Upper bound of second Hankel determinant for bi-univalent functions with respect to symmetric conjugate¹

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Abstract

In this article, our aim is to estimate an upper bounds for the second Hankel determinant $H_2(2)$ of a certain class of analytic and bi-univalent functions with respect to symmetric conjugate defined in the open unit disk U .

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1 Introduction

Let the notation $\mathcal{A}$ be the family of functions f that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and obeying the following normalization

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

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Indicate by S the class of all functions in $\mathcal{A}$ which are univalent in U . Also, let S_{sc}^* be the subclass of S consisting of functions given by (1) satisfying

$$Re \left\{ \frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}} \right\} > 0, \quad z \in U.$$

These functions are called starlike with respect to symmetric conjugate points and were introduced by El-Ashwah and Thomas [10]. The class can be extended to other class in U , namely convex functions with respect to symmetric conjugate points. Let C_{sc} denote the class of convex functions with respect to symmetric conjugate points and satisfy the conditions

$$Re \left\{ \frac{(zf'(z))'}{(f(z) - \overline{f(-\bar{z})})'} \right\} > 0, \quad z \in U.$$

The Koebe one-quarter theorem [9] ensures that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. So every $f \in S$ has an inverse function f^{-1} which satisfies $f^{-1}(f(z)) = z$, ($z \in U$) and $f(f^{-1}(w)) = w$, ($|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$), where

$$(2) \quad g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U . Let Σ denote the class of bi-univalent functions in U satisfying (1).

In fact, Srivastava et al. [21] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Caglar et al. [5], Orhan et al. [17], Altinkaya and Yalcin [3], Srivastava and Wanas [22] and others (see, for example [17, 19, 20, 22, 23, 24, 25, 26]). The problem of finding the coefficient estimates on $|a_n|$ ($n = 3, 4, \dots$) for functions $f \in \Sigma$ is still an open problem.

One of the significant tools in the theory of univalent functions is Hankel determinant which are utility, for example, in showing that a function of bounded characteristic in U , i.e., a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral

coefficients, is rational [6]. Also the Hankel determinant plays an important role in the study of singularities. Noonan and Thomas [16] defined the q^{th} Hankel determinant of $f \in \mathcal{A}$ for $n \geq 0$ and $q \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

The Hankel determinants

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

are well known as Fekete-Szegő and second Hankel determinant functionals, respectively. Fekete and Szegő [11] consider the further generalized functional $a_3 - \mu a_2^2$, where μ is real number. Recently, several authors established upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions (see [1, 2, 7, 13, 14, 15]). On the other hand, Zaprawa [27, 28] extended the study of the Fekete-Szegő problem for some classes of bi-univalent functions. Very recently, the upper bounds of $H_2(2)$ for some classes were discussed by Deniz et al. [8] (see also [4]).

To derive the desired bounds in our study, we shall require the following lemmas.

Lemma 1 [18]. *If the function $p \in \mathcal{P}$ is given by the series $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, then the sharp estimate $|p_k| \leq 2$ ($k = 1, 2, 3, \dots$) holds.*

Lemma 2 [12]. *If the function $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2 Main Results

We begin this section by defining the class $\mathcal{B}_{\Sigma}^{sc}(\delta, \gamma)$ as follows:

Definition 1 A function $f \in \Sigma$ is said to be in the class $\mathcal{B}_{\Sigma}^{sc}(\delta, \gamma)$ ($0 \leq \delta \leq 1$, $0 \leq \gamma < 1$) if it satisfies the conditions:

$$\operatorname{Re} \left\{ \frac{\delta z^3 f'''(z) + (1 + \delta) z^2 f''(z) + z f'(z)}{\delta z^2 \left(f(z) - \overline{f(-\bar{z})} \right)'' + (1 - \delta) z \left(f(z) - \overline{f(-\bar{z})} \right)' + \delta \left(f(z) - \overline{f(-\bar{z})} \right)} \right\} > \frac{\gamma}{2}$$

and

$$\operatorname{Re} \left\{ \frac{\delta w^3 g'''(w) + (1 + \delta) w^2 g''(w) + w g'(w)}{\delta w^2 \left(g(w) - \overline{g(-\bar{w})} \right)'' + (1 - \delta) w \left(g(w) - \overline{g(-\bar{w})} \right)' + \delta \left(g(w) - \overline{g(-\bar{w})} \right)} \right\} > \frac{\gamma}{2},$$

where $z, w \in U$ and $g = f^{-1}$ is given by (2).

Theorem 1 Let $f \in \mathcal{B}_{\Sigma}^{sc}(\delta, \gamma)$ be given by (1). Then

$$\begin{aligned} & |a_2 a_4 - a_3^2| \\ & \leq \begin{cases} \frac{(1-\gamma)^2}{2(9\delta+4)(\delta+2)^3} \left((1-\gamma)^2 [(4\delta+3)(\delta+2) - 2(9\delta+4)] + (\delta+2)^2 \right), \\ \gamma \in \left[0, 1 - \frac{(\delta+2) \left((9\delta+4) + \sqrt{(9\delta+4)^2 - 8[(4\delta+3)(\delta+2) - 2(9\delta+4)] [(9\delta+4)(\delta+2) - (4\delta+3)^2]} \right)}{2(4\delta+3)[(4\delta+3)(\delta+2) - 2(9\delta+4)]} \right]; \\ \frac{(1-\gamma)^2}{8(9\delta+4)(\delta+2)} \frac{(1-\gamma)^2 (9\delta+4) \Psi_1 - 6(1-\gamma) \Psi_2 + (\delta+2)^2 [8(9\delta+4)(\delta+2) - 9(4\delta+3)^2]}{(1-\gamma)^2 (4\delta+3)^2 [(4\delta+3)(\delta+2) - 2(9\delta+4)] - (1-\gamma) \Psi_4 + 2(\delta+2)^2 [(9\delta+4)(\delta+2) - (4\delta+3)^2]}, \\ \gamma \in \left(1 - \frac{(\delta+2) \left((9\delta+4) + \sqrt{(9\delta+4)^2 + 8(4\delta+3)^2 [(4\delta+3)(\delta+2) - 2(9\delta+4)]} \right)}{4(4\delta+3)[(4\delta+3)(\delta+2) - 2(9\delta+4)]}, 1 \right), \end{cases} \end{aligned}$$

where

$$\Psi_1 = 8(4\delta+3)(\delta+2)^2 - (9\delta+4)(16\delta+33),$$

and

$$\Psi_2 = (9\delta+4)(4\delta+3)(\delta+2).$$

Proof. Let $f \in \mathcal{B}_{\Sigma}^{sc}(\delta, \gamma)$. Then there exists $p, q \in \mathcal{P}$ such that

$$(3) \quad \frac{\delta z^3 f'''(z) + (1 + \delta) z^2 f''(z) + z f'(z)}{\delta z^2 \left(f(z) - \overline{f(-\bar{z})} \right)'' + (1 - \delta) z \left(f(z) - \overline{f(-\bar{z})} \right)' + \delta \left(f(z) - \overline{f(-\bar{z})} \right)} = \frac{\gamma}{2} + \frac{1 - \gamma}{2} p(z)$$

and

$$(4) \quad \frac{\delta w^3 g'''(w) + (1 + \delta) w^2 g''(w) + w g'(w)}{\delta w^2 \left(g(w) - \overline{g(-\bar{w})} \right)'' + (1 - \delta) w \left(g(w) - \overline{g(-\bar{w})} \right)' + \delta \left(g(w) - \overline{g(-\bar{w})} \right)} = \frac{\gamma}{2} + \frac{1 - \gamma}{2} q(w),$$

where $g = f^{-1}$ and p, q have the following series representations:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots.$$

By equating the coefficients in (3) and (4), it is obtained that

$$(5) \quad (\delta + 2) a_2 = \frac{1 - \gamma}{2} p_1,$$

$$(6) \quad (4\delta + 3) a_3 = \frac{1 - \gamma}{2} p_2,$$

$$(7) \quad 2(9\delta + 4) a_4 - \left(12\delta^2 + 2(1 - \delta)^2 + (10\delta + 3)(1 - \delta) + 8\delta + 1 \right) a_2 a_3 = \frac{1 - \gamma}{2} p_3,$$

$$(8) \quad -(\delta + 2) a_2 = \frac{1 - \gamma}{2} q_1,$$

$$(9) \quad (4\delta + 3) (2a_2^2 - a_3) = \frac{1 - \gamma}{2} q_2$$

and

$$\begin{aligned}
 (10) \quad & \left(12\delta^2 + 2(1-\delta)^2 + (10\delta+3)(1-\delta) + 8\delta+1 \right) (2a_2^2 - a_3)a_2 \\
 & - 2(9\delta+4)(5a_2^3 - 5a_2a_3 + a_4) \\
 & = \frac{1-\gamma}{2}q_3,
 \end{aligned}$$

In view of (5) and (8), it easy to see that

$$(11) \quad p_1 = -q_1$$

and

$$(12) \quad a_2 = \frac{1-\gamma}{2(\delta+2)}p_1.$$

By subtracting (6) from (9) and using (12), we get

$$(13) \quad a_3 = \frac{(1-\gamma)^2}{4(\delta+2)^2}p_1^2 + \frac{1-\gamma}{4(4\delta+3)}(p_2 - q_2).$$

Also, subtracting (7) from (10), further computations using (12) and (13) lead to

$$\begin{aligned}
 (14) \quad a_4 = & \frac{(4\delta+3)(1-\gamma)^3}{16(9\delta+4)(\delta+2)^2}p_1^3 + \frac{5(1-\gamma)^2}{16(\delta+2)(4\delta+3)}p_1(p_2 - q_2) \\
 & + \frac{1-\gamma}{8(9\delta+4)}(p_3 - q_3).
 \end{aligned}$$

Thus, using (12), (13) and (14), we deduce that

$$\begin{aligned}
 (15) \quad |a_2a_4 - a_3^2| = & \left| \frac{[(4\delta+3)(\delta+2) - 2(9\delta+4)](1-\gamma)^4}{32(9\delta+4)(\delta+2)^3}p_1^4 \right. \\
 & + \frac{(1-\gamma)^3}{32(4\delta+3)(\delta+2)^2}p_1^2(p_2 - q_2) + \frac{(1-\gamma)^2}{16(9\delta+4)(\delta+2)}p_1(p_3 - q_3) \\
 & \left. - \frac{(1-\gamma)^2}{16(4\delta+3)^2}(p_2 - q_2)^2 \right|.
 \end{aligned}$$

According to Lemma 2 and (11), we write

$$(16) \quad p_2 - q_2 = \frac{(4 - p_1^2)(x - y)}{2}$$

and

$$(17) \quad p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1(4 - p_1^2)(x + y)}{2} - \frac{p_1(4 - p_1^2)(x^2 + y^2)}{4} \\ + \frac{(4 - p_1^2) \left[(1 - |x|^2)z - (1 - |y|^2)w \right]}{2},$$

for some x, y, z and w with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$ and $|w| \leq 1$.

Substituting the calculated values from (16) and (17) in the right hand side of (15), we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| \frac{p_1^4(1 - \gamma)^4[(4\delta + 3)(\delta + 2) - 2(9\delta + 4)]}{32(9\delta + 4)(\delta + 2)^3} + \frac{p_1^2(4 - p_1^2)(1 - \gamma)^3(x - y)}{64(4\delta + 3)(\delta + 2)^2} \right. \\ &+ \frac{p_1^4(1 - \gamma)^2}{32(9\delta + 4)(\delta + 2)} + \frac{p_1^2(4 - p_1^2)(1 - \gamma)^2(x + y)}{32(9\delta + 4)(\delta + 2)} - \frac{p_1^2(4 - p_1^2)(1 - \gamma)^2(x^2 + y^2)}{64(9\delta + 4)(\delta + 2)} \\ &+ \left. \frac{p_1(4 - p_1^2)(1 - \gamma)^2 \left[(1 - |x|^2)z - (1 - |y|^2)w \right]}{32(9\delta + 4)(\delta + 2)} - \frac{(4 - p_1^2)^2(1 - \gamma)^2(x - y)^2}{64(4\delta + 3)^2} \right| \\ &\leq \frac{p_1^4(1 - \gamma)^4[(4\delta + 3)(\delta + 2) - 2(9\delta + 4)]}{32(9\delta + 4)(\delta + 2)^3} + \frac{p_1^4(1 - \gamma)^2}{32(9\delta + 4)(\delta + 2)} \\ &+ \frac{p_1(4 - p_1^2)(1 - \gamma)^2}{32(9\delta + 4)(\delta + 2)} + \left[\frac{p_1^2(4 - p_1^2)(1 - \gamma)^3}{64(4\delta + 3)(\delta + 2)^2} + \frac{p_1^2(4 - p_1^2)(1 - \gamma)^2}{32(9\delta + 4)(\delta + 2)} \right] (|x| + |y|) \\ &+ \left[\frac{p_1^2(4 - p_1^2)(1 - \gamma)^2}{64(9\delta + 4)(\delta + 2)} - \frac{p_1(4 - p_1^2)(1 - \gamma)^2}{32(9\delta + 4)(\delta + 2)} \right] (|x|^2 + |y|^2) \\ &+ \frac{(4 - p_1^2)^2(1 - \gamma)^2}{64(4\delta + 3)^2} (|x| + |y|)^2. \end{aligned}$$

Since the function p is in the class $\mathcal{P}$, so $|p_1| \leq 2$. Choosing $p_1 = p$, we can assume without loss of generality that $p \in [0, 2]$. Then, for $\eta_1 = |x| \leq 1$ and $\eta_2 = |y| \leq 1$, we have

$$|a_2a_4 - a_3^2| \leq L_1 + L_2(\eta_1 + \eta_2) + L_3(\eta_1^2 + \eta_2^2) + L_4(\eta_1 + \eta_2)^2 = M(\eta_1, \eta_2),$$

where

$$L_1 = L_1(p) = \frac{(1-\gamma)^2}{32(9\delta+4)(\delta+2)} \times \\ \times \left[\left(\frac{(1-\gamma)^2 [(4\delta+3)(\delta+2) - 2(9\delta+4)]}{(\delta+2)^2} + 1 \right) p^4 - 2p^3 + 8p \right] \geq 0,$$

$$L_2 = L_2(p) = \frac{p^2(4-p^2)(1-\gamma)^2}{64(\delta+2)} \left(\frac{1-\gamma}{(4\delta+3)(\delta+2)} + \frac{2}{9\delta+4} \right) \geq 0,$$

$$L_3 = L_3(p) = \frac{p(p-2)(4-p^2)(1-\gamma)^2}{64(9\delta+4)(\delta+2)} \leq 0,$$

$$L_4 = L_4(p) = \frac{(4-p^2)^2(1-\gamma)^2}{64(4\delta+3)^2} \geq 0.$$

We next maximize the function $M(\eta_1, \eta_2)$ on the closed square $[0, 1] \times [0, 1]$. We must investigate the maximum of $M(\eta_1, \eta_2)$ according to $p \in (0, 2)$, $p = 0$ and $p = 2$ taking into account the sign of $M_{\eta_1\eta_1} \cdot M_{\eta_2\eta_2} - (M_{\eta_1\eta_2})^2$.

Since $L_3 < 0$ and $L_3 + 2L_4 > 0$ for $p \in (0, 2)$, we conclude that

$$M_{\eta_1\eta_1} \cdot M_{\eta_2\eta_2} - (M_{\eta_1\eta_2})^2 < 0.$$

Therefore the function M cannot have a local maximum in the interior of the closed square $[0, 1] \times [0, 1]$.

Now, we investigate the maximum M on the boundary of the closed square $[0, 1] \times [0, 1]$.

When $\eta_1 = 0$ and $0 \leq \eta_2 \leq 1$ (similarly $\eta_2 = 0$ and $0 \leq \eta_1 \leq 1$), we have

$$M(0, \eta_2) = E(\eta_2) = L_1 + L_2\eta_2 + (L_3 + L_4)\eta_2^2.$$

(1) The case $L_3 + L_4 \geq 0$:

In this case for $0 < \eta_2 < 1$ and any fixed p with $0 \leq p \leq 2$, it is easily observed that $E'(\eta_2) = L_2 + 2(L_3 + L_4)\eta_2 > 0$. Therefore $E(\eta_2)$ is increasing function and hence, for fixed $p \in [0, 2)$, the maximum of $E(\eta_2)$ occurs at $\eta_2 = 1$ and

$$\max E(\eta_2) = E(1) = L_1 + L_2 + L_3 + L_4.$$

(2) The case $L_3 + L_4 < 0$:

Since $L_2 + 2(L_3 + L_4) \geq 0$ for $0 < \eta_2 < 1$ and any fixed p with $0 \leq p \leq 2$, it is easily observed that $L_2 + 2(L_3 + L_4) < L_2 + 2(L_3 + L_4)\eta_2 < L_2$. Therefore $E'(\eta_2) > 0$ and hence, for fixed $p \in [0, 2]$, the maximum of $E(\eta_2)$ occurs at $\eta_2 = 1$.

Also, for $p = 2$, we find

$$(18) \quad M(\eta_1, \eta_2) = \frac{(1-\gamma)^2}{2(9\delta+4)(\delta+2)} \left(\frac{(1-\gamma)^2 [(4\delta+3)(\delta+2) - 2(9\delta+3)]}{(\delta+2)^2} + 1 \right).$$

Taking into account the value (18) and the cases 1 and 2, for $0 \leq \eta_2 \leq 1$ and any fixed p with $0 \leq p \leq 2$,

$$\max E(\eta_2) = E(1) = L_1 + L_2 + L_3 + L_4.$$

When $\eta_1 = 1$ and $0 \leq \eta_2 \leq 1$ (similarly $\eta_2 = 1$ and $0 \leq \eta_1 \leq 1$), we have

$$M(1, \eta_2) = K(\eta_2) = L_1 + L_2 + L_3 + L_4 + (L_2 + 2L_4)\eta_2 + (L_3 + L_4)\eta_2^2.$$

Similarly to the above cases of $L_3 + L_4$, we find that

$$\max K(\eta_2) = K(1) = L_1 + 2L_2 + 2L_3 + 4L_4.$$

Since $E(1) \leq K(1)$ for $p \in [0, 2]$, $\max M(\eta_1, \eta_2) = M(1, 1)$ on the boundary of the closed square $[0, 1] \times [0, 1]$. Hence, the maximum of M occurs at $\eta_1 = 1$ and $\eta_2 = 1$ in the closed square $[0, 1] \times [0, 1]$.

Assume that $T : [0, 2] \rightarrow \mathbb{R}$ be defined by

$$(19) \quad T(p) = \max M(\eta_1, \eta_2) = M(1, 1) = L_1 + 2L_2 + 2L_3 + 4L_4.$$

Now, substituting the values of L_1, L_2, L_3 and L_4 in (19), we conclude that

$$\begin{aligned} T(p) = & \frac{(1-\gamma)^2}{32(9\delta+4)(4\delta+3)^2(\delta+2)^3} \left\{ \left[(1-\gamma)^2(4\delta+3)^2[(4\delta+3)(\delta+2) \right. \right. \\ & - 2(9\delta+4)] - (1-\gamma)(9\delta+4)(4\delta+3)(\delta+2) - 2(4\delta+3)^2(\delta+2)^2 \\ & \left. \left. + 2(9\delta+4)(\delta+2)^3 \right] p^4 + 4(\delta+2)[(1-\gamma)(9\delta+4)(4\delta+3) \right. \\ & \left. \left. + 3(4\delta+3)^2(\delta+2) - 4(9\delta+4)(\delta+2)^2 \right] p^2 + 32(9\delta+4)(\delta+2)^3 \right\}. \end{aligned}$$

Suppose that $T(p)$ has a maximum value in an interior of $p \in [0, 2]$, then

$$T'(p) \frac{(1-\gamma)^2}{8(9\delta+4)(4\delta+3)^2(\delta+2)^3} \left\{ \left[(1-\gamma)^2(4\delta+3)^2[(4\delta+3)(\delta+2) - 2(9\delta+4)] - (1-\gamma)(9\delta+4)(4\delta+3)(\delta+2) - 2(4\delta+3)^2(\delta+2)^2 + 2(9\delta+4)(\delta+2)^3 \right] p^3 + 8(\delta+2)[(1-\gamma)(9\delta+4)(4\delta+3) + 3(4\delta+3)^2(\delta+2) - 4(9\delta+4)(\delta+2)^2] p \right\}.$$

After some calculations, we consider the following cases:

Case1: Assume that

$$\begin{aligned} & (1-\gamma)^2(4\delta+3)^2[(4\delta+3)(\delta+2) - 2(9\delta+4)] \\ & - (1-\gamma)(9\delta+4)(4\delta+3)(\delta+2) \\ & + 2(\delta+2)^2[(9\delta+4)(\delta+2) - (4\delta+3)^2] \geq 0. \end{aligned}$$

Thus

$$\gamma \in \left[0, 1 - \frac{(\delta+2)((9\delta+4))}{2(4\delta+3)[(4\delta+3)(\delta+2) - 2(9\delta+4)]} + \frac{\sqrt{(9\delta+4)^2 - 8[(4\delta+3)(\delta+2) - 2(9\delta+4)][(9\delta+4)(\delta+2) - (4\delta+3)^2]}}{2(4\delta+3)[(4\delta+3)(\delta+2) - 2(9\delta+4)]} \right]$$

and so $T'(p) > 0$ for $p \in (0, 2)$. Since T is an increasing function in the interval $(0, 2)$, hence the maximum point of T must be on the boundary of $p \in [0, 2]$. Then, we have

$$\begin{aligned} \max_{0 \leq p \leq 2} T(p) &= T(2) = \frac{(1-\gamma)^2}{2(9\delta+4)(\delta+2)^3} \times \\ &\times \left((1-\gamma)^2[(4\delta+3)(\delta+2) - 2(9\delta+4)] + (\delta+2)^2 \right). \end{aligned}$$

Case2: Assume that

$$\begin{aligned} & (1-\gamma)^2(4\delta+3)^2[(4\delta+3)(\delta+2) - 2(9\delta+4)] \\ & - (1-\gamma)(9\delta+4)(4\delta+3)(\delta+2) \\ & + 2(\delta+2)^2[(9\delta+4)(\delta+2) - (4\delta+3)^2] < 0, \end{aligned}$$

that is,

$$\gamma \in \left(1 - \frac{(\delta+2)((9\delta+4))}{2(4\delta+3)[(4\delta+3)(\delta+2)-2(9\delta+4)]} + \sqrt{\frac{(9\delta+4)^2 - 8[(4\delta+3)(\delta+2)-2(9\delta+4)][(9\delta+4)(\delta+2)-(4\delta+3)^2]}{4(4\delta+3)[(4\delta+3)(\delta+2)-2(9\delta+4)]}}, 1 \right).$$

Therefore $T'(p) = 0$ implies the real critical point $p_{0_1} = 0$ or

$$p_{0_2} = \sqrt{\frac{-2(1-\gamma)\Psi_2 + 2(\delta+2)^2[4(9\delta+4)(\delta+2)-3(4\delta+3)^2]}{(1-\gamma)^2(4\delta+3)^2[(4\delta+3)(\delta+2)-2(9\delta+4)]}}.$$

When

$$\gamma \in \left(1 - \frac{(\delta+2)((9\delta+4))}{2(4\delta+3)[(4\delta+3)(\delta+2)-2(9\delta+4)]} + \sqrt{\frac{(9\delta+4)^2 - 8[(4\delta+3)(\delta+2)-2(9\delta+4)][(9\delta+4)(\delta+2)-(4\delta+3)^2]}{4(4\delta+3)[(4\delta+3)(\delta+2)-2(9\delta+4)]}}, 1 - \frac{(\delta+2)\left((9\delta+4) + \sqrt{(9\delta+4)^2 + 8(4\delta+3)^2[(4\delta+3)(\delta+2)-2(9\delta+4)]}\right)}{4(4\delta+3)[(4\delta+3)(\delta+2)-2(9\delta+4)]} \right),$$

we observe that $p_{0_2} \geq 2$, that is, p_{0_2} is out of the interval $(0, 2)$. Hence the maximum value of $T(p)$ occurs at $p_{0_1} = 0$ or p_{0_2} which contradicts our assumption of having the maximum value at the interior point of $p \in [0, 2]$. Since T is an increasing function in the interval $(0, 2)$, so the maximum point of T must be on the boundary of $p \in [0, 2]$, that is, $p = 2$. Therefore, we obtain

$$\begin{aligned} \max_{0 \leq p \leq 2} T(p) &= T(2) = \frac{(1-\gamma)^2}{2(9\delta+4)(\delta+2)^3} \times \\ &\times \left((1-\gamma)^2[(4\delta+3)(\delta+2)-2(9\delta+4)] + (\delta+2)^2 \right). \end{aligned}$$

When

$$\gamma \in \left(1 - \frac{(\delta+2)\left((9\delta+4) + \sqrt{(9\delta+4)^2 + 8(4\delta+3)^2[(4\delta+3)(\delta+2)-2(9\delta+4)]}\right)}{4(4\delta+3)[(4\delta+3)(\delta+2)-2(9\delta+4)]}, 1 \right),$$

we observe that that $p_{0_2} < 2$, that is, p_{0_2} is an interior of the interval $[0, 2]$. Since $T''(p_{0_2}) < 0$, so the maximum value of $T(p)$ occurs at $p = p_{0_2}$. Therefore, we obtain

$$\begin{aligned} \max_{0 \leq p \leq 2} T(p) &= T(p_{0_2}) = \frac{(1-\gamma)^2}{8(9\delta+4)(\delta+2)} \times \\ &\times \frac{(1-\gamma)^2(9\delta+4)\Psi_1 - 6(1-\gamma)\Psi_2 + (\delta+2)^2 \left[8(9\delta+4)(\delta+2) - 9(4\delta+3)^2 \right]}{(1-\gamma)^2(4\delta+3)^2 [(4\delta+3)(\delta+2) - 2(9\delta+4)] - (1-\gamma)\Psi_2} \\ &\quad \text{---} \\ &\quad + 2(\delta+2)^2 \left[(9\delta+4)(\delta+2) - (4\delta+3)^2 \right]. \end{aligned}$$

This completes the proof of our Theorem.

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