On the reconstruction via Urysohn-Chlodovsky operators¹

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Abstract

The main first goal of this work is to introduce an Urysohn type Chlodovsky operators defined on positive real axis by using the Urysohn type interpolation of the given function f and bounded on every finite subinterval. The basis used in this construction are the Fréchet and Prenter Density Theorems together with Urysohn type operator values instead of the rational sampling values of the function. Afterwards, we will state some convergence results, which are generalization and extension of the theory of classical interpolation of functions to operators.

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1 Introduction

In order to find a positive answer to the reconstruction problem for functions defined on the infinite intervals, in [9] Chlodovsky introduced and studied a sequence $(C_n f)$ of discrete linear positive operators, which are a generalization of the celebrated Bernstein polynomials.

Let f be a real function on $[0, \infty)$, which is bounded on each finite interval. The classical Bernstein-Chlodovsky operators defined as:

$$(1) \quad (C_n f)(x) := \begin{cases} \sum_{k=0}^n f\left(\frac{kb_n}{n}\right) p_{k,n}\left(\frac{x}{b_n}\right), & \text{if } 0 \leq x \leq b_n, \\ f(x), & \text{if } x > b_n \end{cases},$$

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with Bernstein basis polynomials

$$(2) \quad p_{v,n}(y) = \binom{n}{\nu} y^\nu (1-y)^{n-\nu}, \quad 0 \leq y \leq 1, \quad 0 \leq \nu \leq n.$$

Throughout the paper we assume that the sequence (b_n) satisfies

$$(3) \quad \lim_{n \rightarrow \infty} b_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0.$$

Let $\|f\|_{a,b} := \sup_{a \leq x \leq b} |f(x)|$, then Chlodovsky showed that if

$$(4) \quad \lim_{n \rightarrow \infty} \exp\left(-\alpha \frac{n}{b_n}\right) \|f\|_{0,b_n} = 0$$

for every $\alpha > 0$, then $(C_n f)(x)$ converges to $f(x)$ at each point of continuity of f (see Lorentz [18], p.36).

In time, a throughgoing study on these operators was developed, see for instance [7], [1], [2] and [3], and new properties of it have been pointed out. A detailed synthesis of the results on Chlodovsky operators can be found in the survey article [11] by Karsli.

Owing to the relations between positive linear operators and linear singular integrals, we consider the positive linear operators as a kind of singular integrals (See [8]).

The starting point of this study is motivated by the important applications of the integral operators in various fields of applied science and engineering.

In fact, many problems in engineering and mechanics can be transformed into integral equations and corresponding integral operators. Especially the integral operators of Urysohn type are used frequently when describing real problems which arise from different sciences, such as physics, mechanics, engineering, theory of elasticity, signal-image reconstruction and in the applications of mathematical physics.

Now, we propose a modification of Chlodovsky operators (1) in the following form;

$$(C_n f)(x) = \int_0^{b_n} f(t) d_t K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right)$$

with

$$K_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) =: \begin{cases} \sum_{kb_n \leq nt} p_{k,n}\left(\frac{x}{b_n}\right) & , \quad 0 < t \leq b_n \\ 0 & , \quad t = 0 \end{cases}$$

which is constant on $kb_n/n \leq t < (k+1)b_n/n$, $k = 0, 1, \dots, n-1$.

Finally, consider the following Urysohn equation:

$$x(t) = y(t) + \int_a^b k(t, s, x(s))ds, \quad t \in [a, b]$$

and the corresponding Urysohn operator

$$Ux(t) = \int_a^b k(t, s, x(s))ds, \quad t \in [a, b]$$

where k and y are known functions and x is the unknown function to be determined.

By virtue of these relations, the purpose of this study is to find a positive solution to the approximation (or superposition) problem for operators. In other words, we will propose a generalization and extension of the theory of interpolation to operators by introducing an operator, called Urysohn-Chlodovsky operators. More precisely, instead of the rational sampling values of a function, we can use the Urysohn-type operator values. From a mathematical point of view, the aforementioned Urysohn-Chlodovsky operators are more flexible than the previous one and, practically, the convergence problems can be extended to the operators.

2 Preliminaries and auxiliary results

In this section we shall introduce some notation and background material used throughout this paper.

As usual, we denote by $C[0, 1]$ the Banach space of continuous functions $u : [0, 1] \rightarrow \mathbb{R}$ with norm

$$\|u\| = \sup\{|u(x)| : x \in [0, 1]\}.$$

Let us consider the following Urysohn operator

$$(5) \quad Fx(t) = F(t, x(\cdot)) = \int_0^1 f(t, s, x(s))ds, \quad t \in [0, 1], \quad 0 \leq x(\cdot) \leq 1$$

with the kernel f . The kernel of this operator $f(t, s, x)$ is called the Green's function and its properties depend on the function x . For a constant function $x(s) = a$, we set $Fa(t) = F(a)$ (see [22]-[23], [21], [24], [20] and [25]).

Since the Dirac Delta function ($\delta(x)$) is actually a generalized function (distributions), the derivative of the Heaviside function ($H(x)$) is the Dirac Delta function in distributional sense, namely

$$\delta(x) = \frac{dH(x)}{dx}$$

holds true.

Hence, we assume that the following continuous interpolation conditions hold.

$$(6) \quad Fx_i(t) := F(t, x_i(\cdot, \xi)) = \int_0^1 f(t, z, x_i(z, \xi)) dz, \quad t \in [0, 1]$$

where $0 \leq x_i(z, \xi) \leq b_n$ defined as $x_i(z, \xi) = \frac{ib_n}{n} H(z - \xi)$; $\xi \in [0, 1]$; and $i = 0, 1, 2, \dots, n$.

Clearly, one has

$$(7) \quad \frac{\partial F(t, x_i(\cdot, \xi))}{\partial \xi} = \frac{\partial F(t, \frac{ib_n}{n} H(z - \xi))}{\partial \xi} = f(t, \xi, 0) - f(t, \xi, \frac{ib_n}{n}),$$

where $x_i(\cdot, \xi) = \frac{ib_n}{n} H(\cdot - \xi)$; $\xi \in [0, 1]$, and $i = 0, 1, 2, \dots$.

We set

$$(8) \quad F_1\left(t, \xi, \frac{ib_n}{n}\right) := \frac{\partial F(t, x_i(\cdot, \xi))}{\partial \xi} = \frac{\partial Fx_i(t)}{\partial \xi}.$$

From now on, it is possible to construct Chlodovsky type approximation operator by means of the Urysohn operator (6) with unknown function $f : [0, 1]^2 \times [0, \infty) \rightarrow \mathbb{R}$, which converges to the Urysohn operator $Fx(t)$.

Definition 1. Let F be the Urysohn integral operator of f . Then in view of (5) and (6), the Urysohn-Chlodovsky operators is defined as:

$$(9) \quad (UC_n F)x(t) = \int_0^1 \left[\sum_{k=0}^n f\left(t, s, \frac{kb_n}{n}\right) p_{k,n}\left(\frac{x(s)}{b_n}\right) \right] ds.$$

acting on functions $f : [0, 1]^2 \times [0, \infty) \rightarrow \mathbb{R}$, which is bounded on every subinterval of $[0, \infty)$, where $p_{k,n}$ is the Bernstein basis (2). In particular, we will put $Dom (UC_n F) = \bigcap_{n \in \mathbb{N}} Dom UC_n F$, where $Dom (UC_n F)$ is the set of all functions $f : [0, 1]^2 \times [0, \infty) \rightarrow \mathbb{R}$ for which the operator is well defined.

The literature about approximation results by means of integral operators or reconstruction of a function by means of integral operators is extensive. Unlike these approach, we will constructed and investigate this approximation or reconstruction problem for operators instead of functions. Due to the importance of integral operators, unlike previous works, we will investigate the positive solution for the approximation or reconstruction problem in the sense of Urysohn-Chlodovsky operators to the Urysohn operators.

The approximation or reconstruction problem is well-known and much older for the functions, but for the operators it needs to be investigated. We can mention some

of the very recent papers of the author about this problem as [13]-[16], in which the author defined and investigated the Urysohn type nonlinear Bernstein operators as;

$$(NB_nF)x(t) = \int_0^1 \left[\sum_{k=0}^n P_{k,n} \left(x(s), f \left(t, s, \frac{k}{n} \right) \right) \right] ds, \quad 0 \leq x(s) \leq 1,$$

where $P_{k,n}$ satisfy some suitable assumptions. Moreover, in [15], Karsli considered a sequence $NBF = (NB_nF)$ of operators, which represents the Urysohn type nonlinear form of the two dimensional Bernstein operators defined by P.L. Butzer on the square $S = [0, 1] \times [0, 1]$ (see [5], [6]), having the form:

$$\begin{aligned} & (NB_nF)(x(t), y(t)) \\ &= \int_0^1 \int_0^1 \left[\sum_{k=0}^n \sum_{i=0}^n P_{k,i,n} \left(x(s), y(z), f \left(t, s, z, \frac{k}{n}, \frac{i}{n} \right) \right) \right] ds dz, \\ & 0 \leq x(s), y(z) \leq 1, \quad n \in \mathbb{N}. \end{aligned}$$

The central issue of the paper [16] was to give a positive answer to the superposition problem for functionals and operators by introducing the Urysohn nonlinear operators of the two dimensional Stancu operators $(P_n^\alpha f)(x, y)$ defined on the triangle $\triangle := \{(s, z) : s, z \geq 0, s + z \leq 1\}$.

For further reading, we refer to the reader [10], [19], [12] and the very recent paper of the author on the Urysohn type generalized sampling operators [17].

The following results are well-known for classical Chlodovsky operators.

Lemma 1 [9]. For every $x \in [0, b_n]$ and $n \geq 1$, we have

$$\begin{aligned} (10) \quad & (C_n 1)(x) = 1 \\ & (C_n s)(x) = x \\ & (C_n s^2)(x) = x^2 + \frac{x(b_n - x)}{n}. \end{aligned}$$

As a result, one can find the following equalities;

$$(C_n(t-x)^2)(x) = \frac{x(b_n - x)}{n}, \quad (C_n(t-x))(x) = 0.$$

Equalities (10) implies that $(C_n t^2)(x)$ and hence $(C_n(t-x)^2)(x)$ does not converge to x^2 and 0 uniformly on $[0, b_n]$, respectively, as $n \rightarrow \infty$.

Infact, if the point is not fixed then the condition (3) is enough for poinwise convergence and uniform coverage on any compact set. But for an uniform convergence result on $[0, \infty)$, as $n \rightarrow \infty$, this condition is not enough, because x can be $\frac{b_n}{2}$ and hence

$$(C_n(t-x)^2)(x) \leq \frac{\frac{b_n}{2} (b_n - \frac{b_n}{2})}{n} = \frac{b_n^2}{4n},$$

which can tends to infinity, as $n \rightarrow \infty$.

Lemma 2. [7] For $n = 1, 2, 3, \dots$, any given $m \in \mathbb{N}_0$, let

$$T_{n,m}^*(x) := \sum_{k=0}^n \left(\frac{kb_n}{n} - x \right)^m p_{k,n} \left(\frac{x}{b_n} \right).$$

Then for any fixed $x \in [0, \infty)$,

$$(11) \quad T_{n,m}^*(x) \leq A_m(x) \left(\frac{b_n - x}{n} \right)^{[(m+1)/2]} \quad (n \rightarrow \infty),$$

where $A_m(x)$ depends only on x and m ; $[a]$ denotes the integral part of a .

Lemma 3. [7] For $t \in [0, 1]$ the inequality

$$0 \leq z \leq \frac{3}{2} \sqrt{nt(1-t)}$$

implies

$$\sum_{|k-nt| \geq 2z\sqrt{nt(1-t)}} p_{k,n}(t) \leq 2 \exp(-z^2).$$

In particular, for sufficiently large n ,

$$\sum_{\left| \frac{kb_n}{n} - x \right| \geq \delta} p_{k,n} \left(\frac{x}{b_n} \right) \leq 2 \exp \left(-\frac{\delta^2}{4x} \frac{n}{b_n} \right).$$

Remark 1. By (7), $(UC_n F)$ can be written as;

$$(UC_n F)x(t) = F(0) - \int_0^1 \left[\sum_{k=0}^n \frac{\partial F \left(\frac{kb_n}{n} H(t-s) \right)}{\partial s} p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds.$$

In fact;

$$\begin{aligned} (UC_n F)x(t) &= \int_0^1 \left[\sum_{k=0}^n f \left(t, s, \frac{kb_n}{n} \right) p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds \\ &= \int_0^1 \left[\sum_{k=0}^n \left[f(t, s, 0) - \frac{\partial F \left(\frac{kb_n}{n} H(t-s) \right)}{\partial s} \right] p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds \\ &= \int_0^1 \left[\sum_{k=0}^n f(t, s, 0) p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds \\ &\quad - \int_0^1 \left[\sum_{k=0}^n \frac{\partial F \left(\frac{kb_n}{n} H(t-s) \right)}{\partial s} p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds, \end{aligned}$$

Since $\sum_{k=0}^n p_{k,n} \left(\frac{x(s)}{b_n} \right) = 1$, we have

$$\begin{aligned} (UC_n F)x(t) &= \int_0^1 f(t, s, 0) ds - \int_0^1 \left[\sum_{k=0}^n \frac{\partial F \left(\frac{kb_n}{n} H(t-s) \right)}{\partial s} p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds \\ &= F(0) - \int_0^1 \left[\sum_{k=0}^n \frac{\partial F \left(\frac{kb_n}{n} H(t-s) \right)}{\partial s} p_{k,n} \left(\frac{x(s)}{b_n} \right) \right] ds. \end{aligned}$$

3 Convergence property

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems.

Definition 2. Let $f \in C([0, 1]^2 \times [0, \infty))$ and $\delta > 0$ be given. Then the partial modulus of continuity of f is given by;

$$(12) \quad \omega(\delta) := \sup_{(t,s) \in [0,1]^2} \sup_{|u_1 - u_2| \leq \delta} |f(t, s, u_1) - f(t, s, u_2)|.$$

Clearly, if $\delta = |u_1 - u_2|$, one has

$$|f(t, s, u_1) - f(t, s, u_2)| \leq \omega(|u_1 - u_2|) \leq \left(1 + \frac{|u_1 - u_2|}{\delta}\right) \omega(\delta).$$

In order to approximate unbounded functions by using the Chlodovsky operators defined in (9), we take into consideration the exponential weighted space $C_{0,\alpha}[0, \infty)$, which is the set of all real valued functions f continuous on $[0, \infty)$, satisfying the growth condition (4) for every $\alpha > 0$.

We are now ready to establish one of the main results of this study:

Theorem 1. Let the function f , defined on $[0, \infty)$, satisfy the growth condition (4) for every $\alpha > 0$, $\{b_n\}$ being a positive sequence satisfying (3), and suppose that F be the Urysohn integral operator of the function f , with $0 \leq x(s) \leq b_n$. Then,

$$\lim_{n \rightarrow \infty} (UC_n F)x(t) = Fx(t),$$

holds true at each point $x(s)$ of continuity of f .

Proof. In view of the definition of the operator (9) and (6), we have

$$\begin{aligned}
(UC_n F)x(t) - Fx(t) &= F(0) - \int_0^1 \left[\sum_{k=0}^n \frac{\partial F\left(\frac{kb_n}{n}H(t-s)\right)}{\partial s} p_{k,n}\left(\frac{x(s)}{b_n}\right) \right] ds - Fx(t) \\
&= F(0) - Fx(t) - \int_0^1 \left[\sum_{k=0}^n \frac{\partial F\left(\frac{kb_n}{n}H(t-s)\right)}{\partial s} p_{k,n}\left(\frac{x(s)}{b_n}\right) \right] ds \\
&= F(0) - Fx(t) - \int_0^1 \sum_{k=0}^n F_1\left(t, s, \frac{kb_n}{n}\right) p_{k,n}\left(\frac{x(s)}{b_n}\right) ds.
\end{aligned}$$

Note that, by (7) and (8), one has

$$F(0) - Fx(t) = \int_0^1 f(t, s, 0) ds - \int_0^1 f(t, s, x(s)) ds = \int_0^1 F_1(t, s, x(s)) ds.$$

So we obtain

$$(UC_n F)x(t) - Fx(t) = \int_0^1 \sum_{k=0}^n p_{k,n}\left(\frac{x(s)}{b_n}\right) \left[F_1(t, s, x(s)) - F_1\left(t, s, \frac{kb_n}{n}\right) \right] ds.$$

Let us divide the last term into two parts as;

$$(UC_n F)x(t) - Fx(t) = P_1 + P_2,$$

where

$$P_1 = \int_0^1 \sum_{\left|\frac{kb_n}{n} - x(s)\right| < \delta} p_{k,n}\left(\frac{x(s)}{b_n}\right) \left[F_1(t, s, x(s)) - F_1\left(t, s, \frac{kb_n}{n}\right) \right] ds$$

and

$$P_2 = \int_0^1 \sum_{\left|\frac{kb_n}{n} - x(s)\right| \geq \delta} p_{k,n}\left(\frac{x(s)}{b_n}\right) \left[F_1(t, s, x(s)) - F_1\left(t, s, \frac{kb_n}{n}\right) \right] ds.$$

Hence one has

$$|(UC_n F)x(t) - Fx(t)| \leq |P_1| + |P_2|$$

together with

$$|P_1| \leq \int_0^1 \sum_{\left|\frac{kb_n}{n} - x(s)\right| < \delta} p_{k,n}\left(\frac{x(s)}{b_n}\right) \left| F_1(t, s, x(s)) - F_1\left(t, s, \frac{kb_n}{n}\right) \right| ds,$$

and

$$|P_2| \leq \int_0^1 \sum_{\left| \frac{kb_n}{n} - x(s) \right| \geq \delta} p_{k,n} \left(\frac{x(s)}{b_n} \right) \left| F_1(t, s, x(s)) - F_1 \left(t, s, \frac{kb_n}{n} \right) \right| ds.$$

Since $x \in C[0, 1]$ and $0 \leq x(s) \leq b_n$, then clearly

$$\left| F_1(t, s, x(s)) - F_1 \left(t, s, \frac{kb_n}{n} \right) \right| < \epsilon$$

holds true when $\left| \frac{kb_n}{n} - x(s) \right| < \delta$, and

$$\begin{aligned} \left| F_1(t, s, x(s)) - F_1 \left(t, s, \frac{kb_n}{n} \right) \right| &\leq |F_1(t, s, x(s))| + \left| F_1 \left(t, s, \frac{kb_n}{n} \right) \right| \\ &\leq 4 \|f\|_{0, b_n} \end{aligned}$$

holds true, when $\left| \frac{kb_n}{n} - x(s) \right| \geq \delta$.

So

$$\begin{aligned} |P_1| &\leq \int_0^1 \sum_{\left| \frac{kb_n}{n} - x(s) \right| < \delta} p_{k,n} \left(\frac{x(s)}{b_n} \right) \left| F_1(t, s, x(s)) - F_1 \left(t, s, \frac{kb_n}{n} \right) \right| ds \\ &\leq \epsilon \int_0^1 \sum_{\left| \frac{kb_n}{n} - x(s) \right| < \delta} p_{k,n} \left(\frac{x(s)}{b_n} \right) ds \\ &\leq \epsilon, \end{aligned}$$

and

$$\begin{aligned} |P_2| &\leq \int_0^1 \sum_{\left| \frac{kb_n}{n} - x(s) \right| \geq \delta} p_{k,n} \left(\frac{x(s)}{b_n} \right) \left| F_1(t, s, x(s)) - F_1 \left(t, s, \frac{kb_n}{n} \right) \right| ds \\ &\leq 4 \|f\|_{0, b_n} \int_0^1 \sum_{\left| \frac{kb_n}{n} - x(s) \right| \geq \delta} p_{k,n} \left(\frac{x(s)}{b_n} \right) ds \\ &\leq 4 \|f\|_{0, b_n} \int_0^1 2 \exp \left(-\frac{\delta^2}{4x(s)} \frac{n}{b_n} \right) ds \\ &\leq 8 \|f\|_{0, b_n} \exp \left(-\frac{\delta^2}{4x(s)} \frac{n}{b_n} \right). \end{aligned}$$

Collecting these estimates we have

$$\begin{aligned} |(UC_n F)x(t) - Fx(t)| &\leq |P_1| + |P_2| \\ &\leq \epsilon + 8 \|f\|_{0, b_n} \exp \left(-\frac{\delta^2}{4x(s)} \frac{n}{b_n} \right). \end{aligned}$$

That is

$$\lim_{n \rightarrow \infty} (UC_n F)x(t) = Fx(t).$$

This completes the proof.

Theorem 2. Let F be the Urysohn integral operator of the function f satisfies (4), with $0 \leq x(s) \leq b_n$. Then $(C_n F)$ converges to F uniformly in x . That is

$$\lim_{n \rightarrow \infty} \|(UC_n F)x(t) - Fx(t)\|_{C[0,1]} = 0.$$

uniformly on any compact subinterval $[0, \infty)$ if $f \in C_{0,\alpha}[0, \infty)$.

Theorem 3. Let F be the Urysohn integral operator of f with $x \in C[0, 1]$, and $0 \leq x(s) \leq b_n$. Then for every $\varepsilon > 0$

$$|(UC_n F)x(t) - Fx(t)| \leq 2\omega(\delta)$$

holds true at each point $x(s)$ of continuity of f , where $\delta = \sqrt{\frac{x(s)(b_n - x(s))}{n}}$.

Proof. Clearly one has

$$\begin{aligned} |(UC_n F)x(t) - Fx(t)| &\leq \int_0^1 \sum_{k=0}^n p_{k,n} \left(\frac{x(s)}{b_n} \right) \left| F_1(t, s, x(s)) - F_1 \left(t, s, \frac{kb_n}{n} \right) \right| ds \\ (13) \quad &\leq \int_0^1 \sum_{k=0}^n p_{k,n} \left(\frac{x(s)}{b_n} \right) \omega \left(\left| x(s) - \frac{kb_n}{n} \right| \right) ds \\ &: = I_n(x), \end{aligned}$$

say. Since $x \in C[0, 1]$ we can re-write (13) as follows

$$\begin{aligned} I_n(x) &\leq \int_0^1 \sum_{k=0}^n p_{k,n} \left(\frac{x(s)}{b_n} \right) \omega \left(\left| x(s) - \frac{k}{n} \right| \right) ds \\ &\leq \omega(\delta) \left\{ 1 + \delta^{-1} \int_0^1 \sum_{k=0}^n \left| \frac{k}{n} - x(s) \right| p_{k,n} \left(\frac{x(s)}{b_n} \right) ds \right\} \\ &\leq \omega(\delta) \left\{ 1 + \delta^{-2} \int_0^1 \sum_{k=0}^n \left(\frac{k}{n} - x(s) \right)^2 p_{k,n} \left(\frac{x(s)}{b_n} \right) ds \right\} \\ &\leq \omega(\delta) \left\{ 1 + \frac{x(s)(b_n - x(s))}{\delta^2 n} \right\}. \end{aligned}$$

If we set

$$\delta = \sqrt{\frac{x(s)(b_n - x(s))}{n}},$$

then

$$|(UC_n F)x(t) - Fx(t)| \leq 2\omega(\delta),$$

holds true. Thus the proof is now complete.

4 Practical examples and graphical representations

In this section we will apply the theory to the theory of interpolation of functions to functionals or operators by means of Urysohn Chlodovsky operators (9).

We note that in all of the figures, the graph with the red line belongs to the original function, the graph with the green line to the operators with $n = 10$, and finally the graph consisting of blue line to the operators with $n = 30$ and $b_n = n^{\frac{1}{3}}$.

Example 1. Let us consider the operator $Fx(t) = \int_0^1 x^3(t)dt$, and we take its corresponding Urysohn Chlodovsky operator $(UC_n F)x(t)$, then one has for $n = 10$, $n = 30$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_{10}$.

Figure 1: Approximation of $(UC_n F)x(t)$, for $n = 10$, $n = 30$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_{10}$.

Example 2. Let us consider the operator $Fx(t) = \int_0^1 e^{x(z)-x^2(z)}dz$, and we take its corresponding Urysohn Chlodovsky operator $(UC_n F)x(t)$ (9), then one has for $n = 10$, $n = 30$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_{10}$.

Figure 2: Approximation of $(UC_n F)x(t)$ for $n = 10$, $n = 30$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_{10}$.

Example 3. Let us consider the operator $Fx(t) = \int_0^1 \sin [x^3(t) + 1] dt$, and we take its corresponding Urysohn Chlodovsky operator $(UC_n F)x(t)$ (9), then one has for $n = 2$, $n = 20$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_{20}$.

Figure 3: Approximation of $(UC_n F)x(t)$ for $n = 2$, $n = 20$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_{20}$.

Example 4. Let us consider the operator $Fx(t) = \int_0^1 \sin [x^3(t) + 1] dt$, and we take its corresponding Urysohn Chlodovsky operator $(UC_n F)x(t)$ (9), then one has for $n = 2$, $n = 20$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_2$.

Figure 4: Approximation of $(UC_n F)x(t)$, for $n = 2$, $n = 20$ and $b_n = n^{\frac{1}{3}}$ and $0 \leq x(t) \leq b_2$.

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