Differential subordination results for holomorphic functions associated with Wanas Operator and Poisson Distribution series ¹

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Abstract

In the current paper, we introduce a new family for holomorphic functions defined by Wanas operator associated with Poisson distribution series. Also we derive some interesting geometric properties for functions belongs to this family.

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1 Introduction

Denote by $\mathcal{A}$ the family of all functions f that are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and having the form:

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U,$$

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"With a view to recalling the principal of subordination between holomorphic functions, let the functions f and g holomorphic in U , we say that the function f is subordinate to g , written $f \prec g$ or $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function w holomorphic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function g is univalent in U , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [7])."

If $f \in \mathcal{A}$ is given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U,$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

A function $f \in \mathcal{A}$ is said to be starlike of order ρ in U if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \rho, \quad (0 \leq \rho < 1; z \in U).$$

Indicate the class of all starlike functions of order ρ in U by $S^*(\rho)$.

A function $f \in \mathcal{A}$ is said to be prestarlike of order ρ in U if

$$\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in S^*(\rho), \quad (\rho < 1).$$

Indicate the class of all prestarlike functions of order ρ in U by $\mathfrak{R}(\rho)$.

Clearly a function $f \in \mathcal{A}$ is in the class $\mathfrak{R}(0)$ if and only if f is convex univalent in U and $\mathfrak{R}(\frac{1}{2}) = S^*(\frac{1}{2})$.

A variable X is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \dots$ with probabilities $e^{-\lambda}, \lambda \frac{e^{-\lambda}}{1!}, \lambda^2 \frac{e^{-\lambda}}{2!}, \lambda^3 \frac{e^{-\lambda}}{3!}, \dots$ respectively, where λ is called the parameter. Thus

$$P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!}, \quad r = 0, 1, 2, 3, \dots$$

Recently, Porwal [5] introduced a power series whose coefficients are probabilities of Poisson distribution

$$\chi(\lambda, z) = z + \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} z^n, \quad z \in U,$$

where $\lambda > 0$. By ratio test the radius of convergence of above series is infinity.

Very recently, Wanas [8] introduced a new differential operator $W_{\alpha,\beta}^{k,\delta} : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$W_{\alpha,\beta}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^n,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \delta \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.

Now, by making use of the convolution of two series, we define the operator $\Upsilon_{\lambda,\alpha,\beta}^{k,\delta} : \mathcal{A} \rightarrow \mathcal{A}$ as follows:

$$\begin{aligned} (2) \quad \Upsilon_{\lambda,\alpha,\beta}^{k,\delta} f(z) &= \chi(\lambda, z) * W_{\alpha,\beta}^{k,\delta} f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} \left[\sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^n, \end{aligned}$$

It is readily verified from (2) that

$$\begin{aligned} (3) \quad z \left(\Upsilon_{\lambda,\alpha,\beta}^{k,\delta} f(z) \right)' &= \left[\sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \right] \Upsilon_{\lambda,\alpha,\beta}^{k,\delta+1} f(z) \\ &\quad - \left[\sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right] \Upsilon_{\lambda,\alpha,\beta}^{k,\delta} f(z). \end{aligned}$$

Denote by H the class of functions h with $h(0) = 1$, which are holomorphic and convex univalent in U .

To prove our main results, we will require the following Lemmas.

Lemma 1 [3]. *Let g be holomorphic in U and let h be holomorphic and convex univalent in U with $h(0) = g(0)$. If*

$$(4) \quad g(z) + \frac{1}{\tau} z g'(z) \prec h(z),$$

where $\operatorname{Re}(\tau) \geq 0$ and $\tau \neq 0$, then

$$g(z) \prec \check{h}(z) = \tau z^{-\tau} \int_0^z t^{\tau-1} h(t) dt \prec h(z)$$

and $\check{h}$ is the best dominant of (4).

Lemma 2 [6]. Let $\rho < 1$, $f \in S^*(\rho)$ and $g \in \mathfrak{R}(\rho)$. Then, for any holomorphic function F in U

$$\frac{g * (fF)}{g * f}(U) \subset \bar{co}(F(U)),$$

where $\bar{co}(F(U))$ denotes the closed convex hull of $F(U)$.

Such type of study was carried out by various authors for another classes, like, Liu [2], Prajapat and Raina [4] and Atshan and Wanas [1].

2 Main Results

Definition 1 A function $f \in \mathcal{A}$ is said to be in the class $T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ if it satisfies the subordination condition:

$$(5) \quad \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) \right. \\ \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \prec h(z),$$

where $\eta \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \delta \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and $h \in H$.

Theorem 1 Let $0 \leq \eta < \varepsilon$. Then

$$T(\varepsilon, \lambda, \alpha, \beta, k, \delta; h) \subset T(\eta, \lambda, \alpha, \beta, k, \delta; h).$$

Proof. Let $0 \leq \eta < \varepsilon$ and $f \in T(\varepsilon, \lambda, \alpha, \beta, k, \delta; h)$.

Assume that

$$(6) \quad g(z) = \frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z)}{z} \\ = 1 + \sum_{n=2}^{\infty} \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} \left[\sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^{n-1}.$$

It is obvious that the function g is holomorphic in U and $g(0) = 1$. Since $f \in T(\varepsilon, \lambda, \alpha, \beta, k, \delta; h)$, then we deduce that

$$(7) \quad \frac{1}{z} \left[\left(1 - \varepsilon \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) + \varepsilon \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \prec h(z).$$

Differentiating both sides of (6) with respect to z and using (3) and (7), we find that

$$\begin{aligned} & \frac{1}{z} \left[\left(1 - \varepsilon \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) + \varepsilon \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \\ &= g(z) + \varepsilon z g'(z) \prec h(z). \end{aligned}$$

An application of Lemma 1 with $\tau = \frac{1}{\varepsilon}$, yields

$$(8) \quad g(z) \prec h(z).$$

Evidently, $0 \leq \frac{\eta}{\varepsilon} < 1$ and that h is convex univalent in U , it follows from (6), (7) and (8) that

$$\begin{aligned} & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \\ &= \frac{\eta}{\varepsilon z} \left[\left(1 - \varepsilon \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) + \varepsilon \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] + \left(1 - \frac{\eta}{\varepsilon} \right) g(z) \prec h(z). \end{aligned}$$

Hence, $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ and the proof of Theorem 1 is completed.

Theorem 2 Let $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ be defined as in (1). Then the function

$$I(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (\operatorname{Re}(c) > -1),$$

is also in the class $T(\eta, \lambda, \alpha, \beta, k, \delta; h)$.

Proof. Let $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ be defined as in (1). Then, we find that

$$(9) \quad \begin{aligned} & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) \right. \\ & \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \prec h(z), \end{aligned}$$

We can easily see that

$$(10) \quad I(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n.$$

We have from (10) that $I \in \mathcal{A}$ and

$$(11) \quad f(z) = \frac{cI(z) + zI'(z)}{c+1}.$$

Define the function J by

$$(12) \quad \begin{aligned} J(z) = & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} I(z) \right. \\ & \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} I(z) \right]. \end{aligned}$$

Differentiating both sides of (12) with respect to z and using (9) and (11), we

obtain

$$\begin{aligned}
& \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) \right. \\
& \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \\
&= \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} \left(\frac{cI(z) + zI'(z)}{c+1} \right) \right. \\
& \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} \left(\frac{cI(z) + zI'(z)}{c+1} \right) \right] \\
&= \frac{c}{z(c+1)} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} I(z) \right. \\
& \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} I(z) \right] \\
& \quad + \frac{1}{z(c+1)} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} (zI'(z)) \right. \\
& \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} (zI'(z)) \right] \\
&= \frac{c}{c+1} J(z) + \frac{1}{c+1} (zJ'(z) + J(z)) = J(z) + \frac{1}{c+1} zJ'(z) \prec h(z).
\end{aligned}$$

An application of Lemma 1 with $\tau = c+1$, yields $J(z) \prec h(z)$. By using (12), we get

$$\begin{aligned}
& \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} I(z) \right. \\
& \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} I(z) \right] \prec h(z),
\end{aligned}$$

which implies that $I \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$.

Theorem 3 Let $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$, $g \in \mathcal{A}$ and

$$(13) \quad \operatorname{Re} \left\{ \frac{g(z)}{z} \right\} > \frac{1}{2}.$$

Then $f * g \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$.

Proof. Let $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ and $g \in \mathcal{A}$. Then, we have

$$(14) \quad \begin{aligned} & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} (f * g)(z) \right. \\ & \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} (f * g)(z) \right] \\ &= \left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \left(\frac{g(z)}{z} \right) * \left(\frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z)}{z} \right) \\ & \quad + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \left(\frac{g(z)}{z} \right) * \left(\frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z)}{z} \right) \\ &= \left(\frac{g(z)}{z} \right) * \varphi(z), \end{aligned}$$

where

$$(15) \quad \begin{aligned} \varphi(z) = & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) \right. \\ & \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \prec h(z). \end{aligned}$$

In view of (13), we note that the function $\frac{g(z)}{z}$ has the Herglotz representation

$$(16) \quad \frac{g(z)}{z} = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in U),$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U , then we find from (14), (15) and (16) that

$$\begin{aligned} & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} (f * g)(z) \right. \\ & \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} (f * g)(z) \right] \\ & = \int_{|x|=1} \varphi(xz) d\mu(x) \prec h(z). \end{aligned}$$

This shows that $f * g \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$.

Theorem 4 Let $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ and $g \in \mathcal{A}$ be prestarlike of order α , ($\alpha < 1$). Then $f * g \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$.

Proof. For $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; h)$ and $g \in \mathcal{A}$, from (14) (used in the proof of Theorem 3), we can write

$$\begin{aligned} (17) \quad & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} (f * g)(z) \right. \\ & \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} (f * g)(z) \right] \\ & = \frac{g(z) * (z\varphi(z))}{g(z) * z}, \quad (z \in U), \end{aligned}$$

where $\varphi(z)$ is defined as in (15). Since h is convex univalent in U , $\psi(z) \prec h(z)$, $g(z) \in \mathfrak{R}(\alpha)$ and $z \in S^*(\alpha)$, ($\alpha < 1$), it follows from (17) and Lemma 2, we obtain the result.

Theorem 5 Let $\eta > 0, \gamma > 0$ and $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; \gamma h + 1 - \gamma)$. If $\gamma \leq \gamma_0$, where

$$(18) \quad \gamma_0 = \frac{1}{2} \left(1 - \frac{1}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du \right)^{-1},$$

then $f \in T(0, \lambda, \alpha, \beta, k, \delta; h)$. The bound γ_0 is the sharp when $h(z) = \frac{1}{1-z}$.

Proof. Assume that

$$(19) \quad g(z) = \frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z)}{z}.$$

Let $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; \gamma h + 1 - \gamma)$ with $\eta > 0$ and $\gamma > 0$. Then, we obtain

$$\begin{aligned} & g(z) + \eta z g'(z) \\ &= \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) \right. \\ & \quad \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] \prec \gamma h(z) + 1 - \gamma. \end{aligned}$$

Making use of Lemma 1, we observe that

$$(20) \quad g(z) \prec \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z t^{\frac{1}{\eta}-1} h(t) dt + 1 - \gamma = (h * \phi)(z),$$

where

$$(21) \quad \phi(z) = \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \gamma.$$

If $0 < \gamma \leq \gamma_0$, where $\gamma_0 > 1$ is given by (18), then we find from (21) that

$$\begin{aligned} \operatorname{Re}(\phi(z)) &= \frac{\gamma}{\eta} \int_0^1 u^{\frac{1}{\eta}-1} \operatorname{Re} \left(\frac{1}{1-uz} \right) du + 1 - \gamma \\ &> \frac{\gamma}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \gamma \geq \frac{1}{2}. \end{aligned}$$

By using the Herglotz representation for $\phi(z)$, from (19) and (20), we have

$$\frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z)}{z} \prec (h * \phi)(z) \prec h(z).$$

Since h is convex univalent in U , then $f \in T(0, \lambda, \alpha, \beta, k, \delta; h)$.

For $h(z) = \frac{1}{1-z}$ and $f \in \mathcal{A}$ defined by

$$\frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z)}{z} = \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \gamma,$$

we obtain

$$\begin{aligned} & \frac{1}{z} \left[\left(1 - \eta \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m \right) \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z) \right. \\ & \left. + \eta \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \Upsilon_{\lambda, \alpha, \beta}^{k, \delta+1} f(z) \right] = \gamma h(z) + 1 - \gamma. \end{aligned}$$

Thus, $f \in T(\eta, \lambda, \alpha, \beta, k, \delta; \gamma h + 1 - \gamma)$. Also, for $\gamma > \gamma_0$, we have

$$\operatorname{Re} \left\{ \frac{\Upsilon_{\lambda, \alpha, \beta}^{k, \delta} f(z)}{z} \right\} \rightarrow \frac{\gamma}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \gamma < \frac{1}{2}, \quad (z \rightarrow -1),$$

which implies that $f \notin T(0, \lambda, \alpha, \beta, k, \delta; h)$. Thus the bound γ_0 cannot be increased when $h(z) = \frac{1}{1-z}$. This completes the proof of the theorem.

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