Differences and quotients of positive linear operators ¹

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Abstract

We consider the classical *Szász-Mirakyan* and *Szász-Mirakyan-Durrmeyer* operators, as well as a *Kantorovich* modification and a discrete version of it. The images of exponential functions under these operators are determined. We establish estimates involving differences and quotients of these images.

2010 Mathematics Subject Classification: 41A36.

Key words and phrases: differences of positive linear operators, quotients of positive linear operators, images of exponential functions.

1 Introduction

For $n \geq 1$ and $k = 0, 1, \dots$, let $s_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}$, $x \in [0, \infty)$.

Let

$$(1) \quad \mathbb{S}_n(f; x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) s_{n,k}(x),$$

be the Szász-Mirakyan operators, and

$$(2) \quad \mathbb{D}(f; x) := n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} f(t) s_{n,k}(t) dt, \quad x \in [0, \infty),$$

the Szász-Mirakyan-Durrmeyer operators. We consider those functions $f \in C[0, \infty)$ for which the integrals and the series are convergent.

The operators

$$(3) \quad \mathbb{K}_n(f; x) := \sum_{j=0}^{\infty} \left(n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t) dt \right) e^{-nx} \frac{(nx)^j}{j!}, \quad x \geq 0,$$

¹Received 26 July, 2020

Accepted for publication (in revised form) 20 September, 2020

are Kantorovich-type modifications of $\mathbb{K}_n$.
Moreover,

$$(4) \quad D_n(f; x) := \sum_{j=0}^n f\left(\frac{2j+1}{2n}\right) e^{-nx} \frac{(nx)^j}{j!}, \quad x \geq 0,$$

is the sequence of discrete operators associated with $\mathbb{K}_n$.

Details concerning these operators can be found in [1] and [2].

Let $n \geq 1$ be given. In Section 2 we shall determine the images of the functions $f_\lambda(t) := e^{\lambda t}$, $t \geq 0$, $0 < \lambda < n$, under the operators $\mathbb{S}_n$, $\mathbb{D}_n$, $\mathbb{K}_n$ and D_n .

Estimates involving differences and quotients of these images will be presented in Section 3.

2 Images of the function f_λ

Let $\theta := \frac{\lambda}{n} \in (0, 1)$.

Theorem 1 *With the above notation we have for $x \geq 0$,*

$$(5) \quad \textbf{i)} \quad \mathbb{S}_n(f_\lambda; x) = \exp\left(\frac{e^\theta - 1}{\theta} \lambda x\right),$$

$$(6) \quad \textbf{ii)} \quad \mathbb{D}_n(f_\lambda; x) = \frac{1}{1-\theta} \exp\left(\frac{1}{1-\theta} \lambda x\right),$$

$$(7) \quad \textbf{iii)} \quad \mathbb{K}_n(f_\lambda; x) = \frac{e^\theta - 1}{\theta} \exp\left(\frac{e^\theta - 1}{\theta} \lambda x\right),$$

$$(8) \quad \textbf{iv)} \quad D_n(f_\lambda; x) = \left(\exp \frac{\theta}{2}\right) \exp\left(\frac{e^\theta - 1}{\theta} \lambda x\right).$$

Proof. *i)* According to (1) we have

$$\begin{aligned} \mathbb{S}_n(f_\lambda; x) &= \sum_{k=0}^{\infty} e^{\frac{\lambda k}{n}} e^{-nx} \frac{(nx)^k}{k!} \\ &= e^{-nx} \sum_{k=0}^{\infty} \frac{(nxe^\theta)^k}{k!} \\ &= e^{-nx} e^{nxe^\theta} \\ &= \exp(nx(e^\theta - 1)) \\ &= \exp\left(\frac{e^\theta - 1}{\theta} \lambda x\right). \end{aligned}$$

ii) Now we use (1) to get

$$\mathbb{D}(f_\lambda; x) = n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} e^{\lambda t} e^{-nt} \frac{(nt)^k}{k!} dt.$$

The last integral is in fact

$$\begin{aligned} \frac{n^k}{k!} \int_0^{\infty} e^{-(n-\lambda)t} t^k dt &= \frac{n^k}{k!} \int_0^{\infty} e^{-s} \frac{s^k}{(n-\lambda)^{k+1}} ds \\ &= \frac{n^k}{(n-\lambda)^{k+1}} \frac{1}{k!} \Gamma(k+1) \\ &= \frac{n^k}{(n-\lambda)^{k+1}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{D}_n(f_\lambda; x) &= ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{n^k}{(n-\lambda)^{k+1}} \\ &= \frac{n}{n-\lambda} e^{-nx} \exp\left(\frac{n^2 x}{n-\lambda}\right) \\ &= \frac{1}{1-\theta} \exp\left(\frac{n^2 x}{n-\lambda} - nx\right) \\ &= \frac{1}{1-\theta} \exp\left(\frac{1}{1-\theta} \lambda x\right). \end{aligned}$$

iii) Using (3) we obtain

$$\begin{aligned} \mathbb{K}_n(f_\lambda; x) &= \frac{ne^{-nx}}{\lambda} \sum_{j=0}^{\infty} \left(e^{\frac{j+1}{n}\lambda} - e^{\frac{j}{n}\lambda} \right) \frac{(nx)^j}{j!} \\ &= \frac{e^{-nx}}{\theta} \sum_{j=0}^{\infty} (e^{(j+1)\theta} - e^{j\theta}) \frac{(nx)^j}{j!} \\ &= \frac{e^{-nx}}{\theta} (e^\theta - 1) \sum_{j=0}^{\infty} \frac{(nxe^\theta)^j}{j!} \\ &= \frac{e^\theta - 1}{\theta} e^{-nx} e^{nxe^\theta} \\ &= \frac{e^\theta - 1}{\theta} \exp\left(\frac{e^\theta - 1}{\theta} \lambda x\right). \end{aligned}$$

iv) According to (4),

$$\begin{aligned}
 D_n(f_\lambda; x) &= \sum_{j=0}^{\infty} \left(\exp \left(\frac{2j+1}{2n} \lambda - nx \right) \right) \frac{(nx)^j}{j!} \\
 &= \sum_{j=0}^{\infty} \left(\exp \frac{j\lambda}{n} \exp \left(\frac{\lambda}{2n} - nx \right) \right) \frac{(nx)^j}{j!} \\
 &= \left(\exp \left(\frac{\theta}{2} - nx \right) \right) \exp(nxe^\theta) \\
 &= e^{\frac{\theta}{2}} \exp \left(nx(e^\theta - 1) \right) \\
 &= e^{\frac{\theta}{2}} \exp \left(\frac{e^\theta - 1}{\theta} \lambda x \right).
 \end{aligned}$$

3 Differences and quotients of images

Due to the structure of equations (5)-(8), it is more convenient to study quotients of operators and differences of operators acting on exponential functions. For example,

$$(9) \quad \frac{\mathbb{K}_n(f_\lambda; x)}{\mathbb{S}_n(f_\lambda; x)} = \frac{n}{\lambda} (e^{\frac{\lambda}{n}} - 1), \quad x \geq 0,$$

and

$$(10) \quad \frac{D_n(f_\lambda; x)}{\mathbb{S}_n(f_\lambda; x)} = e^{\frac{\lambda}{2n}}, \quad x \geq 0.$$

For a fixed $\lambda > 0$, both quotients tend to 1 when $n \rightarrow \infty$.

Let us mention also the following result.

Proposition 1 For $x \geq 0$ and $0 < \lambda < n$,

$$(11) \quad 0 \leq \frac{\mathbb{K}_n(f_\lambda; x) - D_n(f_\lambda; x)}{\mathbb{S}_n(f_\lambda; x)} \leq (e - \sqrt{e} - 1) \left(\frac{\lambda}{n} \right)^2,$$

and

$$(12) \quad 1 \leq \frac{\mathbb{D}_n(f_\lambda; x)}{\mathbb{K}_n(f_\lambda; x)} \leq \left(\frac{n}{n - \lambda} \right)^{\frac{n\lambda x}{n - \lambda} + 1}.$$

Proof. Using (9) and (10) we have

$$\begin{aligned}
 \frac{\mathbb{K}_n(f_\lambda; x) - D_n(f_\lambda; x)}{\mathbb{S}_n(f_\lambda; x)} &= \frac{e^\theta - 1}{\theta} - e^{\frac{\theta}{2}} \\
 &= \sum_{j=2}^{\infty} \left(\frac{1}{(j+1)!} - \frac{1}{j!2^j} \right) \theta^j \\
 &\leq \theta^2 \sum_{j=2}^{\infty} \left(\frac{1}{(j+1)!} - \frac{1}{j!2^j} \right) \\
 &= \theta^2 (e - \sqrt{e} - 1),
 \end{aligned}$$

and so (11) is proved.

To prove (12), let us use the notation

$$a := \frac{e^\theta - 1}{\theta} = 1 + \frac{\theta}{2!} + \frac{\theta^2}{3!} + \dots ,$$

$$b := \frac{1}{1 - \theta} = 1 + \theta + \theta^2 + \dots .$$

Then $1 < a < b$ and

$$\begin{aligned} \frac{\mathbb{D}_n(f_\lambda; x)}{\mathbb{K}_n(f_\lambda; x)} &= \frac{a}{b} \exp \left((b - a)\lambda x \right) \\ &= \frac{b}{a} \exp \left(\frac{b - a}{\log \frac{a}{b}} \lambda x \log \frac{b}{a} \right) \\ &\leq \frac{b}{a} \exp \left(b\lambda x \log \frac{b}{a} \right) \\ &= \left(\frac{b}{a} \right)^{b\lambda x + 1} \leq b^{b\lambda x + 1} \\ &= \left(\frac{1}{1 - \theta} \right)^{\frac{\lambda x}{1 - \theta} + 1} \\ &= \left(\frac{n}{n - \lambda} \right)^{\frac{n\lambda x}{n - \lambda} + 1} . \end{aligned}$$

Remark 1 According to (11) and (12), the corresponding quotients tend to 0, respectively to 1, when n goes to infinity.

References

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