SOME SRIVASTAVA-BRAFMAN TYPE GENERATING RELATIONS FOR A GENERAL CLASS OF MULTI-INDEX AND MULTI-VARIABLE GOULD-HOPPER AND DATTOLI TYPE HYPERGEOMETRIC POLYNOMIALS

M. I. QURESHI, KALEEM A. QURAISHI AND RAM PAL

Abstract

In this article, we first introduce and study a new family of the multi-index and multi-variable Gould-Hopper and Dattoli type polynomials $\{H_n^{(c_m, c_{m-1}, \dots, c_3, c_2)}(a_1, a_2, \dots, a_m)\}$ defined by (2.1), which are an extension of different types of Hermite polynomials defined in section 1. We next consider multi-variable linear, bilinear and bilateral generating relations of the newly defined hypergeometric polynomials, using series iteration techniques. Further, we generalize these generating relations in the forms of multiple series identities involving bounded multiple sequences, Fox-Wright hypergeometric function and Srivastava-Daoust multi-variable hypergeometric function.

Mathematics Subject Classification 2010: Primary 33C45, 33C47, 33C50; Secondary 33C20, 33C70, 33C80, 42C05.

Keywords and Phrases: Generalized hypergeometric functions; Srivastava-Daoust multi-variable hypergeometric functions; Gould-Hopper polynomials; Classical Hermite polynomials; Extended family of the generalized Hermite polynomials; Generating functions; Multi-index and Multi-variable Dattoli type polynomials; Generating relations of Brafman, Mehler, Srivastava-Lavoie, Dattoli-Torre-Lorenzutta.

1. INTRODUCTION AND PRELIMINARIES

The Pochhammer symbol or generalized factorial function or shifted factorial is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)\cdots(a+n-1) & ; \text{ if } n = 1, 2, 3, \dots \\ 1 & ; \text{ if } n = 0 \\ n! & ; \text{ if } a = 1, n = 1, 2, 3, \dots \end{cases}$$

where $a \neq 0, -1, -2, \dots$ and the notation Γ stands for Gamma function.

If $m = 1, 2, 3, 4, \dots$ and $n = 0, 1, 2, 3, 4, \dots$ then

$$(b)_{mn} = m^{mn} \left(\frac{b}{m}\right)_n \left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-2}{m}\right)_n \left(\frac{b+m-1}{m}\right)_n$$

In 1933, E. M. Wright defined a more interesting generalized hypergeometric function of one variable [23, p.21(1.2.38, 1.2.40); 25, pp.50-51(1.5.21), p.179(34iii), p.395(23)] in the following forms:

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), (\alpha_2, A_2), \dots, (\alpha_p, A_p) & ; \\ (\beta_1, B_1), (\beta_2, B_2), \dots, (\beta_q, B_q) & ; \end{matrix} \right. \left. \begin{matrix} z \\ \end{matrix} \right] \\ = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \Gamma(\alpha_2 + nA_2) \cdots \Gamma(\alpha_p + nA_p)}{\Gamma(\beta_1 + nB_1) \Gamma(\beta_2 + nB_2) \cdots \Gamma(\beta_q + nB_q)} \frac{z^n}{n!}$$

where the coefficients $A_1, A_2, A_3, \dots, A_p, B_1, B_2, B_3, \dots, B_q$ are positive rational numbers and $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p, \beta_1, \beta_2, \beta_3, \dots, \beta_q$ are complex parameters.

$${}_p\Psi_q^* \left[\begin{matrix} (\alpha_1, A_1), (\alpha_2, A_2), \dots, (\alpha_p, A_p) & ; \\ (\beta_1, B_1), (\beta_2, B_2), \dots, (\beta_q, B_q) & ; \end{matrix} \quad z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{nA_1} (\alpha_2)_{nA_2} \cdots (\alpha_p)_{nA_p} z^n}{(\beta_1)_{nB_1} (\beta_2)_{nB_2} \cdots (\beta_q)_{nB_q} n!}$$

The above function makes sense when $\delta \equiv (1 + B_1 + B_2 + \cdots + B_q) - (A_1 + A_2 + \cdots + A_p) > 0$ and $0 < |z| < \infty; z \neq 0$

The equality holds only for suitably constrained values of $|z|$ or appropriately bounded $|z|$ i.e. $\delta = 0$ and $0 < |z| < R \equiv A_1^{-A_1} A_2^{-A_2} \cdots A_p^{-A_p} B_1^{B_1} B_2^{B_2} \cdots B_q^{B_q}$.

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), (\alpha_2, 1), \dots, (\alpha_p, 1) & ; \\ (\beta_1, 1), (\beta_2, 1), \dots, (\beta_q, 1) & ; \end{matrix} \quad z \right] = \frac{\prod_{j=1}^p \Gamma(\alpha_j)}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p & ; \\ \beta_1, \beta_2, \dots, \beta_q & ; \end{matrix} \quad z \right]$$

In 1969, H. M. Srivastava and M. C. Daoust defined extremely generalized hypergeometric function [23,p.37(21,22,23) see also pp.270-272;25,pp.64-65(18,19,20)] of n variables

$$\begin{aligned} & F_{D:E^{(1)};\dots;E^{(n)}}^{A:B^{(1)};\dots;B^{(n)}} \left(\begin{matrix} [(a_A) : \theta^{(1)}, \dots, \theta^{(n)}] : [(b_{B^{(1)}}^{(1)}) : \phi^{(1)}]; \dots; [(b_{B^{(n)}}^{(n)}) : \phi^{(n)}]; \\ [(d_D) : \psi^{(1)}, \dots, \psi^{(n)}] : [(e_{E^{(1)}}^{(1)}) : \delta^{(1)}]; \dots; [(e_{E^{(n)}}^{(n)}) : \delta^{(n)}]; \end{matrix} \quad z_1, \dots, z_n \right) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \Xi(m_1, \dots, m_n) \frac{z_1^{m_1}}{(m_1)!} \cdots \frac{z_n^{m_n}}{(m_n)!} \end{aligned}$$

where for the sake of convenience

$$\Xi(m_1, \dots, m_n) = \frac{\prod_{j=1}^A (a_j)_{m_1 \theta_j^{(1)} + \dots + m_n \theta_j^{(n)}} \prod_{j=1}^{B^{(1)}} (b_j^{(1)})_{m_1 \phi_j^{(1)} + \dots} \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n \phi_j^{(n)}}}{\prod_{j=1}^D (d_j)_{m_1 \psi_j^{(1)} + \dots + m_n \psi_j^{(n)}} \prod_{j=1}^{E^{(1)}} (e_j^{(1)})_{m_1 \delta_j^{(1)} + \dots} \prod_{j=1}^{E^{(n)}} (e_j^{(n)})_{m_n \delta_j^{(n)}}}$$

where the coefficients $\theta_j^{(k)}, j = 1, 2, \dots, A; \phi_j^{(k)}, j = 1, 2, \dots, B^{(k)}; \psi_j^{(k)}, j = 1, 2, \dots, D; \delta_j^{(k)}, j = 1, 2, \dots, E^{(k)}; \forall k \in \{1, 2, \dots, n\}$ are zero and real constants (positive or negative) and $(b_{B^{(k)}}^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}, j = 1, 2, \dots, B^{(k)}; \forall k \in \{1, 2, \dots, n\}$ with similar interpretations for others.

For notational purpose the coefficients in Srivastava-Daoust multiple hypergeometric function may be negative real numbers and zero [23,p.37(last three lines),p.270(last two lines)].

The Gould-Hopper polynomials $g_n^m(x, y)$ [8,p.512(18); see also 16,p.58(6.2)] defined in the year 1962, are the generalization of classical Hermite polynomials $H_n(x)$ [18,p.187(2)]. The notation

$H_n^{(m)}(x, y)$ for $g_n^m(x, y)$ was given by Dattoli et al.[8] and it is defined by [8;16;25,p.76(6)] in the form:

$$g_n^m(x, y) = H_n^{(m)}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{n!}{k! (n - mk)!} y^k x^{n-mk} \quad (1.1)$$

$$= x^n {}_mF_0 \left[\begin{array}{c} \Delta(m; -n) \quad ; \\ \hline \quad \quad \quad ; \end{array} \quad y \left(-\frac{m}{x} \right)^m \right] \quad (1.2)$$

where $n \in \mathbb{N}_0$, $m \in \{2, 3, 4, \dots\}$ and $\Delta(m; -n)$ abbreviates the array of m parameters given by

$$\frac{-n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}$$

and $[x]$ denotes the greatest integer function. It is being understood that the set $\Delta(0; \lambda)$ is empty.

When $m = 2$, $y = -1$ and replacing x by $2x$ in (1.2), these polynomials reduce to the classical Hermite polynomials given by

$$\begin{aligned} H_n(x) &= g_n^2(2x, -1) = H_n^{(2)}(2x, -1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n! (2x)^{n-2k}}{k! (n-2k)!} \\ &= (2x)^n {}_2F_0 \left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2} \quad ; \\ \hline \quad \quad \quad ; \end{array} \quad -\frac{1}{x^2} \right] \end{aligned} \quad (1.3)$$

$$H_n^{(2)}\left(x, -\frac{1}{4}\right) = \frac{H_n(x)}{2^n} \quad (1.4)$$

The Gould-Hopper polynomials are, infact, contained in certain general hypergeometric polynomials considered earlier by Brafman

$$\begin{aligned} &B_n^m[\alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \beta_2, \dots, \beta_s; x] \\ &= {}_{m+r}F_s[\Delta(m; -n), \alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \beta_2, \dots, \beta_s; x] \end{aligned}$$

where α_j and β_j are arbitrary complex parameters independent of n .

The equation (1.1) can be derived from the following generating relation[8,p.512(19)]:

$$\sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!} = \exp(xt + yt^m) \quad (1.5)$$

and reduces to Hermite polynomial of two variables[8,p.511(11)]:

$$H_n(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n! (2x)^{n-2k} y^k}{k! (n-2k)!} \quad (1.6)$$

$$= H_n^{(2)}(2x, -y) \quad (1.7)$$

which can be derived from the generating relation of Dattoli et al.[8,pp.510-511(8,11)]:

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \exp(2xt - yt^2) \quad (1.8)$$

$$H_n(x, 1) = H_n(x) \quad (1.9)$$

Dattoli et al.[14,p.403(25,26)] defined the following generating relation

$$\sum_{n=0}^{\infty} H_n^{(3,2)}(x, y, z) \frac{t^n}{n!} = \exp(xt + yt^2 + zt^3) \quad (1.10)$$

where

$$\begin{aligned} H_n^{(3,2)}(x, y, z) &= \sum_{p,q=0}^{2p+3q \leq n} \frac{n! x^{n-2p-3q} y^p z^q}{p! q! (n-2p-3q)!} \\ &= x^n F_{0;0;0}^{1;0;0} \left(\begin{matrix} [-n : 2, 3] : \text{---} ; \text{---} ; \\ \text{---} : \text{---} ; \text{---} ; \end{matrix} ; \begin{matrix} \frac{y}{(-x)^2}, \frac{z}{(-x)^3} \end{matrix} \right) \end{aligned} \quad (1.11)$$

Dattoli, Torre and Lorenzutta[14,pp.402,403,405(14,24,34)] gave the following generating relations

$$\sum_{m=0}^{\infty} H_{n+m}^{(3,2)}(w, x, y) \frac{t^m}{m!} = \exp(wt + xt^2 + yt^3) H_n^{(3,2)}(w + 2xt + 3yt^2, x + 3yt, y) \quad (1.12)$$

$$\sum_{m=0}^{\infty} H_{n+m}^{(3)}(w, y) \frac{t^m}{m!} = \exp(wt + yt^3) H_n^{(3,2)}(w + 3yt^2, 3yt, y) \quad (1.13)$$

In 1957, Brafman gave the following bilateral generating relation[24,p.306; 25,p.445-446(Q.No.11(i))]

$$\begin{aligned} &\sum_{n=0}^{\infty} H_n(x) B_n^q[\alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \beta_2, \dots, \beta_s : y] \frac{t^n}{n!} \\ &= \exp(2xt - t^2) \sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_m \cdots (\alpha_r)_m ((\frac{-t}{q})^q y)^m}{(\beta_1)_m (\beta_2)_m \cdots (\beta_s)_m m!} H_{qm}(x - t) \\ &= \exp(2xt - t^2) F_q^{(1)} \left[x - t, y \left(-\frac{t}{q} \right)^q \right] \end{aligned} \quad (1.14)$$

where

$$F_q^{(1)}[x, t] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^r (\alpha_j)_n}{\prod_{j=1}^s (\beta_j)_n} H_{qn}(x) \frac{t^n}{n!}$$

Srivastava and Brafman linear generating relation[5,p.948(28);20,p.67(3.6); see also 2,pp.2-3;4(Chapter 6);25,p.138(6,7)]

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} t^k H_k^{(m)}(x, h) \cong (1 - xt)^{-\lambda} {}_mF_0 \left[\begin{matrix} \Delta(m; \lambda) & ; \\ \text{---} & ; \end{matrix} \begin{matrix} h(\frac{mt}{1-xt})^m \end{matrix} \right] \quad (1.15)$$

Mehler's bilinear generating relation[3,p.166(4);18,p.198(2);25,p.83(13)]

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{n!} = (1 - 4t^2)^{-\frac{1}{2}} \exp \left[\frac{4xyt - 4(x^2 + y^2)t^2}{1 - 4t^2} \right] \quad (1.16)$$

Brafman's bilateral generating relation[18,p.198(last equation)]

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_2F_0 \left[\begin{matrix} -n, c & ; \\ \text{---} & ; \end{matrix} y \right] H_n(x) \frac{t^n}{n!} \\ & \cong \exp(2xt - t^2) \{1 + 2yt(x - t)\}^{-c} {}_2F_0 \left[\begin{matrix} \frac{c}{2}, \frac{c+1}{2} & ; \\ \text{---} & ; \end{matrix} -\frac{4y^2t^2}{(1+2xyt-2yt^2)^2} \right] \end{aligned} \quad (1.17)$$

Srivastava-Lavoie bilateral generating relation[24,p.306;25,pp.445-446(Q.No.11(ii))]

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(x) g_n^q(y, \lambda) \frac{t^n}{n!} &= \exp(2xyt - y^2t^2) \sum_{k=0}^{\infty} H_{qk}(x - yt) \frac{(\lambda t^q)^k}{k!} \\ &= \exp(2xyt - y^2t^2) F_q^{(2)}[x - yt, \lambda t^q] \end{aligned} \quad (1.18)$$

where

$$F_q^{(2)}[x, t] = \exp(x^2) \exp \left(t(-D_x)^q \right) \left\{ \exp(-x^2) \right\}$$

Additional generating relations

[18,p.197(1);25,p.419(13)]

$$\sum_{n=0}^{\infty} H_{n+k}(x) \frac{t^n}{n!} = \exp(2xt - t^2) H_k(x - t) \quad (1.19)$$

[1,p.125(6.3.1)]

$$\begin{aligned} & \sum_{m=0}^{\infty} H_{n+m}^{(4,3,2)}(w, x, y, z) \frac{t^m}{m!} = \exp(wt + xt^2 + yt^3 + zt^4) \times \\ & \times H_n^{(4,3,2)}(w + 2xt + 3yt^2 + 4zt^3, x + 3yt + 6zt^2, y + 4zt, z) \end{aligned} \quad (1.20)$$

[1,p.128(6.4.2)]

$$\sum_{m=0}^{\infty} H_{n+m}^{(4,2)}(w, x, z) \frac{t^m}{m!} = \exp(wt + xt^2 + zt^4) H_n^{(4,3,2)}(w + 2xt + 4zt^3, x + 6zt^2, 4zt, z) \quad (1.21)$$

[1,p.128(6.4.3)]

$$\sum_{m=0}^{\infty} H_{n+m}^{(4,3)}(w, y, z) \frac{t^m}{m!} = \exp(wt + yt^3 + zt^4) H_n^{(4,3,2)}(w + 3yt^2 + 4zt^3, 3yt + 6zt^2, y + 4zt, z) \quad (1.22)$$

[1,p.128(6.4.4)]

$$\sum_{m=0}^{\infty} H_{n+m}^{(4)}(w, z) \frac{t^m}{m!} = \exp(wt + zt^4) H_n^{(4,3,2)}(w + 4zt^3, 6zt^2, 4zt, z) \quad (1.23)$$

[1,p.128(6.4.5)]

$$\sum_{m=0}^{\infty} H_{n+m}^{(2)}(w, x) \frac{t^m}{m!} = \exp(wt + xt^2) H_n^{(2)}(w + 2xt, x) \quad (1.24)$$

Closely related and motivated by the work of Carlitz-Srivastava[6], Dattoli-Srivastava-Khan[10], Dattoli-Srivastava-Sacchetti[11], Dattoli-Srivastava-Zhukovsky[12;13], Djordjević-Srivastava[15] and Srivastava[21;22] on different types of hypergeometric polynomials and corresponding generating relations given by (1.1)-(1.24), we define a multi-index, multi-variable Gould-Hopper and Dattoli type polynomials in section 2.

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers, therefore numerator and denominator parameters are adjusted in such a way that each term is completely well defined and meaningful then without any loss of convergence, we have the following applications.

2. MULTI-INDEX, MULTI-VARIABLE GOULD-HOPPER AND DATTOLI TYPE POLYNOMIALS

Definition:

If the indices $c_m > c_{m-1} > c_{m-2} > \cdots > c_4 > c_3 > c_2 \geq 2$; $c_2, c_3, \dots, c_m \in \mathbb{N}$, then we define

$$\begin{aligned} & H_n^{(c_m, c_{m-1}, \dots, c_3, c_2)}(a_1, a_2, \dots, a_m) \\ &= (a_1)^n F_{0:0;0;\dots;0}^{1:0;0;\dots;0} \left(\begin{array}{c} [-n : c_2, c_3, \dots, c_m] : \\ \hline \end{array} \right. \\ & \quad \left. \begin{array}{c} \text{---} ; \text{---} ; \cdots ; \text{---} ; \\ \text{---} ; \text{---} ; \cdots ; \text{---} ; \end{array} \quad (-a_1)^{-c_2} a_2, (-a_1)^{-c_3} a_3, \dots, (-a_1)^{-c_m} a_m \right) \end{aligned} \quad (2.1)$$

where $a_1 \neq 0$ and $m \geq 2$.

Generating relation:

If the indices $c_m > c_{m-1} > c_{m-2} > \cdots > c_2 \geq 2$; $c_2, c_3, \dots, c_m \in \mathbb{N}$; the complex variable $a_1 \neq 0$, $v \neq 0$ and at least one of the remaining complex variables $a_2, a_3, \dots, a_m$ is non-zero; then

$$\exp(a_1 v + a_2 v^{c_2} + a_3 v^{c_3} + \cdots + a_m v^{c_m})$$

$$= \sum_{n=0}^{\infty} H_n^{(c_m, c_{m-1}, \dots, c_3, c_2)}(a_1, a_2, \dots, a_m) \frac{v^n}{n!} \quad (2.2)$$

Using series iteration technique, we can derive (2.2)

3. GENERALIZATION OF GENERATING RELATIONS (1.13), (1.19), (1.23) AND (1.24)

Any values of parameters and variables leading to the result which do not make sense are tacitly excluded, then

$$\sum_{n=0}^{\infty} H_{n+k}^{(m)}(x, y) \frac{t^n}{n!} = \exp(xt + yt^m) \times \\ \times H_k^{(m, m-1, \dots, 3, 2)} \left(x + myt^{m-1}, \binom{m}{2} yt^{m-2}, \binom{m}{3} yt^{m-3}, \dots, \binom{m}{m-2} yt^2, \binom{m}{m-1} yt, y \right) \quad (3.1)$$

where k is a non-negative integer and $\binom{m}{r}$ is the Binomial coefficient.

Proof:

In view of the definitions of Gould-Hopper polynomials (1.1), multi-index, multi-variable Dattoli type polynomials (2.1), generating relations (1.5), (2.2) and applying the method of Rainville [18, pp.196-197(1)], we can obtain (3.1).

4. GENERALIZATION OF SRIVASTAVA AND BRAFMAN LINEAR GENERATING RELATION (1.15)

Any values of parameters and variables leading to the result which do not make sense are tacitly excluded, then

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} H_k^{(m, m-1, \dots, 3, 2)}(a_1, a_2, \dots, a_m) \cong (1 - a_1 t)^{-\lambda} F_{0;0;0;\dots;0}^{1;0;0;\dots;0} \left(\begin{array}{c} [\lambda : 2, 3, \dots, m] : \\ \hline \text{---; ---; } \cdots \text{; ---;} \\ \text{---; ---; } \cdots \text{; ---;} \end{array} ; \frac{a_2 t^2}{(1-a_1 t)^2}, \frac{a_3 t^3}{(1-a_1 t)^3}, \dots, \frac{a_m t^m}{(1-a_1 t)^m} \right) \quad (4.1)$$

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} H_k^{(c_m, c_{m-1}, \dots, c_3, c_2)}(a_1, a_2, \dots, a_m) \cong (1 - a_1 t)^{-\lambda} F_{0;0;0;\dots;0}^{1;0;0;\dots;0} \left(\begin{array}{c} [\lambda : c_2, c_2, \dots, c_m] : \\ \hline \text{---; ---; } \cdots \text{; ---;} \\ \text{---; ---; } \cdots \text{; ---;} \end{array} ; \frac{a_2 t^{c_2}}{(1-a_1 t)^{c_2}}, \frac{a_3 t^{c_3}}{(1-a_1 t)^{c_3}}, \dots, \frac{a_m t^{c_m}}{(1-a_1 t)^{c_m}} \right) \quad (4.2)$$

Proofs:

Generating relations (4.1) and (4.2) can be obtained by the methods of Rainville [18, p.190(1)];

see also 2, pp.2-3;4, Chapter 6] and Srivastava-Manocha[25, p.138(6,7)].

5. BILINEAR AND BILATERAL GENERATING RELATIONS

Any values of parameters and variables leading to the result which do not make sense are tacitly excluded, then

Generalization of Brafman, Mehler, Srivastava-Lavoie relations (1.14), (1.16), (1.18)

$$\sum_{n=0}^{\infty} H_n^{(p)}(x, g) H_n^{(q)}(y, h) \frac{t^n}{n!}$$

$$= \exp\{xyt + h(xt)^q\} \sum_{n=0}^{\infty} \frac{(gt^p)^n}{n!} H_{pn}^{(q, q-1, \dots, 3, 2)} \left(y + qh(xt)^{q-1}, \binom{q}{2} h(xt)^{q-2}, \binom{q}{3} h(xt)^{q-3}, \dots, h \right) \quad (5.1)$$

It is to be noted that R.H.S. cannot be simplified further using series iteration technique.

Generalization of Brafman's bilateral relation (1.17)

$$\sum_{n=0}^{\infty} {}_2F_0 \left[\begin{matrix} -n, g & ; \\ & z \end{matrix} \right] H_n^{(m)}(x, y) \frac{t^n}{n!}$$

$$\cong \exp(xt + yt^m) \{1 + tz(x + myt^{m-1})\}^{-g} F_{0;0;0;\dots;0}^{1;0;0;\dots;0} \left(\begin{matrix} [g : 2, 3, \dots, m] : \text{---} ; \text{---} ; \dots ; \\ \text{---} : \text{---} ; \text{---} ; \dots ; \end{matrix} \right)$$

$$\text{---} ; \frac{\binom{m}{2} yt^m (-z)^2}{\{1 + tz(x + myt^{m-1})\}^2}, \frac{\binom{m}{3} yt^m (-z)^3}{\{1 + tz(x + myt^{m-1})\}^3}, \dots, \frac{\binom{m}{m-1} yt^m (-z)^{m-1}}{\{1 + tz(x + myt^{m-1})\}^{m-1}}, \frac{yt^m (-z)^m}{\{1 + tz(x + myt^{m-1})\}^m} \text{---} \quad (5.2)$$

Proofs:

In view of the definitions of Gould-Hopper polynomials (1.1), Srivastava-Daoust function, multiple series identity of Srivastava[25, p.101(Lemma 3(6))] using linear generating relations (3.1), (4.1) and applying the series iteration technique of Rainville[18, pp.197-198; see also 17], we can derive (5.1) and (5.2).

6. UNIFICATION OF GENERATING RELATIONS (1.12), (1.13), (1.19)-(1.24) AND (3.1)

Any values of parameters and variables leading to the result which do not make sense are tacitly excluded, then

$$\sum_{n=0}^{\infty} H_{n+k}^{(m, m-1, \dots, 3, 2)}(a_1, a_2, \dots, a_m) \frac{t^n}{n!} = \exp(a_1 t + a_2 t^2 + \dots + a_m t^m) \times$$

$$\times H_k^{(m, m-1, \dots, 3, 2)}(b_1, b_2, \dots, b_m) \quad (6.1)$$

where

$$b_p = \sum_{j=p}^m a_j \binom{j}{j-p} t^{j-p} \quad (6.2)$$

k is a non-negative integer; $m \geq 2$.

Proof:

Using series iteration technique and applying the method of Rainville[18,p.196], we can derive (6.1).

7. FURTHER GENERALIZATIONS

Any values of parameters and variables leading to the result which do not make sense are tacitly excluded, then

General theorem: Suppose $\{\Omega(n_1; n_2, n_3, \dots, n_m)\}$, $\forall n_i \in \mathbb{N}_0$ is a bounded multiple sequence of arbitrary real or complex numbers (which is independent of the complex variables $z_1, z_2, \dots, z_m$ and t); indices $c_2, c_3, \dots, c_m$ are arbitrary positive integers, then

$$\begin{aligned} & \sum_{n_1, n_2, \dots, n_m=0}^{\infty} \Omega(n_1; n_2, n_3, \dots, n_m) \frac{(z_1 t)^{n_1}}{n_1!} \prod_{j=2}^m \frac{(z_j t^{c_j})^{n_j}}{n_j!} \\ &= \sum_{n_1=0}^{\infty} \left[\frac{z_1^{n_1}}{n_1!} \sum_{\substack{M \leq n_1 \\ n_2, n_3, \dots, n_m=0}}^{M \leq n_1} (-n_1)_M \Omega(n_1 - M; n_2, n_3, \dots, n_m) \prod_{j=2}^m \left\{ \frac{z_j}{(-z_1)^{c_j}} \right\}^{n_j} \frac{1}{n_j!} \right] t^{n_1} \quad (7.1) \\ & \quad \left(c_m > c_{m-1} > \dots > c_3 > c_2 \geq 2; M = c_2 n_2 + c_3 n_3 + \dots + c_m n_m; z_1 \neq 0; t \neq 0 \right) \end{aligned}$$

provided that each of the series involved is absolutely convergent.

Proof:

Suppose left hand side of (7.1) is denoted by S , then

$$S = \sum_{n_1, n_2, \dots, n_m=0}^{\infty} \Omega(n_1; n_2, n_3, \dots, n_m) \frac{z_1^{n_1}}{n_1!} \frac{z_2^{n_2}}{n_2!} \dots \frac{z_m^{n_m}}{n_m!} t^{n_1 + c_2 n_2 + c_3 n_3 + \dots + c_m n_m}$$

Now replacing n_1 by $n_1 - c_2 n_2 - c_3 n_3 - \dots - c_m n_m$, applying multiple series manipulation formula[25,p.102(Lemma 4)] and properties of factorial function, after simplification we get the right hand side of (7.1).

By suitably specializing the bounded coefficients $\Omega(n_1; n_2, n_3, \dots, n_m)$, complex variables $z_1, z_2, \dots, z_m$ and arbitrary positive integers $c_2, c_3, \dots, c_m$ in above general theorem and in view of the series identity of Srivastava[19,p.4(12); see also 25,p.52(3)], we get the following multi-variable hypergeometric generating relations (7.2)-(7.5).

Throughout this paper the figure occurring below or above the curly brackets at any place indicate the total number of similar quantities or pairs covered by it.

Special Cases:**Case I:**

If the parameters $b_1, b_2, \dots, b_B, d_1, d_2, \dots, d_D \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \dots\}$; indices $c_m > c_{m-1} > c_{m-2} > \dots > c_2 \geq 2$; $c_2, c_3, \dots, c_m \in \mathbb{N}$; the complex variable $a_1 \neq 0, v \neq 0$ and at least one of the remaining complex variables $a_2, a_3, \dots, a_m$ is non-zero; the coefficients $\beta_1, \beta_2, \dots, \beta_B, \delta_1, \delta_2, \dots, \delta_D \notin \mathbb{N}$, then

$$\begin{aligned}
 {}_B\Psi_D^* & \left[\begin{array}{l} (b_1, \beta_1), (b_2, \beta_2), \dots, (b_B, \beta_B) \quad ; \\ (d_1, \delta_1), (d_2, \delta_2), \dots, (d_D, \delta_D) \quad ; \end{array} \quad a_1 v + a_2 v^{c_2} + a_3 v^{c_3} + \dots + a_m v^{c_m} \right] \\
 &= \sum_{n=0}^{\infty} \frac{(a_1)^n \prod_{j=1}^B (b_j)_{n\beta_j}}{\prod_{j=1}^D (d_j)_{n\delta_j}} F_{D+1: \underbrace{0; 0; \dots; 0}_{(m-1)}}^{\underbrace{0; 0; \dots; 0}_{(m-1)}} \left(\begin{array}{l} [-n : c_2, c_3, \dots, c_m], \\ [b_1 + n\beta_1 : -\beta_1(c_2 - 1), -\beta_1(c_3 - 1), \dots, -\beta_1(c_m - 1)], \dots, \\ [d_1 + n\delta_1 : -\delta_1(c_2 - 1), -\delta_1(c_3 - 1), \dots, -\delta_1(c_m - 1)], \dots, \\ [b_B + n\beta_B : -\beta_B(c_2 - 1), -\beta_B(c_3 - 1), \dots, -\beta_B(c_m - 1)] : \\ [d_D + n\delta_D : -\delta_D(c_2 - 1), -\delta_D(c_3 - 1), \dots, -\delta_D(c_m - 1)] : \\ \text{---} ; \dots ; \text{---} ; \\ a_2 (-a_1)^{-c_2}, a_3 (-a_1)^{-c_3}, \dots, a_m (-a_1)^{-c_m} \end{array} \right) \frac{v^n}{n!} \quad (7.2) \\
 & \text{---} ; \dots ; \text{---} ;
 \end{aligned}$$

provided that each of the series involved is absolutely convergent.

- (i) $1 + (\delta_1 + \delta_2 + \dots + \delta_D) - (\beta_1 + \beta_2 + \dots + \beta_B) > 0$, then $0 < |a_1 v + a_2 v^{c_2} + a_3 v^{c_3} + \dots + a_m v^{c_m}| < \infty$
- (ii) $1 + (\delta_1 + \delta_2 + \dots + \delta_D) - (\beta_1 + \beta_2 + \dots + \beta_B) = 0$, then $0 < |a_1 v + a_2 v^{c_2} + a_3 v^{c_3} + \dots + a_m v^{c_m}| < \beta_1^{-\beta_1} \beta_2^{-\beta_2} \dots \beta_B^{-\beta_B} \delta_1^{-\delta_1} \delta_2^{-\delta_2} \dots \delta_D^{-\delta_D}$

Case II:

If the parameters $b_1, b_2, \dots, b_B, d_1, d_2, \dots, d_D \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \dots\}$; indices $c_m > c_{m-1} > c_{m-2} > \dots > c_2 \geq 2$; $c_2, c_3, \dots, c_m \in \mathbb{N}$; the complex variable $a_1 \neq 0, v \neq 0$ and at least one of the remaining complex variables $a_2, a_3, \dots, a_m$ is non-zero; the coefficients $\beta_1, \beta_2, \dots, \beta_B, \delta_1, \delta_2, \dots, \delta_D \in \mathbb{N}$, then

$$\begin{aligned}
 {}_B\Psi_D^* & \left[\begin{array}{l} (b_1, \beta_1), (b_2, \beta_2), \dots, (b_B, \beta_B) \quad ; \\ (d_1, \delta_1), (d_2, \delta_2), \dots, (d_D, \delta_D) \quad ; \end{array} \quad a_1 v + a_2 v^{c_2} + a_3 v^{c_3} + \dots + a_m v^{c_m} \right] \\
 &= \sum_{n=0}^{\infty} \frac{(a_1)^n \prod_{j=1}^B (b_j)_{n\beta_j}}{\prod_{j=1}^D (d_j)_{n\delta_j}} F_{B+1: \underbrace{0; 0; \dots; 0}_{(m-1)}}^{\underbrace{0; 0; \dots; 0}_{(m-1)}} \left(\begin{array}{l} [-n : c_2, c_3, \dots, c_m], \\ [b_1 + n\beta_1 : -\beta_1(c_2 - 1), -\beta_1(c_3 - 1), \dots, -\beta_1(c_m - 1)], \dots, \\ [d_1 + n\delta_1 : -\delta_1(c_2 - 1), -\delta_1(c_3 - 1), \dots, -\delta_1(c_m - 1)], \dots, \\ [b_B + n\beta_B : -\beta_B(c_2 - 1), -\beta_B(c_3 - 1), \dots, -\beta_B(c_m - 1)] : \\ [d_D + n\delta_D : -\delta_D(c_2 - 1), -\delta_D(c_3 - 1), \dots, -\delta_D(c_m - 1)] : \\ \text{---} ; \dots ; \text{---} ; \\ a_2 (-a_1)^{-c_2}, a_3 (-a_1)^{-c_3}, \dots, a_m (-a_1)^{-c_m} \end{array} \right) \frac{v^n}{n!}
 \end{aligned}$$

$$\begin{aligned}
& [1 - d_1 - n\delta_1 : \delta_1(c_2 - 1), \delta_1(c_3 - 1), \dots, \delta_1(c_m - 1)], \dots, \\
& [1 - b_1 - n\beta_1 : \beta_1(c_2 - 1), \beta_1(c_3 - 1), \dots, \beta_1(c_m - 1)], \dots, \\
& [1 - d_D - n\delta_D : \delta_D(c_2 - 1), \delta_D(c_3 - 1), \dots, \delta_D(c_m - 1)] : \\
& [1 - b_B - n\beta_B : \beta_B(c_2 - 1), \beta_B(c_3 - 1), \dots, \beta_B(c_m - 1)] : \\
& \text{---} ; \dots ; \text{---} ; \\
& \text{---} ; \dots ; \text{---} ; \left(\frac{a_2 (-1)^{(c_2-1)(\beta_1+\beta_2+\dots+\beta_B)}}{(-1)^{(c_2-1)(\delta_1+\delta_2+\dots+\delta_D)} (-a_1)^{c_2}}, \dots, \frac{a_m (-1)^{(c_m-1)(\beta_1+\beta_2+\dots+\beta_B)}}{(-1)^{(c_m-1)(\delta_1+\delta_2+\dots+\delta_D)} (-a_1)^{c_m}} \right) \frac{v^n}{n!} \quad (7.3)
\end{aligned}$$

provided that each of the series involved is absolutely convergent.

Case III:

If the parameters $b_1, b_2, \dots, b_B, d_1, d_2, \dots, d_D \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \dots\}$; indices $c_m > c_{m-1} > c_{m-2} > \dots > c_2 \geq 2$; $c_2, c_3, \dots, c_m \in \mathbb{N}$; the complex variable $a_1 \neq 0, v \neq 0$ and at least one of the remaining complex variables $a_2, a_3, \dots, a_m$ is non-zero, then

$$\begin{aligned}
& {}_B F_D \left[\begin{matrix} (b_B) & ; \\ & a_1 v + a_2 v^{c_2} + a_3 v^{c_3} + \dots + a_m v^{c_m} \\ (d_D) & ; \end{matrix} \right] \\
& = \sum_{n=0}^{\infty} \frac{(a_1)^n \prod_{j=1}^B (b_j)_n}{\prod_{j=1}^D (d_j)_n} F_{B+1}^{D+1} \left(\begin{matrix} \overbrace{0; 0; \dots; 0}^{(m-1)} \\ \underbrace{0; 0; \dots; 0}_{(m-1)} \end{matrix} \right) \left(\begin{matrix} [-n : c_2, c_3, \dots, c_m], [1 - d_1 - n : c_2 - 1, c_3 - 1, \dots, c_m - 1], \\ [1 - b_1 - n : c_2 - 1, c_3 - 1, \dots, c_m - 1], \\ [1 - d_2 - n : c_2 - 1, c_3 - 1, \dots, c_m - 1], \dots, [1 - d_D - n : c_2 - 1, c_3 - 1, \dots, c_m - 1] : \text{---} ; \\ [1 - b_2 - n : c_2 - 1, c_3 - 1, \dots, c_m - 1], \dots, [1 - b_B - n : c_2 - 1, c_3 - 1, \dots, c_m - 1] : \text{---} ; \\ \text{---} ; \dots ; \text{---} ; \\ \text{---} ; \dots ; \text{---} ; \end{matrix} \right) \frac{v^n}{n!} \quad (7.4)
\end{aligned}$$

provided that each of the series involved is absolutely convergent.

Case IV:

If the parameters $b_1, b_2, \dots, b_B, d_1, d_2, \dots, d_D \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \dots\}$; the complex variable $a_1 \neq 0, v \neq 0$ and at least one of the remaining complex variables $a_2, a_3, \dots, a_m$ is non-zero, then

$$\begin{aligned}
& {}_B F_D \left[\begin{matrix} (b_B) & ; \\ & a_1 v + a_2 v^2 + a_3 v^3 + \dots + a_m v^m \\ (d_D) & ; \end{matrix} \right] \\
& = \sum_{n=0}^{\infty} \frac{(a_1)^n \prod_{j=1}^B (b_j)_n}{\prod_{j=1}^D (d_j)_n} F_{B+1}^{D+1} \left(\begin{matrix} \overbrace{0; 0; \dots; 0}^{(m-1)} \\ \underbrace{0; 0; \dots; 0}_{(m-1)} \end{matrix} \right) \left(\begin{matrix} [-n : 2, 3, \dots, m], [1 - d_1 - n : 1, 2, 3, \dots, (m-1)], \\ [1 - b_1 - n : 1, 2, 3, \dots, (m-1)], \end{matrix} \right)
\end{aligned}$$

$$\begin{aligned}
& [1 - d_2 - n : 1, 2, 3, \dots, (m-1)], \dots, [1 - d_D - n : 1, 2, 3, \dots, (m-1)] : \text{---}; \\
& [1 - b_2 - n : 1, 2, 3, \dots, (m-1)], \dots, [1 - b_B - n : 1, 2, 3, \dots, (m-1)] : \text{---}; \\
& \text{---}; \dots; \text{---}; \\
& \text{---}; \dots; \text{---}; \left((-1)^{2+(B-D)}(a_1)^{-2}a_2, (-1)^{3+2(B-D)}(a_1)^{-3}a_3, \dots, (-1)^{m+(m-1)(B-D)}(a_1)^{-m}a_m \right) \frac{v^n}{n!}
\end{aligned}
\tag{75}$$

provided that each of the series involved is absolutely convergent.

ACKNOWLEDGEMENT

Authors are thankful to the referee for their valuable suggestions and comments for the improvement of this paper.

REFERENCES

1. Ahmad, Nadeem; *The Study of Certain Multivariable Gaussian Hypergeometric Series*, Ph.D. Thesis, Jamia Millia Islamia(A Central University), New Delhi, India, 2006.
2. Andhare, P. G. and Chaudhary, M. S.; Generating Function Involving Generalized Hermite Polynomials, *Bull. Marathwada Math. Soc.*, **4** (2003), 1-3.
3. Bell, W. W.; *Special Functions for Scientists and Engineers*, D. Van Nostrand Company Ltd., London, 1968.
4. Bhadana, K. G.; *A Study of Hypergeometric Function, I-Function of N-Variable with Applications*, Ph.D. Thesis, University of Kota, Kota, Rajasthan, India, 2007.
5. Brafman, F.; Generating Functions of Jacobi and Related Polynomials, *Proc. Amer. Math. Soc.*, **2** (1951), 942-949.
6. Carlitz, L. and Srivastava, H. M.; Some New Generating Functions for the Hermite Polynomials, *J. Math. Anal. Appl.*, **149**(2) (1990), 513-520.
7. Dattoli, G.; Differential Identities, Generalized Polynomials and Applications in Physics and Mathematics, *Centre de Recherches Mathematiques CRM, Proceedings and Lecture Notes*, **25** (2000).
8. Dattoli, G., Chiccoli, C., Lorenzutta, S., Maino, G. and Torre, A.; Generalized Bessel Functions and Generalized Hermite Polynomials, *J. Math. Anal. Appl.*, **178** (1993), 509-516.
9. Dattoli, G., Lorenzutta, S. and Sacchetti, D.; A Note on Operational Rules for Hermite and Laguerre Polynomials, Operational Rules and Special Polynomials, *Internat. J. Math. Stat. Sci.*, **9**(2) (2000) 227-238.
10. Dattoli, G., Srivastava, H. M. and Khan, S.; Operational Versus Lie-algebraic Methods and the Theory of Multi-variable Hermite polynomials, *Integral Transforms Spec. Funct.*, **16**(1) (2005), 81-91.
11. Dattoli, G., Srivastava, H. M. and Sacchetti, D.; The Hermite Polynomials and the Bessel Functions from a General Point of View, *Internat. J. Math. and Math. Sci.*, **57** (2003), 3633-3642.

12. Dattoli, G., Srivastava, H. M. and Zhukovsky, K; Orthogonality Properties of the Hermite and Related Polynomials, *J. Comput. Appl. Math.*, **182**(1) (2005), 165-172.
13. Dattoli, G., Srivastava, H. M. and Zhukovsky, K; Operational Methods and Differential Equations with Applications to initial-Value Problems, *Appl. Math. Comput.*, **184**(2) (2007), 979-1001.
14. Dattoli, G., Torre, A. and Lorenzutta, S.; Operational Identities and Properties of Ordinary and Generalized Special Functions, *J. Math. Anal. Appl.*, **236** (1999), 399-414.
15. Djordjević, G. B. and Srivastava, H. M.; Some Generating Functions and Other Properties Associated with the Generalized Hermite and Related Polynomials, *Integral Transforms Spec. Funct.*, **22**(12) (2011), 895-906.
16. Gould, H. W. and Hopper, A. T.; Operational Formulas Connected with Two Generalizations of Hermite Polynomials, *Duke Math. J.*, **29** (1962), 51-63.
17. McBride, E. B.; *Obtaining Generating Functions*, (Springer Tracts in Natural Philosophy) Springer-Verlag, New York, Heidelberg, Berlin, 1971.
18. Rainville, E. D.; *Special Functions*, The Macmillan Co. Inc., New York, 1960; Reprinted by Chelsea Publ. Co. Bronx, New York, 1971.
19. Srivastava, H. M.; Certain Double Integrals Involving Hypergeometric Functions, *Jñānābha Sect. A*, **1** (1971), 1-10.
20. Srivastava, H. M.; On q -Generating Functions and Certain Formulas of David Zeitlin, *Illinois J. Math.*, **15** (1971), 64-72.
21. Srivastava, H. M.; A Note on a Generating Function for the Generalized Hermite Polynomials, *Nederl. Akad. Wetensch. Proc. Ser. A*, **79** = *Indag. Math.*, **38** (1976), 457-461.
22. Srivastava, H. M.; Some Bilateral Generating Functions for a Certain Class of Special Functions. I and II, *Nederl. Akad. Wetensch. Proc. Ser. A* **83** = *Indag. Math.*, **42** (1980), 221-233 and 234-246.
23. Srivastava, H. M. and Karlsson, P. W.; *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
24. Srivastava, H. M. and Lavoie, J.-L.; A Certain Method of Obtaining Bilateral Generating Functions, *Nederl. Akad. Wetensch. Proc. Ser. A* **78**=*Indag Math.*, **37** (1975), 304-320.
25. Srivastava, H. M. and Manocha, H. L.; *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

M. I. Qureshi

Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025 (India)

E-mail: miqureshi_delhi@yahoo.co.in

Kaleem A. Quraishi

Mathematics Section, Mewat Engineering College (Waqf),
Palla, Nuh, Mewat-122107, Haryana (India)
E-mail: kaleemspn@yahoo.co.in

Ram Pal

Department of Applied Sciences and Humanities, Aryabhat Polytechnic,
G. T. Karnal Road, Delhi-110033 (India)
E-mail: rampal1966@rediffmail.com