Certain subclasses of univalent and bi-univalent functions related to shell-like curves connected with Fibonacci numbers ¹

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Abstract

This paper is concerned with certain subclasses of univalent and bi-univalent functions related to shell-like curves connected with Fibonacci numbers. We find estimates of the initial coefficients $|a_2|$ and $|a_3|$ for the functions in these classes. Also we investigate upper bounds for the Fekete-Szegő functional and second Hankel determinant for these classes.

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1 Introduction

Let A be the class of functions of the form

$$(1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. By S , we denote the class of functions $f(z) \in A$ and univalent in U .

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Let us denote by B , the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the open unit disc U and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in U.$$

Consider two functions f and g analytic in U . We say that f is subordinate to g (symbolically $f \prec g$) if there exists a Schwarz function $u(z) \in B$ for which $f(z) = g(u(z))$. This result is known as principle of subordination.

By S^* , we denote the class of starlike functions $f \in S$ which satisfies the following condition:

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \text{ or } \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, (z \in U).$$

By K , we denote the class of convex functions $f \in S$ which satisfies the following condition:

$$\operatorname{Re} \left(\frac{(zf'(z))'}{f'(z)} \right) > 0 \text{ or } \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, (z \in U).$$

A function $f \in S$ is said to be α -convex if it satisfies the inequality

$$\operatorname{Re} \left((1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

The class of α -convex functions is denoted by $M(\alpha)$ and was introduced by Mocanu [13]. In particular $M(0) \equiv S^*$ and $M(1) \equiv K$.

For $f \in A$, Sălăgean [17] introduced the following operator:

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z),$$

and in general,

$$D^n f(z) = D(D^{n-1} f(z)), n \in N$$

or equivalent to

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in N_0 = N \cup \{0\}.$$

The inverse functions of the functions in the class S may not be defined on the entire unit disc U although the functions in the class S are invertible. However using Koebe-one quarter theorem [4] it is obvious that the image of U under every function $f \in S$ contains a disc of radius $\frac{1}{4}$. Hence every univalent function f has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z (z \in U)$$

and

$$f(f^{-1}(w)) = w \left(|w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right)$$

where

$$(2) \quad g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U . By Σ , we denote the class of bi-univalent functions in U defined by (1).

Lewin [10] discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [1] investigated certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients.

In 1976, Noonan and Thomas [14] stated the q th Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q+1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q+1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [15] determined the rate of growth of $H_q(n)$ as $n \rightarrow \infty$ for functions given by Eq. (1) with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials and the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman [9]. Also Hankel determinant was studied by various authors including Hayman [8], Mehrotra and Singh [12], Singh and Singh [19] and Singh [18].

Easily, one can observe that the Fekete-Szegő functional is $H_2(1)$. Fekete and Szegő [6] then further generalised the estimate of $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

Sokół [20] introduced the class SL of shell-like functions $f \in A$ defined as below:

Definition 1 A function $f \in A$ given by (1) is said to be in the class SL of starlike shell-like functions if it satisfies the following condition:

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

where $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

It should be observed that SL is a subclass of the class S^* of starlike functions.

Later Dziok et al. [2] introduced the class KSL of convex functions related to a shell-like curve as under:

Definition 2 A function $f \in A$ given by (1) is said to be in the class KSL of convex shell-like functions if it satisfies the condition that

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

where $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Again Dziok et al. [3] defined the following class of α -convex shell-like functions:

Definition 3 A function $f \in A$ given by (1) is said to be in the class SLM_α of α -convex shell-like functions if it satisfies the condition that

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

where $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Obviously $SLM_0 \equiv SL$ and $SLM_1 \equiv KSL$.

Motivated by the above defined classes, we introduce the following subclasses of univalent functions related to shell-like curves connected with Fibonacci numbers as below:

Definition 4 A function $f \in A$ given by (1) is said to be in the class SLM^α if it satisfies the condition that

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

where $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Obviously SLM^α is a subclass of the class of α -convex functions. Also $SLM^0 \equiv SL$ and $SLM^1 \equiv KSL$.

Definition 5 A function $f \in A$ given by (1) is said to be in the class $SLM_\alpha(n)$ if it satisfies the condition that

$$(1 - \alpha) \frac{D^{n+1}f(z)}{D^n f(z)} + \alpha \frac{(D^{n+1}f(z))'}{(D^n f(z))'} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

where $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

We have the following observations:

- (i) $SLM_\alpha(0) \equiv SLM_\alpha$.
- (ii) $SLM_0(0) \equiv SL$.
- (iii) $SLM_1(0) \equiv KSL$.

The function $\tilde{p}$ is not univalent in U , but it is univalent in the disc $|z| < \frac{3 - \sqrt{5}}{2} \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}\left(\frac{-1}{2\tau}\right) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \frac{\sqrt{5}}{5}$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|},$$

which shows that the number $|\tau|$ divides $[0, 1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = \frac{3 - \sqrt{5}}{2} \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n :

$$\tau^n = u_n \tau + u_{n-1}.$$

Also the subclasses of bi-univalent functions related to shell-like curves were studied by various authors [11, 21, 22].

The earlier work on bi-univalent functions related to shell-like curves connected with Fibonacci numbers motivate us to define the following subclasses:

To avoid repetition, throughout the paper we assume that $0 \leq \alpha \leq 1$, $\tau = \frac{1 - \sqrt{5}}{2} \approx -0.618$ and $z \in U$.

Definition 6 A function $f \in \Sigma$ given by (1) is said to be in the class $SLM^{(\alpha, \Sigma)}(\tilde{p}(z))$ if it satisfies the following conditions:

$$(3) \quad \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

and

$$(4) \quad \frac{wg'(w) + \alpha w^2 g''(w)}{(1 - \alpha)g(w) + \alpha w g'(w)} \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}$$

where $g = f^{-1}$ and $z, w \in U$.

In particular

(i) $SLM^{(0, \Sigma)}(\tilde{p}(z)) \equiv SL_{\Sigma}(\tilde{p}(z))$, the class of bi-univalent starlike shell-like functions studied by Güney et al.[7].

(ii) $SLM^{(1, \Sigma)}(\tilde{p}(z)) \equiv KSL_{\Sigma}(\tilde{p}(z))$, the class of bi-univalent convex shell-like functions studied by Güney et al.[7].

Definition 7 A function $f \in \Sigma$ given by (1) is said to be in the class $SLM_{\alpha, \Sigma}(n, \tilde{p}(z))$ if it satisfies the following conditions:

$$(5) \quad (1 - \alpha) \frac{D^{n+1}f(z)}{D^n f(z)} + \alpha \frac{(D^{n+1}f(z))'}{(D^n f(z))'} \prec \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

and

$$(6) \quad (1 - \alpha) \frac{D^{n+1}h(w)}{D^n h(w)} + \alpha \frac{(D^{n+1}h(w))'}{(D^n h(w))'} \prec \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}$$

where $h = f^{-1}$ and $z, w \in U$.

The following observations are obvious:

- (i) $SLM_{\alpha, \Sigma}(0, \tilde{p}(z)) \equiv SLM_{\alpha, \Sigma}(\tilde{p}(z))$, the class of bi-univalent α -convex shell-like functions studied by Güney et al.[7].
- (ii) $SLM_{0, \Sigma}(0, \tilde{p}(z)) \equiv SL_{\Sigma}(\tilde{p}(z))$.
- (iii) $SLM_{1, \Sigma}(0, \tilde{p}(z)) \equiv KSL_{\Sigma}(\tilde{p}(z))$.

In this paper we introduce certain subclasses of univalent and bi-univalent functions related to shell-like curves connected with Fibonacci numbers. We find estimates of the initial coefficients $|a_2|$ and $|a_3|$ and upper bounds for the Fekete-Szegő functional and second Hankel determinant for the functions in these classes.

2 Coefficient bounds for the function classes $SLM^{(\alpha, \Sigma)}(\tilde{p}(z))$ and $SLM_{\alpha, \Sigma}(n, \tilde{p}(z))$

For deriving our results, we need the following lemma:

Lemma 1 [16] If $p \in P$ be family of all functions p analytic in U for which $Re[p(z)] > 0$ and have the form $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ for $z \in U$, then $|p_n| \leq 2$ for each n .

Theorem 1 If $f \in SLM^{(\alpha, \Sigma)}(\tilde{p}(z))$, then

$$(7) \quad |a_2| \leq \frac{|\tau|}{\sqrt{(1 + \alpha)^2 - (2 + 4\alpha + 4\alpha^2)\tau}}$$

and

$$(8) \quad |a_3| \leq \frac{|\tau|[(1 + \alpha)^2 - (4 + 8\alpha + 4\alpha^2)\tau]}{2(1 + 2\alpha)[(1 + \alpha)^2 - (2 + 4\alpha + 4\alpha^2)\tau]}.$$

Proof. As $f \in SLM^{(\alpha, \Sigma)}(\tilde{p}(z))$, so by Definition 6 and using the concept of subordination, there exists Schwarz functions $r(z)$ and $s(z)$ such that

$$(9) \quad \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} = \tilde{p}(r(z))$$

and

$$(10) \quad \frac{wg'(w) + \alpha w^2 g''(w)}{(1 - \alpha)g(w) + \alpha wg'(w)} = \tilde{p}(s(w))$$

where $r(z) = 1 + r_1 z + r_2 z^2 + \dots$ and $s(w) = 1 + s_1 w + s_2 w^2 + \dots$

On expanding, it yields

$$(11) \quad \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} = 1 + (1 + \alpha)a_2 z + (2(1 + 2\alpha)a_3 - (1 + \alpha)^2 a_2^2)z^2 + \dots$$

and

$$(12) \quad \frac{wg'(w) + \alpha w^2 g''(w)}{(1 - \alpha)g(w) + \alpha wg'(w)} = 1 - (1 + \alpha)a_2 w + [-2(1 + 2\alpha)a_3 + (3(1 + 2\alpha) - \alpha^2)a_2^2]w^2 + \dots$$

Again

$$(13) \quad \tilde{p}(r(z)) = 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left[\frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{c_1^2}{4} \tilde{p}_2 \right] z^2 + \dots$$

and

$$(14) \quad \tilde{p}(s(z)) = 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left[\frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) + \frac{d_1^2}{4} \tilde{p}_2 \right] w^2 + \dots$$

Using (11) and (13) in (9) and equating the coefficients of z and z^2 , we get

$$(15) \quad (1 + \alpha)a_2 = \frac{c_1 \tau}{2}$$

and

$$(16) \quad 2(1 + 2\alpha)a_3 - (1 + \alpha)^2 a_2^2 = \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) \tau + \frac{c_1^2}{4} 3\tau^2.$$

Again using (12) and (14) in (10) and equating the coefficients of w and w^2 , we get

$$(17) \quad -(1 + \alpha)a_2 = \frac{d_1 \tau}{2}$$

and

$$(18) \quad -2(1 + 2\alpha)a_3 + [3(1 + 2\alpha) - \alpha^2]a_2^2 = \frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) \tau + \frac{d_1^2}{4} 3\tau^2.$$

From (15) and (16), it is clear that

$$(19) \quad c_1 = -d_1$$

and

$$(20) \quad 2a_2^2 = \frac{(c_1^2 + d_1^2)\tau^2}{4(1 + \alpha)^2}.$$

Adding (16) and (18), it yields

$$(21) \quad 2(1 + 2\alpha - \alpha^2)a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2.$$

Putting (20) in (21), we get:

$$(22) \quad 2[(1 + \alpha)^2 + \tau(-2 - 4\alpha - 4\alpha^2)]a_2^2 = \frac{1}{2}(c_2 + d_2)\tau^2.$$

Using Lemma 1 and on applying triangle inequality in (22), (7) can be easily obtained.

Now subtracting (18) from (16), we obtain

$$(23) \quad 4(1 + 2\alpha)a_3 - (8\alpha + 4)a_2^2 = \frac{1}{2}(c_2 - d_2)\tau.$$

Applying triangle inequality and using Lemma 1 and (22) in (23), it yields

$$(24) \quad 4(1 + 2\alpha)|a_3| \leq 2|\tau| + (8\alpha + 4)|a_2|^2.$$

From (24), result (8) is obvious.

For $\alpha = 0$, Theorem 1 agrees with the following result proved by Güney et al.[7](Corollary 1):

Corollary 1 *If $f(z) \in SL_\Sigma(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{1 - 2\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1 - 4\tau)}{2(1 - 2\tau)}.$$

For $\alpha = 1$, Theorem 1 agrees with the following result proved by Güney et al.[7](Corollary 2):

Corollary 2 *If $f(z) \in KSL_\Sigma(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4 - 10\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1 - 4\tau)}{3(2 - 5\tau)}.$$

Theorem 2 If $f \in SLM_{\alpha, \Sigma}(n, \tilde{p}(z))$, then

$$(25) \quad |a_2| \leq \frac{|\tau|}{\sqrt{4^n(1+\alpha)^2 + [2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n]\tau}}$$

and

$$(26) \quad |a_3| \leq \frac{|\tau|4^n[(1+\alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1+2\alpha)3^n[4^n(1+\alpha)^2 + (2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau]}.$$

Proof. As $f \in SLM_{\alpha, \Sigma}(n, \tilde{p}(z))$, so by Definition 7 and using the concept of subordination, there exists Schwarz functions $r(z)$ and $s(z)$ such that

$$(27) \quad (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} = \tilde{p}(r(z))$$

and

$$(28) \quad (1-\alpha)\frac{D^{n+1}h(w)}{D^n h(w)} + \alpha\frac{(D^{n+1}h(w))'}{(D^n h(w))'} = \tilde{p}(s(w))$$

where $r(z) = 1 + r_1 z + r_2 z^2 + \dots$ and $s(w) = 1 + s_1 w + s_2 w^2 + \dots$

On expanding, it yields

$$(29) \quad (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} = 1 + (1+\alpha)2^n a_2 z + (2(1+2\alpha)3^n a_3 - (1+3\alpha)4^n a_2^2)z^2 + \dots$$

and

$$(30) \quad (1-\alpha)\frac{D^{n+1}h(w)}{D^n h(w)} + \alpha\frac{(D^{n+1}h(w))'}{(D^n h(w))'} = 1 - (1+\alpha)2^n a_2 w + [-2(1+2\alpha)3^n a_3 + (4(1+2\alpha)3^n - (1+3\alpha)4^n)a_2^2]w^2 + \dots$$

Again

$$(31) \quad \tilde{p}(r(z)) = 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left[\frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{c_1^2}{4} \tilde{p}_2 \right] z^2 + \dots$$

and

$$(32) \quad \tilde{p}(s(z)) = 1 + \frac{\tilde{p}_1 d_1 w}{2} + \left[\frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) + \frac{d_1^2}{4} \tilde{p}_2 \right] w^2 + \dots$$

Using (29) and (31) in (27) and equating the coefficients of z and z^2 , we get

$$(33) \quad (1+\alpha)2^n a_2 = \frac{c_1 \tau}{2}$$

and

$$(34) \quad 2(1+2\alpha)3^n a_3 - (1+3\alpha)4^n a_2^2 = \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) \tau + \frac{c_1^2}{4} 3\tau^2.$$

Again using (30) and (32) in (28) and equating the coefficients of w and w^2 , we get

$$(35) \quad -(1+\alpha)2^n a_2 = \frac{d_1 \tau}{2}$$

and

$$(36) \quad -2(1+2\alpha)3^n a_3 + [4(1+2\alpha)3^n - (1+3\alpha)4^n] a_2^2 = \frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) \tau + \frac{d_1^2}{4} 3\tau^2.$$

From (33) and (35), it is clear that

$$(37) \quad c_1 = -d_1$$

and

$$(38) \quad 2a_2^2 = \frac{(c_1^2 + d_1^2)\tau^2}{4^{n+1}(1+\alpha)^2}.$$

Adding (34) and (36), it yields

$$(39) \quad [4(1+2\alpha)3^n - 2(1+3\alpha)4^n] a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2.$$

Putting (38) in (39), we get:

$$(40) \quad 2[(2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau + 4^n(1+\alpha)^2] a_2^2 = \frac{1}{2}(c_2 + d_2)\tau^2.$$

Using Lemma 1 and on applying triangle inequality in (40), (25) can be easily obtained.

Now subtracting (36) from (34), we obtain

$$(41) \quad 4(1+2\alpha)3^n a_3 - 4(2\alpha+1)3^n a_2^2 = \frac{1}{2}(c_2 - d_2)\tau.$$

Applying triangle inequality and using Lemma 1 and (40) in (41), it yields

$$(42) \quad 4(1+2\alpha)3^n |a_3| \leq 2|\tau| + 4(2\alpha+1)3^n |a_2|^2.$$

From (42), result (26) is obvious.

For $n = 0$, Theorem 2 gives the following result due to Güney et al.[7]:

Corollary 3 If $f(z) \in SLM_{\alpha, \Sigma}(\tilde{p}(z))$, then

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1+\alpha)^2 - (1+\alpha)(2+3\alpha)\tau}}$$

and

$$|a_3| \leq \frac{|\tau|[(1+\alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1+2\alpha)(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}.$$

Also for $n = 0, \alpha = 0$, Theorem 2 agrees with Corollary 1 and for $n = 0, \alpha = 1$, Theorem 2 agrees with Corollary 2.

3 Fekete-Szegő inequality for the function classes $SLM^{(\alpha, \Sigma)}(\tilde{p}(z))$ and $SLM_{\alpha, \Sigma}(n, \tilde{p}(z))$

Theorem 3 Let $f(z) \in SLM^{(\alpha, \Sigma)}(\tilde{p}(z))$, then

$$(43) \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & |\mu - 1| \leq \frac{[(1+\alpha)^2 + \tau(-2-4\alpha-4\alpha^2)]}{2(1+2\alpha)|\tau|}; \\ \frac{|1-\mu|\tau^2}{[(1+\alpha)^2 + \tau(-2-4\alpha-4\alpha^2)]}, & |\mu - 1| \geq \frac{[(1+\alpha)^2 + \tau(-2-4\alpha-4\alpha^2)]}{2(1+2\alpha)|\tau|}. \end{cases}$$

Proof. From (22) and (23), it yields

$$(44) \quad a_3 - \mu a_2^2 = (1-\mu) \frac{\tau^2(c_2 + d_2)}{4[(1+\alpha)^2 + \tau(-2-4\alpha-4\alpha^2)]} + \frac{\tau(c_2 - d_2)}{8(1+2\alpha)}.$$

(44) can be expressed as

$$(45) \quad a_3 - \mu a_2^2 = \left[h(\mu) + \frac{\tau}{8(1+2\alpha)} \right] c_2 + \left[h(\mu) - \frac{\tau}{8(1+2\alpha)} \right] d_2,$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4[(1+\alpha)^2 + \tau(-2-4\alpha-4\alpha^2)]}.$$

Taking modulus, we obtain

$$(46) \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{8(1+2\alpha)}; \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{8(1+2\alpha)}. \end{cases}$$

So result (43) can be easily obtained from (46).

For $\alpha = 0$, Theorem 3 agrees with the following result proved by Güney et al.[7] (Corollary 4):

Corollary 4 If $f(z) \in SL_{\Sigma}(\tilde{p}(z))$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2}, & |\mu - 1| \leq \frac{1-2\tau}{2|\tau|}; \\ \frac{|1-\mu|\tau^2}{1-2\tau}, & |\mu - 1| \geq \frac{1-2\tau}{2|\tau|}. \end{cases}$$

For $\alpha = 1$, Theorem 3 agrees with the following result proved by Güney et al.[7] (Corollary 5):

Corollary 5 If $f(z) \in KSL_{\Sigma}(\tilde{p}(z))$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{6}, |\mu - 1| \leq \frac{2 - 5\tau}{3|\tau|}; \\ \frac{|1 - \mu|\tau^2}{2(2 - 5\tau)}, |\mu - 1| \geq \frac{2 - 5\tau}{3|\tau|}. \end{cases}$$

Theorem 4 Let $f(z) \in SLM_{\alpha, \Sigma}(n, \tilde{p}(z))$, then

$$(47) \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)3^n}, |\mu - 1| \leq \frac{[(2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau + (1+\alpha)^2 4^n]}{2|\tau|(1+2\alpha)3^n}; \\ \frac{|1 - \mu|\tau^2}{[(2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau + (1+\alpha)^2 4^n]}, \\ |\mu - 1| \geq \frac{[(2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau + (1+\alpha)^2 4^n]}{2|\tau|(1+2\alpha)3^n}. \end{cases}$$

Proof. From (40) and (41), it yields

(48)

$$a_3 - \mu a_2^2 = (1 - \mu) \frac{\tau^2(c_2 + d_2)}{4[(2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau + (1+\alpha)^2 4^n]} + \frac{\tau(c_2 - d_2)}{8(1+2\alpha)3^n}.$$

(48) can be expressed as

$$a_3 - \mu a_2^2 = \left[h(\mu) + \frac{\tau}{8(1+2\alpha)3^n} \right] c_2 + \left[h(\mu) - \frac{\tau}{8(1+2\alpha)3^n} \right] d_2$$

where

$$h(\mu) = \frac{(1 - \mu)\tau^2}{4[(2(1+2\alpha)3^n - (3\alpha^2 + 9\alpha + 4)4^n)\tau + (1+\alpha)^2 4^n]}.$$

Taking modulus, we obtain

$$(49) \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)3^n}, 0 \leq |h(\mu)| \leq \frac{|\tau|}{8(1+2\alpha)3^n}; \\ 4|h(\mu)|, |h(\mu)| \geq \frac{|\tau|}{8(1+2\alpha)3^n}. \end{cases}$$

So (47) can be easily obtained from (49).

For $n = 0$, Theorem 4 gives the following result due to Güney et al.[7]:

Corollary 6 Let $f(z) \in SLM_{\alpha, \Sigma}(\tilde{p}(z))$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, |\mu - 1| \leq \frac{(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}{2|\tau|(1+2\alpha)}; \\ \frac{|1 - \mu|\tau^2}{(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}, |\mu - 1| \geq \frac{(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}{2|\tau|(1+2\alpha)}. \end{cases}$$

Also for $n = 0, \alpha = 0$, Theorem 4 agrees with Corollary 4 and for $n = 0, \alpha = 1$, Theorem 4 agrees with Corollary 5.

4 Second Hankel determinant for the function classes $SLM^\alpha(\tilde{p}(z))$ and $SLM_\alpha(n, \tilde{p}(z))$

For deriving our results, we need the following lemmas:

Lemma 2 [21] *If $p(z) = 1 + p_1z + p_2z^2 + \dots$, and $p \prec \tilde{p}$, then*

$$|p_1| \leq |\tau|$$

and

$$|p_2| \leq 3\tau^2.$$

Lemma 3 [21] *If $p(z) = 1 + p_1z + p_2z^2 + \dots$, and $p \prec \tilde{p}$, then*

$$|p_3| \leq 4\tau^3.$$

Theorem 5 *Let $f(z) \in SLM^\alpha(\tilde{p}(z))$, then*

$$|a_2a_4 - a_3^2| \leq \frac{41\alpha^2 + 44\alpha + 11}{3(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\tau^4.$$

Proof. *As $f(z) \in SLM^\alpha(\tilde{p}(z))$, so by definition 4, we have*

$$(50) \quad \frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} = p(z) = 1 + p_1z + p_2z^2 + \dots$$

On expanding and equating the coefficients of z , z^2 and z^3 in (50), we get

$$a_2 = \frac{p_1}{1+\alpha},$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{p_1^2}{2(1+2\alpha)},$$

and

$$a_4 = \frac{p_3}{3(1+3\alpha)} + \frac{p_1p_2}{2(1+3\alpha)} + \frac{p_1^3}{6(1+3\alpha)}.$$

So, using Lemma 2 and Lemma 3, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{41\alpha^2 + 44\alpha + 11}{3(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\tau^4.$$

For $\alpha = 0$, Theorem 5 gives the following result due to Sokól et al.[21]:

Corollary 7 *If $f(z) \in SL$, then*

$$|a_2a_4 - a_3^2| \leq \frac{11}{3}\tau^4.$$

For $\alpha = 1$, Theorem 5 gives the following result due to Sokól et al.[21]:

Corollary 8 If $f(z) \in KSL$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}\tau^4.$$

Theorem 6 Let $f(z) \in SLM_\alpha(n, \tilde{p}(z))$, then

$$(51) \quad |a_2a_4 - a_3^2| \leq \frac{4(1+2\alpha)^2(16\alpha^2+19\alpha+5)}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)8^n}\tau^4 \\ + \frac{3(1+3\alpha)(3\alpha^2+3\alpha+2)^28^n+6(1+2\alpha)(1+5\alpha)(3\alpha^2+3\alpha+2)3^n}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)8^n}\tau^4.$$

Proof. As $f(z) \in SLM_\alpha(n, \tilde{p}(z))$, so by definition 5, we have

$$(52) \quad (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{(D^{n+1}f(z))'}{(D^n f(z))'} = p(z) = 1 + p_1z + p_2z^2 + \dots$$

On expanding and equating the coefficients of z , z^2 and z^3 in (52), we get

$$a_2 = \frac{p_1}{2^n(1+\alpha)},$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{(1+3\alpha)p_1^2}{2(1+2\alpha)(1+\alpha)^2},$$

and

$$a_4 = \frac{p_3}{3(1+3\alpha)4^n} + \frac{(1+5\alpha)3^n p_1 p_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)4^n} \\ + \frac{[(1+3\alpha)(1+5\alpha)3^{n+1} - 2(1+2\alpha)(1+7\alpha)]p_1^3}{6(1+\alpha)^3(1+2\alpha)(1+3\alpha)4^n}.$$

So, using Lemma 2 and Lemma 3, we can easily obtain result (51).

For $n = 0$, Theorem 6 gives the following result due to Sokól et al.[21]:

Corollary 9 If $f(z) \in SLM_\alpha(n)$, then

$$|a_2a_4 - a_3^2| \leq \frac{145\alpha^5 + 625\alpha^4 + 1061\alpha^3 + 867\alpha^2 + 330\alpha + 44}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}\tau^4.$$

Also for $n = 0, \alpha = 0$, Theorem 6 agrees with Corollary 7 and for $n = 0, \alpha = 1$, Theorem 6 agrees with Corollary 8.

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