Ulam stability for Thunsdorff and Cauchy-Schwarz equations¹

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Abstract

We consider the equality case in Thunsdorff inequality and Cauchy-Schwarz inequality. For these two equations we prove Ulam stability.

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1 Introduction

Briefly speaking, an equation is called Ulam stable if for each "approximate solution" x , there exists an exact solution x_0 "near x ". The precise definitions, in various contexts, can be found, e.g., in [3].

We consider Thunsdorff's inequality (see [9], [8], [7]) and Cauchy-Schwarz inequality. The corresponding equality cases provide two equations for which we prove specific forms of Ulam stability.

2 Ulam stability for Thunsdorff equation

We start with Thunsdorff inequality:

Theorem 1 *Let $f \in C[0, 1]$ be convex, $f \geq 0$, $f(0) = 0$. If $0 < p < q$, then*

$$\left(\int_0^1 f^p \right)^{\frac{1}{p}} \leq (q+1)^{\frac{1}{q}} (p+1)^{-\frac{1}{p}} \left(\int_0^1 f^q \right)^{\frac{1}{q}}.$$

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We consider the particular case $p = 1$, $q = 2$. According to Th.1, if $f \in C[0, 1]$ is convex, $f \geq 0$ and $f(0) = 0$, then

$$2 \int_0^1 f(x) dx \leq \sqrt{3} \sqrt{\int_0^1 f^2(x) dx},$$

i.e.,

$$(1) \quad 3 \int_0^1 f^2(x) dx - 4 \left(\int_0^1 f(x) dx \right)^2 \geq 0.$$

Obviously, this inequality is stronger than the Cauchy-Schwarz inequality applied to f , namely

$$(2) \quad \int_0^1 f^2(x) dx - \left(\int_0^1 f(x) dx \right)^2 \geq 0.$$

We associate to (1) the Thunsdorff equation

$$(3) \quad 3 \int_0^1 f^2(x) dx - 4 \left(\int_0^1 f(x) dx \right)^2 = 0$$

and to (2) the Cauchy-Schwarz equation

$$(4) \quad \int_0^1 f^2(x) dx - \left(\int_0^1 f(x) dx \right)^2 = 0.$$

For each $\lambda \in \mathbb{R}$, the function $f(x) := \lambda x$, $x \in [0, 1]$, is a solution of (3), and $f(x) := \lambda$, $x \in [0, 1]$, is a solution of (4).

The following theorem expresses the Ulam stability of Thunsdorff's equation.

Theorem 2 *Let $\varepsilon > 0$ and $g \in C[0, 1]$ convex, $g \geq 0$, $g(0) = 0$, such that*

$$(5) \quad 3 \int_0^1 g^2(x) dx - 4 \left(\int_0^1 g(x) dx \right)^2 \leq 3\varepsilon.$$

Then there exists an exact solution of (3) such that

$$(6) \quad \int_0^1 (g(x) - f(x))^2 dx \leq \varepsilon.$$

Proof. Let $\lambda \in \mathbb{R}$. According to (5) we have

$$\begin{aligned} \int_0^1 (g(x) - \lambda x)^2 dx &= \int_0^1 g^2(x) dx - 2\lambda \int_0^1 xg(x) dx + \frac{1}{3}\lambda^2 \\ &\leq \frac{4}{3} \left(\int_0^1 g(x) dx \right)^2 + \varepsilon - 2\lambda \int_0^1 xg(x) dx + \frac{1}{3}\lambda^2. \end{aligned}$$

i.e.,

$$(7) \quad \int_0^1 (g(x) - \lambda x)^2 dx \leq \frac{1}{3} \left(\lambda^2 - 6\lambda \int_0^1 xg(x)dx + 4 \left(\int_0^1 g(x)dx \right)^2 \right) + \varepsilon.$$

Consider the equation

$$(8) \quad \lambda^2 - 6\lambda \int_0^1 xg(x)dx + 4 \left(\int_0^1 g(x)dx \right)^2 = 0.$$

Its discriminant is

$$\begin{aligned} \Delta &= 36 \left(\int_0^1 xg(x)dx \right)^2 - 16 \left(\int_0^1 g(x)dx \right)^2 \\ &= 4 \left(3 \int_0^1 xg(x)dx + 2 \int_0^1 g(x)dx \right) \left(3 \int_0^1 xg(x)dx - 2 \int_0^1 g(x)dx \right). \end{aligned}$$

Since $g \geq 0$, we have

$$(9) \quad 3 \int_0^1 xg(x)dx + 2 \int_0^1 g(x)dx \geq 0.$$

Moreover,

$$(10) \quad 3 \int_0^1 xg(x)dx - 2 \int_0^1 g(x)dx = \int_0^1 (3x - 2)g(x)dx.$$

The function g is convex, $g \geq 0$, $g(0) = 0$; it follows that it can be uniformly approximated on $[0, 1]$ with linear combinations of functions of the form $\varphi(x) := \max\{0, x - a\}$, $a \in [0, 1]$, all the coefficients of the linear combinations being positive.

Since

$$\int_0^1 (3x - 2) \max\{0, x - a\} dx = \frac{1}{2} a(1 - a)^2 \geq 0, \quad a \in [0, 1],$$

it follows that

$$(11) \quad \int_0^1 (3x - 2)g(x)dx \geq 0.$$

Now (10) and (11) show that

$$(12) \quad 3 \int_0^1 xg(x)dx - 2 \int_0^1 g(x)dx \geq 0.$$

From (9) and (12) we infer that $\Delta \geq 0$, and therefore the equation (8) has a real root λ_0 .

According to (7) we have

$$\int_0^1 (g(x) - \lambda_0 x)^2 dx \leq \varepsilon,$$

and so (6) is satisfied with $f(x) = \lambda_0 x$, $x \in [0, 1]$, which is an exact solution of (3).

3 Ulam stability for Cauchy-Schwarz equation

The Cauchy-Schwarz equation (4) has a similar property of Ulam stability.

Theorem 3 *Let $\varepsilon > 0$ and $g \in C[0, 1]$ such that*

$$(13) \quad \int_0^1 g^2(x)dx - \left(\int_0^1 g(x)dx \right)^2 \leq \varepsilon.$$

Then there exists an exact solution f of (4) such that

$$(14) \quad \int_0^1 (g(x) - f(x))^2 dx \leq \varepsilon.$$

Proof. Let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \int_0^1 (g(x) - \lambda)^2 dx &= \int_0^1 g^2(x)dx - 2\lambda \int_0^1 g(x)dx + \lambda^2 \\ &\leq \left(\int_0^1 g(x)dx \right)^2 + \varepsilon - 2\lambda \int_0^1 g(x)dx + \lambda^2 \\ &= \left(\lambda - \int_0^1 g(x)dx \right)^2 + \varepsilon. \end{aligned}$$

Taking $\lambda_0 = \int_0^1 g(x)dx$, we see that $\int_0^1 (g(x) - \lambda_0)^2 dx \leq \varepsilon$ and so (14) is satisfied with $f(x) = \lambda_0$, $x \in [0, 1]$, which is an exact solution of (4).

Remark 1 *The Ulam stability for some classical functional equations (Cauchy, Jensen, Quadratic equations) was investigated in [6], [3, Chapter 4]. In this case the best (i.e., smallest) Ulam constant was determined. For the pertinent definitions see e.g., [3, Chapter 4]. The Ulam stability can be investigated for other equations associated with classical inequalities which can be found, e.g., in [1], [2], [4], [5].*

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