, 7-10.
- [14] T. Rosy, K. G. Subramanian, G. Murugusundaramoorthy, *Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives*, J. Inequal. Pure and Appl. Math., vol. 4, no. 4, 2003, 1-8.
- [15] S. Shams, S. R. Kulkarni, *A class of univalent functions with negative and fixed finitely many coefficients*, Acta Cienica Indica, XXXIXM, vol. 3, 2003, 587-594.
- [16] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., vol. 51, 1975, 109-116.
- [17] H. M. Srivastava, *Distortion Inequalities for analytic and univalent functions associated with certain fractional calculus and other linear operators*, (In Analytic and Geometric Inequalities and Applications eds. T. M. Rassias and H. M. Srivastava), Kluwar Academic Publishers, vol. 478, 1999, 349-374.

- [18] H. M. Srivastava, R. K. Saxena, *Operators of fractional integration and their applications*, Applied Mathematics and Computation, vol. 118, 2001, 1-52.
- [19] A. Tehranchi, S. R. Kulkarni, *Study of the class of univalent functions with negative coefficients defined by Ruscheweyh derivatives (II)*, J. Rajasthan Academy of Physical Sciences, vol. 5, no. 1, 2006, 105-118.

Waggas Galib Atshan

University of Al-Qadisiyah

College of Science

Department of Mathematics

Diwaniya, Iraq

e-mail:waggashnd@gmail.com, waggas.galib@qu.edu.iq

S. R. Kulkarni

University of Pune

Fergusson College

Department of Mathematics

Pune - 411004, India

e-mail:kulkarni_ferg@yahoo.com