Some Bullen-type inequalities for conformable fractional integrals ¹

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Abstract

In this study, the author has established a new lemma and Bullen-type inequalities for conformable fractional integrals. Also, it is given some applications involving Bullen type integral inequalities for differentiable functions to show the results.

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1 Introduction

The Hermite-Hadamard inequality: Let $\Lambda : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $u_1, u_2 \in J$ with $u_1 < u_2$.

$$(H) \quad \Lambda\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa \leq \frac{\Lambda(u_1) + \Lambda(u_2)}{2}$$

If Λ is concave, this double inequality hold in the inverse way. See [7, 11, 17, 19] for details.

The Bullen inequality:

$$(B) \quad \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa \leq \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda\left(\frac{u_1 + u_2}{2}\right) \right],$$

provided that $\Lambda : [u_1, u_2] \rightarrow \mathbb{R}$ is a convex function on $[u_1, u_2]$ (see for example [12, 13, 20, 27]) for details. In [14], s-convexity is defined by Hudzik and Maligranda

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as follows: a function $\Lambda : \mathbb{R}_+ \rightarrow \mathbb{R}$, where $\mathbb{R}_+ = [0, \infty)$, is said to be s -convex in the second sense if

$$\Lambda(\alpha x + (1 - \alpha)y) \leq \alpha^s \Lambda(x) + (1 - \alpha)^s \Lambda(y)$$

for all $x, y \in [0, \infty)$, $\alpha \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Lemma 1 [10] Let $\Lambda : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable function on J° , $u_1, u_2 \in J$, with $u_1 < u_2$. If $\Lambda' \in L[u_1, u_2]$ and $t \in [0, 1]$ then

$$\begin{aligned} (1) \quad & \frac{u_2 - u_1}{8} \left[\int_0^1 (1 - 2t) \Lambda' \left(\frac{1+t}{2} u_1 + \frac{1-t}{2} u_2 \right) dt \right. \\ & \left. + \int_0^1 (2t - 1) \Lambda' \left(\frac{1+t}{2} u_2 + \frac{1-t}{2} u_1 \right) dt \right] \\ & = \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda \left(\frac{u_1 + u_2}{2} \right) \right] - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa. \end{aligned}$$

Theorem 1 [10] Let $\Lambda : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable function on J° , $u_1, u_2 \in J$, and $\Lambda' \in L[u_1, u_2]$. If $|\Lambda'(\varkappa)|^q$ is s -convex function in the second sense on $[u_1, u_2]$ for some fixed $s \in (0, 1]$, $p, q > 1$, $1/p + 1/q = 1$, then

$$\begin{aligned} (2) \quad & \left| \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda \left(\frac{u_1 + u_2}{2} \right) \right] - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa \right| \\ & \leq \frac{u_2 - u_1}{8} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(\frac{2 - (\frac{1}{2})^s}{s+1} |\Lambda'(u_1)|^q + \frac{(\frac{1}{2})^s}{s+1} |\Lambda'(u_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(\frac{1}{2})^s}{s+1} |\Lambda'(u_1)|^q + \frac{2 - (\frac{1}{2})^s}{s+1} |\Lambda'(u_2)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

If taking $s = 1$ in Theorem 1, the following corollary derived.

Corollary 1 [10] Let $\Lambda : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable function on J° , $u_1, u_2 \in J$ and $\Lambda' \in L[u_1, u_2]$. If $|\Lambda'(\varkappa)|^q$ is convex function on $[u_1, u_2]$ and $1/p + 1/q = 1$, then

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda \left(\frac{u_1 + u_2}{2} \right) \right] - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa \right| \\ & \leq \frac{u_2 - u_1}{8(p+1)^{1/p} 4^{1/q}} \times \\ & \quad \left[(3|\Lambda'(u_1)|^q + |\Lambda'(u_2)|^q)^{\frac{1}{q}} + (|\Lambda'(u_1)|^q + 3|\Lambda'(u_2)|^q)^{\frac{1}{q}} \right]. \end{aligned}$$

Theorem 2 [10] Let $\Lambda : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable function on J° , $u_1, u_2 \in J$, and $\Lambda' \in L[u_1, u_2]$. If $|\Lambda'(\varkappa)|^q$ is s -convex function in the second sense on $[u_1, u_2]$ for $q \geq 1$, then

$$(3) \quad \left| \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda\left(\frac{u_1 + u_2}{2}\right) \right] - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa \right|$$

$$\leq \frac{u_2 - u_1}{8} \left(\frac{1}{2}\right)^{1-1/q} \times$$

$$\left\{ \left[\frac{(9\left(\frac{3}{2}\right)^s - 4 \times 2^s + 2 \times 2^s s - s - 4)}{2^s (s+1)(s+2)} |\Lambda'(u_1)|^q + \frac{s + \frac{1}{2^s}}{2^s (s+1)(s+2)} |\Lambda'(u_2)|^q \right]^{1/q} \right.$$

$$\left. + \left[\frac{(9\left(\frac{3}{2}\right)^s - 4 \times 2^s + 2 \times 2^s s - s - 4)}{2^s (s+1)(s+2)} |\Lambda'(u_2)|^q + \frac{s + \frac{1}{2^s}}{2^s (s+1)(s+2)} |\Lambda'(u_1)|^q \right]^{1/q} \right\}$$

where

$$\frac{1}{2^s} \int_0^1 |1 - 2t| (1+t)^s dt = \frac{(9\left(\frac{3}{2}\right)^s - 4 \times 2^s + 2 \times 2^s s - s - 4)}{2^s (s+1)(s+2)}$$

$$\frac{1}{2^s} \int_0^1 |1 - 2t| (1-t)^s dt = \frac{s + \frac{1}{2^s}}{2^s (s+1)(s+2)}$$

and

$$\int_0^1 |1 - 2t| dt = \frac{1}{2}.$$

If taking $s = 1$ in Theorem 2, we derive the following corollary.

Corollary 2 [10] Let $\Lambda : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable function on J° , $u_1, u_2 \in J$ and $\Lambda' \in L[u_1, u_2]$. If $|\Lambda'(\varkappa)|^q$ is convex function on $[u_1, u_2]$ and $q \geq 1$, then

$$(4) \quad \left| \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda\left(\frac{u_1 + u_2}{2}\right) \right] - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(\varkappa) d\varkappa \right|$$

$$\leq \frac{u_2 - u_1}{8} \left(\frac{1}{2}\right)^{1-1/q} \times$$

$$\left\{ \left[\frac{3}{8} |\Lambda'(u_1)|^q + \frac{1}{8} |\Lambda'(u_2)|^q dt \right]^{1/q} + \left[\frac{3}{8} |\Lambda'(u_2)|^q + \frac{1}{8} |\Lambda'(u_1)|^q dt \right]^{1/q} \right\}.$$

2 Definition and Properties of Conformable Fractional Derivative and Integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in [1]-[6] and [8, 9, 15, 16, 21, 22, 23, 24, 25].

Definition 1 (*Conformable fractional derivative*) Given a function $\Lambda : [0, \infty) \rightarrow \mathbb{R}$. Then the "conformable fractional derivative" of Λ of order α is defined by

$$D_\alpha (\Lambda) (t) = \lim_{\varepsilon \rightarrow 0} \frac{\Lambda (t + \varepsilon t^{1-\alpha}) - \Lambda (t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0, 1]$. If Λ is α -differentiable in some $(0, u_1)$, $\alpha > 0$, $\lim_{t \rightarrow 0^+} \Lambda^{(\alpha)} (t)$ exist, then define

$$\Lambda^{(\alpha)} (0) = \lim_{t \rightarrow 0^+} \Lambda^{(\alpha)} (t).$$

We can write $\Lambda^{(\alpha)} (t)$ for $D_\alpha (\Lambda) (t)$ to denote the conformable fractional derivatives of Λ of order α . In addition, if the conformable fractional derivative of Λ of order α exists, then we simply say Λ is α -differentiable.

Theorem 3 Let $\alpha \in (0, 1]$ and Λ, g be α -differentiable at a point $t > 0$. Then,

- 1) $D_\alpha (u_1 \Lambda + u_2 g) = u_1 D_\alpha (\Lambda) + u_2 D_\alpha (g)$, for all $u_1, u_2 \in \mathbb{R}$,
- 2) $D_\alpha (\lambda) = 0$, for all constant functions $\Lambda (t) = \lambda$,
- 3) $D_\alpha (\Lambda g) = \Lambda D_\alpha (g) + g D_\alpha (\Lambda)$,
- 4) $D_\alpha \left(\frac{\Lambda}{g} \right) = \frac{D_\alpha (\Lambda) g + D_\alpha (g) \Lambda}{g^2}$,
- 5) If Λ is differentiable, then

$$D_\alpha (\Lambda) (t) = t^{1-\alpha} \frac{d\Lambda}{dt} (t).$$

Also,

- a) $D_\alpha (1) = 0$
- b) $D_\alpha (e^{u_1 t}) = u_1 t^{1-\alpha} e^{u_1 t}$, $u_1 \in \mathbb{R}$
- c) $D_\alpha (\sin (u_1 t)) = u_1 t^{1-\alpha} \cos (u_1 t)$, $u_1 \in \mathbb{R}$
- d) $D_\alpha (\cos (u_1 t)) = -u_1 t^{1-\alpha} \sin (u_1 t)$, $u_1 \in \mathbb{R}$
- e) $D_\alpha \left(\frac{1}{\alpha} t^\alpha \right) = 1$
- f) $D_\alpha \left(\sin \left(\frac{t^\alpha}{\alpha} \right) \right) = \cos \left(\frac{t^\alpha}{\alpha} \right)$
- g) $D_\alpha \left(\cos \left(\frac{t^\alpha}{\alpha} \right) \right) = -\sin \left(\frac{t^\alpha}{\alpha} \right)$
- h) $D_\alpha \left(e^{\frac{t^\alpha}{\alpha}} \right) = e^{\frac{t^\alpha}{\alpha}}$.

Theorem 4 (*Mean value theorem for conformable fractional differentiable functions*) Let $\alpha \in (0, 1]$ and $\Lambda : [0, \infty) \rightarrow \mathbb{R}$ be a continuous on $[u_1, u_2]$ and an α -fractional differentiable mapping on (u_1, u_2) with $0 \leq u_1 < u_2$. Then, there exist $c \in (u_1, u_2)$, such that

$$D_\alpha (\Lambda) (c) = \frac{\Lambda (u_2) - \Lambda (u_1)}{\frac{u_2^\alpha}{\alpha} - \frac{u_1^\alpha}{\alpha}}.$$

Definition 2 (*Conformable fractional integral*) Let $\alpha \in (0, 1]$ and $0 \leq u_1 < u_2$. A function $\Lambda : [0, \infty) \rightarrow \mathbb{R}$ is α -fractional integrable on $[u_1, u_2]$ if the integral

$$\int_{u_1}^{u_2} \Lambda (x) d_\alpha x := \int_{u_1}^{u_2} \Lambda (x) x^{\alpha-1} dx,$$

exists and is finite. All α -fractional integrable on $[u_1, u_2]$ is indicated by $L_\alpha^1([u_1, u_2])$.

Remark 1

$$I_\alpha^{u_1}(\Lambda)(t) = I_1^{u_1}(t^{\alpha-1}\Lambda) = \int_{u_1}^t \frac{\Lambda(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1]$.

Theorem 5 Let $\Lambda : (u_1, u_2) \rightarrow \mathbb{R}$ be differentiable and $\alpha \in (0, 1]$. Then, for all $t > u_1$ we have

$$I_\alpha^{u_1} D_\alpha^{u_1}(\Lambda)(t) = \Lambda(t) - \Lambda(u_1).$$

Theorem 6 (Integration by parts). Let $\Lambda, g : [u_1, u_2] \rightarrow \mathbb{R}$ be two functions such that Λg is differentiable. Then,

$$\begin{aligned} & \int_{u_1}^{u_2} \Lambda(x) D_\alpha^{u_1}(g)(x) d_\alpha x \\ &= \Lambda g|_{u_1}^{u_2} - \int_{u_1}^{u_2} g(x) D_\alpha^{u_1}(\Lambda)(x) d_\alpha x. \end{aligned}$$

Theorem 7 Assume that $\Lambda : [u_1, \infty) \rightarrow \mathbb{R}$ such that $\Lambda^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > u_1$ we have

$$D_\alpha^{u_1}(\Lambda)(t) I_\alpha^{u_1} = \Lambda(t).$$

Theorem 8 Let $\alpha \in (0, 1]$ and $\Lambda : [u_1, u_2] \rightarrow \mathbb{R}$ be a continuous on $[u_1, u_2]$ with $0 \leq u_1 < u_2$. Then,

$$|I_\alpha^{u_1}(\Lambda)(x)| \leq I_\alpha^{u_1}|\Lambda|(x).$$

For more details and properties concerning the conformable integral operators, we refer, for example, to the works [1]-[6] and [8, 9, 15, 16, 21, 22, 23, 24, 25]. In this paper, we establish the Bullen type inequalities for conformable fractional integral and we will investigate some integral inequalities connected with Bullen-type inequalities for conformable fractional integrals. The results presented here would provide generalizations of those given in earlier works. In this study, some new identity and Bullen type integral inequalities for differentiable functions are established and are applied to produce some inequalities of special means.

3 Bullen Type Inequalities for Conformable Fractional Integral

With the help of the following lemma, we will give some integral inequalities connected with Bullen-type inequalities for conformable fractional integral.

Lemma 2 Let $\alpha \in (0, 1]$ and $\Lambda : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be an α -fractional differentiable function on (u_1, u_2) with $0 \leq u_1 < u_2$. If $D_\alpha(\Lambda)$ be an α -fractional integrable function on $[u_1^\alpha, u_2^\alpha]$, then the following identity for conformable fractional integral holds:

$$(5) \quad \begin{aligned} & \frac{1}{4} \int_0^1 (1 - 2t^\alpha) \left[D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_1^\alpha + \frac{1-t^\alpha}{2} u_2^\alpha \right) \right. \\ & \quad \left. + D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_2^\alpha + \frac{1-t^\alpha}{2} u_1^\alpha \right) \right] d_\alpha t \\ &= \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right]. \end{aligned}$$

Proof. Integrating by parts

$$\begin{aligned} & \int_0^1 (1 - 2t^\alpha) D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_1^\alpha + \frac{1-t^\alpha}{2} u_2^\alpha \right) d_\alpha t \\ & + \int_0^1 (1 - 2t^\alpha) D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_2^\alpha + \frac{1-t^\alpha}{2} u_1^\alpha \right) d_\alpha t \\ &= (1 - 2t^\alpha) \Lambda \left(\frac{1+t^\alpha}{2} u_1^\alpha + \frac{1-t^\alpha}{2} u_2^\alpha \right) \Big|_0^1 \\ & + 2\alpha \int_0^1 \Lambda \left(\frac{1+t^\alpha}{2} u_1^\alpha + \frac{1-t^\alpha}{2} u_2^\alpha \right) d_\alpha t \\ & + (1 - 2t^\alpha) \Lambda \left(\frac{1+t^\alpha}{2} u_2^\alpha + \frac{1-t^\alpha}{2} u_1^\alpha \right) \Big|_0^1 \\ & + 2\alpha \int_0^1 \Lambda \left(\frac{1+t^\alpha}{2} u_2^\alpha + \frac{1-t^\alpha}{2} u_1^\alpha \right) d_\alpha t \\ &= -\Lambda(u_1^\alpha) - \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) + \frac{4\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{\frac{u_1^\alpha + u_2^\alpha}{2}} \Lambda(x^\alpha) d_\alpha x \\ & -\Lambda(u_2^\alpha) - \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) + \frac{4\alpha}{u_2^\alpha - u_1^\alpha} \int_{\frac{u_1^\alpha + u_2^\alpha}{2}}^{u_2} \Lambda(x^\alpha) d_\alpha x \\ &= -\left[\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha) + 2\Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \\ & + \frac{4\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x. \end{aligned}$$

Thus, proof is completed.

Theorem 9 Let $\alpha \in (0, 1]$ and $\Lambda : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be an α -fractional differentiable function on I° and $D_\alpha(\Lambda)$ be an α -fractional integrable function on I with $0 \leq u_1 < u_2$. If $|\Lambda'|$ be a convex function on I , then the following inequality for conformable fractional integral holds:

$$\begin{aligned}
& \left| \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \right| \\
& \leq \frac{5(u_2^\alpha - u_1^\alpha)}{128} \left((u_1^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)| + (u_2^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)| \right).
\end{aligned}$$

Proof. Since $|\Lambda'|$ is a convex function on I , by the using the properties $D_\alpha(\Lambda \circ g)(t) = \Lambda'(g(t)) D_\alpha(g(t))$ and $D_\alpha(\Lambda(t)) = t^{1-\alpha} \Lambda'(t)$ and by Lemma 2 and using the well known absolute value inequality, and we have

$$\begin{aligned}
& \left| \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \right| \\
& \leq \frac{1}{4} \int_0^1 |1 - 2t^\alpha| \left| D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_1^\alpha + \frac{1-t^\alpha}{2} u_2^\alpha \right) \right| d_\alpha t \\
& \quad + \frac{1}{4} \int_0^1 |1 - 2t^\alpha| \left| D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_2^\alpha + \frac{1-t^\alpha}{2} u_1^\alpha \right) \right| d_\alpha t \\
& \leq \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left((u_1^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)| \int_0^1 |1 - 2t^\alpha| \frac{1+t^\alpha}{2} d_\alpha t \right) \\
& \quad + \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left((u_2^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)| \int_0^1 |1 - 2t^\alpha| \frac{1-t^\alpha}{2} d_\alpha t \right) \\
& \quad + \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left((u_2^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)| \int_0^1 |1 - 2t^\alpha| \frac{1+t^\alpha}{2} d_\alpha t \right) \\
& \quad + \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left((u_1^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)| \int_0^1 |1 - 2t^\alpha| \frac{1-t^\alpha}{2} d_\alpha t \right) \\
& = \frac{5(u_2^\alpha - u_1^\alpha)}{128} \left((u_1^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)| + (u_2^\alpha)^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)| \right).
\end{aligned}$$

where

$$\begin{aligned}
& \int_0^1 |1 - 2t^\alpha| \frac{1+t^\alpha}{2} d_\alpha t \\
& = \int_0^1 |1 - 2t^\alpha| \frac{1+t^\alpha}{2} t^{\alpha-1} dt \\
& = \int_0^{\frac{1}{2^{1/\alpha}}} (1 - 2t^\alpha) \frac{1+t^\alpha}{2} t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1) \frac{1+t^\alpha}{2} t^{\alpha-1} dt \\
& = \frac{3}{16\alpha}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 |1 - 2t^\alpha| \frac{1 - t^\alpha}{2} d_\alpha t \\
&= \int_0^1 |1 - 2t^\alpha| \frac{1 - t^\alpha}{2} t^{\alpha-1} dt \\
&= \int_0^{\frac{1}{2^{1/\alpha}}} (1 - 2t^\alpha) \frac{1 - t^\alpha}{2} t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1) \frac{1 - t^\alpha}{2} t^{\alpha-1} dt \\
&= \frac{1}{8\alpha}.
\end{aligned}$$

Remark 2 If we choose $\alpha = 1$ in Theorem 9, then we obtain following inequality:

$$\begin{aligned}
& \left| \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(x) dx - \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda\left(\frac{u_1 + u_2}{2}\right) \right] \right| \\
&\leq \frac{5(u_2 - u_1)}{128} (|\Lambda'(u_1)| + |\Lambda'(u_2)|).
\end{aligned}$$

Theorem 10 Let $\alpha \in (0, 1]$ and $\Lambda : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be an α -fractional differentiable function on I° and $D_\alpha(\Lambda)$ be an α -fractional integrable function on I with $0 \leq u_1 < u_2$ and $p, q > 1$, $1/p + 1/q = 1$. If $|\Lambda'|^q$ be a convex function on I , then the following inequality for conformable fractional integral holds:

$$\begin{aligned}
& \left| \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \right| \\
&\leq \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{\alpha(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{4\alpha} \right)^{\frac{1}{q}} \\
&\quad \times \left[\left(3(u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left((u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + 3(u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Proof. Since $|\Lambda'|^q$ is a convex function on I , by the using the properties $D_\alpha(\Lambda \circ g)(t) = \Lambda'(g(t)) D_\alpha(g(t))$, $D_\alpha(\Lambda(t)) = t^{1-\alpha} \Lambda'(t)$, by Lemma 2 and using the well known Hölder inequality, we have

$$\begin{aligned}
& \left| \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \right| \\
&\leq \frac{1}{4} \left(\int_0^1 (1 - 2t^\alpha)^p d_\alpha t \right)^{\frac{1}{p}} \left(\int_0^1 \left(D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_1^\alpha + \frac{1-t^\alpha}{2} u_2^\alpha \right) \right)^q d_\alpha t \right)^{\frac{1}{q}} \\
&\quad + \frac{1}{4} \left(\int_0^1 (1 - 2t^\alpha)^p d_\alpha t \right)^{\frac{1}{p}} \left(\int_0^1 \left(D_\alpha(\Lambda) \left(\frac{1+t^\alpha}{2} u_2^\alpha + \frac{1-t^\alpha}{2} u_1^\alpha \right) \right)^q d_\alpha t \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{\alpha(1+p)} \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{3}{4\alpha} (u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + \frac{1}{4\alpha} (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \\
&\quad + \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{\alpha(1+p)} \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{3}{4\alpha} (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q + \frac{1}{4\alpha} (u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q \right)^{\frac{1}{q}} \\
&= \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{\alpha(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{4\alpha} \right)^{\frac{1}{q}} \\
&\quad \times \left[\left(3(u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left((u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + 3(u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-2t^\alpha)^p d_\alpha t &= \int_0^1 (1-2t^\alpha)^p t^{\alpha-1} dt \\
&= \int_0^{\frac{1}{2^{1/\alpha}}} (1-2t^\alpha)^p t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha-1)^p t^{\alpha-1} dt \\
&= \frac{1}{\alpha(1+p)}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \frac{1+t^\alpha}{2} d_\alpha t &= \int_0^1 \frac{1+t^\alpha}{2} t^{\alpha-1} dt = \frac{3}{4\alpha} \\
\int_0^1 \frac{1-t^\alpha}{2} d_\alpha t &= \int_0^1 \frac{1-t^\alpha}{2} t^{\alpha-1} dt = \frac{1}{4\alpha}.
\end{aligned}$$

Remark 3 If we choose $\alpha = 1$ in Theorem 10, then we obtain following inequality:

$$\begin{aligned}
&\left| \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(x) dx - \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda\left(\frac{u_1 + u_2}{2}\right) \right] \right| \\
&\leq \frac{(u_2 - u_1)}{8} \left(\frac{1}{(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{4} \right)^{\frac{1}{q}} \\
&\quad \times \left[\left(3(\Lambda'(u_1))^q + (\Lambda'(u_2))^q \right)^{\frac{1}{q}} + \left((\Lambda'(u_1))^q + 3(\Lambda'(u_2))^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Remark 4 If we choose $\alpha = 1$ in Theorem 10, then we obtain Corollary 1.

Theorem 11 Let $\alpha \in (0, 1]$ and $\Lambda : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be an α -fractional differentiable function on I° and $D_\alpha(\Lambda)$ be an α -fractional integrable function on I with $0 \leq u_1 < u_2$ and $q \geq 1$. If $|\Lambda'|^q$ be a convex function on I , then the following inequality for conformable fractional integral holds:

$$\begin{aligned} & \left| \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \right| \\ & \leq \frac{(u_2^\alpha - u_1^\alpha)}{16} \left(\frac{1}{4}\right)^{\frac{1}{q}} \\ & \quad \times \left[\left(3(u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(3(u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q + (u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Since $|\Lambda'|^q$ is a convex function on I , by the using the properties $D_\alpha(\Lambda \circ g)(t) = \Lambda'(g(t)) D_\alpha(g(t))$ and $D_\alpha(\Lambda(t)) = t^{1-\alpha} \Lambda'(t)$ and assume that $q \geq 1$, by Lemma 2 and using the well known power-mean inequality, we have

$$\begin{aligned} & \left| \frac{\alpha}{u_2^\alpha - u_1^\alpha} \int_{u_1}^{u_2} \Lambda(x^\alpha) d_\alpha x - \frac{1}{2} \left[\frac{\Lambda(u_1^\alpha) + \Lambda(u_2^\alpha)}{2} + \Lambda\left(\frac{u_1^\alpha + u_2^\alpha}{2}\right) \right] \right| \\ & \leq \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\int_0^1 |1-2t^\alpha| d_\alpha t \right)^{1-\frac{1}{q}} \left((u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q \int_0^1 |1-2t^\alpha| \frac{1+t^\alpha}{2} d_\alpha t \right. \\ & \quad \left. + (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \int_0^1 |1-2t^\alpha| \frac{1-t^\alpha}{2} d_\alpha t \right)^{\frac{1}{q}} \\ & \quad + \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\int_0^1 |1-2t^\alpha| d_\alpha t \right)^{1-\frac{1}{q}} \left((u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \int_0^1 |1-2t^\alpha| \frac{1+t^\alpha}{2} d_\alpha t \right. \\ & \quad \left. + (u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q \int_0^1 |1-2t^\alpha| \frac{1-t^\alpha}{2} d_\alpha t \right)^{\frac{1}{q}} \\ & = \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{2\alpha}\right)^{1-\frac{1}{q}} \left(\frac{2}{16\alpha}\right)^{\frac{1}{q}} \\ & \quad \cdot \left(3(u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \\ & \quad + \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{2\alpha}\right)^{1-\frac{1}{q}} \left(\frac{2}{16\alpha}\right)^{\frac{1}{q}} \\ & \quad \cdot \left(3(u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q + (u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&= \frac{(u_2^\alpha - u_1^\alpha)}{16} \left(\frac{1}{4}\right)^{\frac{1}{q}} \times \\
&\quad \left[\left((u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q + (u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(3(u_2^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_2^\alpha)|^q + (u_1^{q\alpha})^{\alpha-1} |D_\alpha(\Lambda)(u_1^\alpha)|^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

where

$$\int_0^1 |1 - 2t^\alpha| d_\alpha t = \int_0^{\frac{1}{2^{1/\alpha}}} (1 - 2t^\alpha) t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1) t^{\alpha-1} dt = \frac{1}{2\alpha}$$

and

$$\begin{aligned}
&\int_0^1 |1 - 2t^\alpha| \frac{1+t^\alpha}{2} d_\alpha t \\
&= \int_0^{\frac{1}{2^{1/\alpha}}} (1 - 2t^\alpha) \frac{1+t^\alpha}{2} t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1) \frac{1+t^\alpha}{2} t^{\alpha-1} dt \\
&= \frac{3}{8\alpha},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 |1 - 2t^\alpha| \frac{1-t^\alpha}{2} d_\alpha t \\
&= \int_0^{\frac{1}{2^{1/\alpha}}} (1 - 2t^\alpha) \frac{1-t^\alpha}{2} t^{\alpha-1} dt + \int_{\frac{1}{2^{1/\alpha}}}^1 (2t^\alpha - 1) \frac{1-t^\alpha}{2} t^{\alpha-1} dt \\
&= \frac{1}{8\alpha}.
\end{aligned}$$

Remark 5 If we choose $\alpha = 1$ in Theorem 11, then we obtain following inequality:

$$\begin{aligned}
&\left| \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \Lambda(x) dx - \frac{1}{2} \left[\frac{\Lambda(u_1) + \Lambda(u_2)}{2} + \Lambda\left(\frac{u_1 + u_2}{2}\right) \right] \right| \\
&\leq \frac{(u_2 - u_1)}{16} \left(\frac{1}{4}\right)^{\frac{1}{q}} \times \\
&\quad \left[\left((\Lambda'(u_1))^q + (\Lambda'(u_2))^q \right)^{\frac{1}{q}} + \left((\Lambda'(u_1))^q + 3(\Lambda'(u_2))^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Remark 6 If we choose $\alpha = 1$ in Theorem 11, then we obtain Corollary 2.

4 Applications

Let

$$\begin{aligned} A(u_1, u_2) &= \frac{u_1 + u_2}{2}, \\ G(u_1, u_2) &= \sqrt{u_1 u_2}, \\ L_p(u_1, u_2) &= \left(\frac{u_2^{p+1} - u_1^{p+1}}{(p+1)(u_2 - u_1)} \right)^{1/p}, \quad u_1 \neq u_2, \quad p \in \mathbb{R}, \quad p \neq -1, 0 \end{aligned}$$

be the arithmetic mean, geometric mean, generalized logarithmic mean for $u_1, u_2 > 0$ respectively.

Proposition 1 *Let $s \in (0, 1]$, $u_1, u_2 > 0$, then*

$$\begin{aligned} & \left| \frac{\alpha (u_2^{na+\alpha} - u_1^{na+\alpha})}{(nu_1 + \alpha)(u_2^\alpha - u_1^\alpha)} - \frac{1}{2} [A(u_1^{n\alpha}, u_2^{n\alpha}) + A^n(u_1^\alpha, u_2^\alpha)] \right| \\ & \leq \frac{5(u_2^\alpha - u_1^\alpha)}{128} \left(n(u_1^\alpha)^{n-1} + n(u_2^\alpha)^{n-1} \right). \end{aligned}$$

Proof. The claim follows from Theorem 9 applied to convex function $\Lambda(\varkappa) = \varkappa^n$ where $n \in \mathbb{N}$.

Example 1 *If we take $\alpha = 1, n = 2$ in Proposition 1, then we can obtain following inequality:*

$$\left| \frac{2A(u_1^2, u_2^2) + G^2(u_1, u_2)}{(2u_1 + 1)} - A[A(u_1^2, u_2^2), A^2(u_1, u_2)] \right| \leq \frac{5(u_2^2 - u_1^2)}{64}.$$

Proposition 2 *Let $\alpha \in (0, 1]$, $u_1, u_2 > 0$, $p, q > 1$, then*

$$\begin{aligned} & \left| \frac{\alpha (u_2^{na+\alpha} - u_1^{na+\alpha})}{(nu_1 + \alpha)(u_2^\alpha - u_1^\alpha)} - \frac{1}{2} [A(u_1^{n\alpha}, u_2^{n\alpha}) + A^n(u_1^\alpha, u_2^\alpha)] \right| \\ & \leq \frac{\alpha(u_2^\alpha - u_1^\alpha)}{8} \left(\frac{1}{\alpha(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{4\alpha} \right)^{\frac{1}{q}} \times \\ & \quad \left[\left(3(nu_1^{\alpha(n-1)})^q + (nu_2^{\alpha(n-1)})^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left((nu_1^{\alpha(n-1)})^q + 3(nu_2^{\alpha(n-1)})^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The claim follows from Theorem 10 applied to convex $\Lambda(\varkappa) = \varkappa^n$ where $n \in \mathbb{N}$.

Example 2 If we take $\alpha = 1$, $p = q = n = 2$ in Proposition 2, then we can obtain following inequality:

$$\begin{aligned} & \left| \frac{2A(u_1^2, u_2^2) + G^2(u_1, u_2)}{(2u_1 + 1)} - A[A(u_1^2, u_2^2), A^2(u_1, u_2)] \right| \\ & \leq \frac{(u_2 - u_1)}{4} \left(\frac{1}{12} \right)^{\frac{1}{2}} \times \left[(3u_1^2 + u_2^2)^{\frac{1}{2}} + (3u_2^2 + u_1^2)^{\frac{1}{2}} \right]. \end{aligned}$$

Proposition 3 Let $\alpha \in (0, 1]$, $u_1, u_2 > 0$, $q \geq 1$, then

$$\begin{aligned} & \left| \frac{\alpha(u_2^{n\alpha+\alpha} - u_1^{n\alpha+\alpha})}{(nu_1 + \alpha)(u_2^\alpha - u_1^\alpha)} - \frac{1}{2} [A(u_1^{n\alpha}, u_2^{n\alpha}) + A^n(u_1^\alpha, u_2^\alpha)] \right| \\ & \leq \frac{u_2^\alpha - u_1^\alpha}{16} \left(\frac{1}{4} \right)^{\frac{1}{q}} \times \\ & \quad \left[\left(3 \left(nu_1^{\alpha(n-1)} \right)^q + \left(nu_2^{\alpha(n-1)} \right)^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\left(nu_1^{\alpha(n-1)} \right)^q + 3 \left(nu_2^{\alpha(n-1)} \right)^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The claim follows from Theorem 11 applied to convex $\Lambda(\varkappa) = \varkappa^n$ where $n \in \mathbb{N}$.

Example 3 If we take $\alpha = 1, q = 2$, $n = 2$ in Proposition 3, then we can obtain following inequality:

$$\begin{aligned} & \left| \frac{2A(u_1^2, u_2^2) + G^2(u_1, u_2)}{(2u_1 + 1)} - A[A(u_1^2, u_2^2), A^2(u_1, u_2)] \right| \\ & \leq \frac{u_2 - u_1}{32} \times \left[(3u_1^2 + u_2^2)^{\frac{1}{2}} + (u_1^2 + 3u_2^2)^{\frac{1}{2}} \right]. \end{aligned}$$

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