The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results

E. SET, M. Z. SARIKAYA, M. E. ÖZDEMİR AND H. YILDIRIM

Abstract

In this paper, we establish Hermite-Hadamard type inequalities for s -convex functions in the second sense and m -convex functions via fractional integrals. The analysis used in the proofs is fairly elementary.

Mathematics Subject Classification 2000: 26D07, 26D10, 26D15, 47A63

Additional Key Words and Phrases: s -convex functions, Hermite-Hadamard type inequalities, Riemann-Liouville fractional integral.

1. INTRODUCTION

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$.

The function f is said to be convex on I if inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

In [10], Hudzik and Maligranda considered, among others, the class of functions which are s -convex in the second sense. This class of functions is defined as the following:

DEFINITION 1. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. This class of s -convex functions is usually denoted by K_s^2 .

It can be easily seen that for $s=1$, s -convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [12], G. Toader considered the class of m -convex functions: another intermediate between the usual convexity and starshaped convexity.

DEFINITION 2. *The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be m -convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m -concave if $(-f)$ is m -convex.

Obviously, for $m=1$ Definition D.1 recaptures the concept of standard convex functions on $[a, b]$ and for $m=0$ the concept starshaped functions.

One of the most famous inequalities for convex functions is Hermite-Hadamard inequality. This double inequality is stated as follows (see for example [5] and [14]): Let f be a convex function on some nonempty interval $[a, b]$ of real line $\mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[16]).

In [8], Hermite-Hadamard inequality for s -convex functions in the second sense is proved by Dragomir *et. al.*

THEOREM 1. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}. \quad (1.2)$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

In [11], Kirmaci et. al. established a new Hermite-Hadamard type inequality which holds for s -convex functions in the second sense. It is given in the next theorem.

THEOREM 2. Let $f : I \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° such that $f' \in L_1([a, b])$, where $a, b \in I$, $a < b$. If $|f'|^q$ is s -convex on $[a, b]$ for some fixed $s \in (0, 1)$ and $q \geq 1$, then:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left[\frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \right]^{\frac{1}{q}} \left[|f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{q}}. \end{aligned} \quad (1.3)$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

DEFINITION 3. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. Here is $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found ([24],[25] and [26]).

For some recent results connected with fractional integral inequalities see ([18]-[29])

In [28] Sarkaya *et. al.* proved a variant of the identity established by Dragomir *et. al.* in [6] for fractional integrals which is stated in the following lemma.

LEMMA 1. Let $f : [a, b] \rightarrow \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &= \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) dt. \end{aligned} \quad (1.4)$$

The aim of this paper is to establish Hermite-Hadamard inequality and Hermite-Hadamard type inequalities for s -convex functions in the second sense and m -conex functions via Riemann-Liouville fractional integral.

2. HERMITE-HADAMARD TYPE INEQUALITIES FOR SOME CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

2.1. For s -convex functions

Hermite-Hadamard inequality can be represented for s -convex functions in fractional integral forms as follows:

THEOREM 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a s -convex mapping in the second sense on $[a, b]$, then the following inequalities for fractional integrals with $\alpha > 0$ and $s \in (0, 1)$ hold:*

$$\begin{aligned} 2^{s-1} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[\frac{J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)}{2} \right] \\ &\leq \left[\frac{1}{(\alpha+s)} + \beta(\alpha, s+1) \right] \frac{f(a) + f(b)}{2} \end{aligned} \quad (2.1)$$

where β is Euler Beta function.

PROOF. Since f is a s -convex mapping in the second sense on $[a, b]$, we have for $x, y \in [a, b]$ with $\lambda = \frac{1}{2}$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2^s}. \quad (2.2)$$

Now, let $x = ta + (1-t)b$ and $y = (1-t)a + tb$ with $t \in [0, 1]$. Then, we get by (2.2) that:

$$2^s f\left(\frac{a+b}{2}\right) \leq f(ta + (1-t)b) + f((1-t)a + tb) \quad (2.3)$$

for all $t \in [0, 1]$

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
& \frac{2^s}{\alpha} f\left(\frac{a+b}{2}\right) \\
& \leq \int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt + \int_0^1 t^{\alpha-1} f((1-t)a + tb) dt \\
& = \frac{1}{(b-a)^\alpha} \int_a^b (b-u)^{\alpha-1} f(u) du - \frac{1}{(a-b)^\alpha} \int_a^b (a-v)^{\alpha-1} f(v) dv \\
& = \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)]
\end{aligned}$$

i.e.

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[\frac{J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)}{2} \right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a s -convex mapping in the second sense, then, for $t \in [0, 1]$, it yields

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

and

$$f((1-t)a + tb) \leq (1-t)^s f(a) + t^s f(b).$$

By adding these inequalities we have

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq [t^s + (1-t)^s] (f(a) + f(b)). \quad (2.4)$$

Thus, multiplying both sides of (2.4) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
& \int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt + \int_0^1 t^{\alpha-1} f((1-t)a + tb) dt \\
& \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} [t^s + (1-t)^s] dt
\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq [1 + (\alpha + s)\beta(\alpha, s + 1)] \frac{f(a) + f(b)}{(\alpha + s)}$$

where the proof is completed.

REMARK 1. If we choose $\alpha = 1$ in Theorem 3, then the inequalities (2.1) become the inequalities (1.2) of Theorem 1.

Using Lemma 1, we can obtain the following fractional integral inequality for s -convex in the second sense:

THEOREM 4. Let $f : [a, b] \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $|f'|^q$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1)$ and $q \geq 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\ & \leq \frac{b-a}{2} \left[\frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha} \right) \right]^{\frac{q-1}{q}} \left\{ \beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; \alpha+1, s+1\right) \right. \\ & \quad \left. + \frac{2^{\alpha+s} - 1}{(\alpha+s+1)2^{\alpha+s}} \right\} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}. \end{aligned} \quad (2.5)$$

PROOF. Suppose that $q = 1$. From Lemma 1 and using the properties of modulus, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\ & \leq \frac{b-a}{2} \int_0^1 \left| (1-t)^\alpha - t^\alpha \right| |f'(ta + (1-t)b)| dt. \end{aligned} \quad (2.6)$$

Since $|f'|$ is s -convex on $[a, b]$, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\
 & \leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| [t^s |f'(a)| + (1-t)^s |f'(b)|] dt \\
 & = \frac{b-a}{2} \left\{ \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] [t^s |f'(a)| + (1-t)^s |f'(b)|] dt \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] [t^s |f'(a)| + (1-t)^s |f'(b)|] dt \right\} \\
 & = \frac{b-a}{2} \left\{ |f'(a)| \left[\int_0^{\frac{1}{2}} t^s (1-t)^\alpha dt - \int_0^{\frac{1}{2}} t^{s+\alpha} dt \right] \right. \\
 & \quad \left. + |f'(b)| \left[\int_0^{\frac{1}{2}} (1-t)^{s+\alpha} dt - \int_0^{\frac{1}{2}} (1-t)^s t^\alpha dt \right] \right. \\
 & \quad \left. + |f'(a)| \left[\int_{\frac{1}{2}}^1 t^{\alpha+s} dt - \int_{\frac{1}{2}}^1 t^s (1-t)^\alpha dt \right] \right. \\
 & \quad \left. + |f'(b)| \left[\int_{\frac{1}{2}}^1 (1-t)^s t^\alpha dt - \int_{\frac{1}{2}}^1 (1-t)^{s+\alpha} dt \right] \right\}. \tag{2.7}
 \end{aligned}$$

Since

$$\int_0^{\frac{1}{2}} t^s (1-t)^\alpha dt = \int_{\frac{1}{2}}^1 (1-t)^s t^\alpha dt = \beta\left(\frac{1}{2}; s+1, \alpha+1\right),$$

$$\int_0^{\frac{1}{2}} (1-t)^s t^\alpha dt = \int_{\frac{1}{2}}^1 t^s (1-t)^\alpha dt = \beta\left(\frac{1}{2}; \alpha+1, s+1\right),$$

$$\int_0^{\frac{1}{2}} t^{s+\alpha} dt = \int_{\frac{1}{2}}^1 (1-t)^{s+\alpha} dt = \frac{1}{2^{s+\alpha+1}(s+\alpha+1)}$$

and

$$\int_0^{\frac{1}{2}} (1-t)^{s+\alpha} dt = \int_{\frac{1}{2}}^1 t^{s+\alpha} dt = \frac{1}{(s+\alpha+1)} - \frac{1}{2^{s+\alpha+1}(s+\alpha+1)}.$$

We obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\ & \leq \frac{b-a}{2} [|f'(a)| + |f'(b)|] \\ & \quad \times \left\{ \beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; \alpha+1, s+1\right) + \frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}} \right\} \end{aligned}$$

which completes the proof for this case. Suppose now that $q > 1$. Since

$|f'|^q$ is s -convex on $[a, b]$, we know that for every $t \in [0, 1]$

$$|f'(ta + (1-t)b)|^q \leq t^s |f'(a)|^q + (1-t)^s |f'(b)|^q, \quad (2.8)$$

so using well know Hölder's inequality (see for example [17]) for $\frac{1}{p} + \frac{1}{q} = 1, (q > 1)$ and (2.8) in (2.6), we have successively

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
& \leq \frac{b-a}{2} \int_0^1 \left| (1-t)^\alpha - t^\alpha \right| |f'(ta + (1-t)b)| dt \\
& = \frac{b-a}{2} \int_0^1 \left| (1-t)^\alpha - t^\alpha \right|^{1-\frac{1}{q}} \left| (1-t)^\alpha - t^\alpha \right|^{\frac{1}{q}} |f'(ta + (1-t)b)| dt \\
& \leq \frac{b-a}{2} \left(\int_0^1 \left| (1-t)^\alpha - t^\alpha \right| dt \right)^{\frac{q-1}{q}} \left(\int_0^1 \left| (1-t)^\alpha - t^\alpha \right| |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{b-a}{2} \left[\frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha} \right) \right]^{\frac{q-1}{q}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} \\
& \quad \times \left\{ \beta \left(\frac{1}{2}; s+1, \alpha+1 \right) - \beta \left(\frac{1}{2}; s+1, \alpha+1 \right) + \frac{2^{\alpha+s} - 1}{(\alpha+s+1)2^{\alpha+s}} \right\}
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
\int_0^1 \left| (1-t)^\alpha - t^\alpha \right| dt &= \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha \right] dt + \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha \right] dt \\
&= \frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha} \right)
\end{aligned}$$

which completes the proof.

REMARK 2. If we take $\alpha=1$ in Theorem 4, then the inequality (2.5) becomes the inequality (1.3) of Theorem 2.

2.2. For m -convex functions

We start with the following theorem:

THEOREM 5. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is m -convex mapping on $[a, b]$, then the following inequalities for fractional integral with $\alpha > 0$ and $m \in (0, 1]$ hold:*

$$\begin{aligned} \frac{2}{\Gamma(\alpha+1)} f\left(\frac{m(a+b)}{2}\right) &\leq \frac{1}{(mb-ma)^\alpha} J_{(ma)^+}^\alpha f(mb) + \frac{m}{(b-a)^\alpha} J_{b^-}^\alpha f(a) \\ &\leq \frac{f(ma) + m^2 f\left(\frac{b}{m}\right)}{(\alpha+1)} + m \frac{f(a) + f(b)}{\alpha(\alpha+1)} \end{aligned} \quad (2.9)$$

PROOF. Since f is m -convex functions, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

and if we choose $t = \frac{1}{2}$, we get

$$f\left(\frac{1}{2}(x + my)\right) \leq \frac{f(x) + mf(y)}{2}.$$

Now, let $x = mta + m(1-t)b$ and $y = (1-t)a + tb$ with $t \in [0, 1]$. Then we get

$$f\left(\frac{1}{2}m(a+b)\right) \leq \frac{f(mta + m(1-t)b) + mf((1-t)a + tb)}{2}. \quad (2.10)$$

Multiplying both sides of (2.10) by $t^{\alpha-1}$, then integrating the resulting inequalities with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
& \frac{1}{\alpha} f\left(\frac{1}{2}m(a+b)\right) \\
& \leq \frac{1}{2} \int_0^1 t^{\alpha-1} f(mta + m(1-t)b) dt + \frac{m}{2} \int_0^1 t^{\alpha-1} f((1-t)a + tb) dt \\
& = \frac{1}{2} \int_{mb}^{ma} \left(\frac{u-mb}{ma-mb}\right)^{\alpha-1} f(u) \frac{du}{m(a-b)} + \frac{m}{2} \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v) \frac{dv}{b-a} \\
& = \frac{1}{2(mb-ma)^\alpha} \int_{ma}^{mb} (mb-u)^{\alpha-1} f(u) du + \frac{m}{2} \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{b^-}^\alpha f(a)
\end{aligned}$$

which the first inequality is proved.

By the m -convexity of f , we also have

$$\begin{aligned}
& \frac{1}{2} [f(mta + m(1-t)b) + mf((1-t)a + tb)] \\
& \leq \frac{1}{2} \left[mtf(a) + m(1-t)f(b) + m(1-t)f(a) + m^2 f\left(\frac{b}{m}\right) \right]
\end{aligned}$$

for all $t \in [0,1]$. Multiplying both sides of above inequality by $t^{\alpha-1}$ and integrating over $t \in [0,1]$, we get

$$\begin{aligned}
& \frac{1}{(mb-ma)^\alpha} \int_{ma}^{mb} (mb-u)^{\alpha-1} f(u) du + \frac{m}{(b-a)^\alpha} \int_a^b (v-a)^{\alpha-1} f(v) dv \\
& \leq \frac{f(ma) + m^2 f\left(\frac{b}{m}\right)}{(\alpha+1)} + m \frac{f(a) + f(b)}{\alpha(\alpha+1)}
\end{aligned}$$

which this gives the second part of (2.9).

COROLLARY 1. *Under the conditions in Theorem 5 with $\alpha = 1$, then the following inequality hold:*

$$\begin{aligned}
f\left(\frac{m(a+b)}{2}\right) & \leq \frac{1}{(b-a)} \int_a^b \frac{f(mx) + mf(x)}{2} dx \\
& \leq \frac{1}{2} \left[\frac{f(ma) + m^2 f\left(\frac{b}{m}\right)}{2} + m \frac{f(a) + f(b)}{2} \right].
\end{aligned} \tag{2.11}$$

REMARK 3. If we take $m=1$ in Corollary kk, then the inequalities (2.11) become the inequalities (1.1).

THEOREM 6. Let $f : [0, \infty) \rightarrow \mathbb{R}$, be m -convex functions with $m \in (0, 1]$, $0 \leq a < b$ and $f \in L_1[a, b]$. $F(x, y)_{(t)} : [0, 1] \rightarrow \mathbb{R}$ are defined as the following:

$$F(x, y)_{(t)} = \frac{1}{2} [f(tx + m(1-t)y) + f((1-t)x + mty)].$$

Then, we have

$$\frac{1}{(b-a)^\alpha} \int_a^b (b-u)^{\alpha-1} F\left(u, \frac{a+b}{2}\right)_{\left(\frac{b-u}{b-a}\right)} du \leq \frac{\Gamma(\alpha)}{2(b-a)^\alpha} J_{a^+}^\alpha f(b) + \frac{m}{2\alpha} f\left(\frac{a+b}{2}\right)$$

for all $t \in [0, 1]$.

PROOF. Since f and g are m -convex functions, we have

$$\begin{aligned} F(x, y)_{(t)} &\leq \frac{1}{2} [tf(x) + m(1-t)f(y) + (1-t)f(x) + mtf(y)] \\ &= \frac{1}{2} [f(x) + mf(y)] \end{aligned}$$

and so,

$$F\left(x, \frac{a+b}{2}\right)_{(t)} \leq \frac{1}{2} \left[f(x) + mf\left(\frac{a+b}{2}\right) \right].$$

If we choose $x = ta + (1-t)b$, we have

$$F\left(ta + (1-t)b, \frac{a+b}{2}\right)_{(t)} \leq \frac{1}{2} \left[f(ta + (1-t)b) + mf\left(\frac{a+b}{2}\right) \right]. \quad (2.12)$$

Thus multiplying both sides of (2.12) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} &\int_0^1 t^{\alpha-1} F\left(ta + (1-t)b, \frac{a+b}{2}\right)_{(t)} dt \\ &\leq \frac{1}{2} \left[\int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt + \int_0^1 t^{\alpha-1} mf\left(\frac{a+b}{2}\right) dt \right]. \end{aligned}$$

Thus, if we use the change of the variable $u = ta + (1-t)b$, $t \in [0,1]$, then have the conclusion.

REFERENCES

- [1] Alomari, M., Darus, M., *On the Hadamard's inequality for log-convex functions on the coordinates*, Journal of Inequalities and Applications, vol. 2009, Article ID 283147, 13 pages, 2009.
- [2] Azpeitia, A.G., *Convex functions and the Hadamard inequality*, Rev. Colombiana Math., 28 (1994), 7-12.
- [3] Bakula, M.K., Özdemir, M.E., Pečarić, J., *Hadamard type inequalities for m -convex and (α, m) -convex functions*, J. Ineq. Pure and Appl. Math., 9(4) (2008), Art. 96.
- [4] Bakula, M. K., Pečarić, J., *Note on some Hadamard-type inequalities*, Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.
- [5] Dragomir, S. S., Pearce, C. E. M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
- [6] Dragomir, S. S., Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., 11(5) (1998), 91-95.
- [7] Dragomir, S. S., *On some new inequalities of Hermite-Hadamard type for m -convex functions*, Tamkang J. Math., 3(1) (2002).
- [8] Dragomir, S. S., Fitzpatrick, S., *The Hadamard's inequality for S -convex functions in the second sense*, Demonstratio Math. 32(4), (1999), 687-696.
- [9] Gill, P. M., Pearce, C. E. M., Pečarić, J., *Hadamard's inequality for r -convex functions*, Journal of Mathematical Analysis and Applications, vol. 215, no. 2, pp. 461--470, 1997.
- [10] Hudzik, H., Maligranda, L., *Some remarks on S -convex functions*, Aequationes Math. 48 (1994), 100-111.
- [11] Kirmaci, U.S., Bakula, M.K., Özdemir, M.E., Pečarić, J., *Hadamard-type inequalities for S -convex functions*, Appl. Math. and Comp., 193 (2007), 26-35.
- [12] Toader, G., *Some generalizations of the convexity*, Proceedings of The Colloquium On Approximation And Optimization, Univ. Cluj-Napoca, Cluj-Napoca, 1985, 329-338.
- [13] Özdemir, M. E., Avcı, M., Set, E., *On some inequalities of Hermite-Hadamard type via m -convexity*, Applied Mathematics Letters, vol. 23, no. 9, pp. 1065--1070, 2010.
- [14] Pečarić, J.E., Proschan, F., Tong, Y.L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
- [15] Set, E., Özdemir, M. E., Dragomir, S. S., *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, Journal of Inequalities and Applications, Article ID 148102, 9 pages, 2010.
- [16] Set, E., Özdemir, M. E., Dragomir, S. S., *On Hadamard-Type inequalities involving several kinds of convexity*, Journal of Inequalities and Applications, Article ID 286845, 12 pages, 2010.
- [17] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993, p. 106.
- [18] Anastassiou, G., Hooshmandasl, M.R., Ghasemi, A., Moftakharzadeh, F., *Montgomery identities for fractional integrals and related fractional inequalities*, J. Ineq. Pure and Appl. Math., 10(4) (2009), Art. 97.
- [19] Belarbi, S., Dahmani, Z., *On some new fractional integral inequalities*, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.
- [20] Dahmani, Z., *New inequalities in fractional integrals*, International Journal of Nonlinear Science, 9(4) (2010), 493-497.
- [21] Dahmani, Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1(1) (2010), 51-58.
- [22] Dahmani, Z., Tabharit, L., Taf, S., *Some fractional integral inequalities*, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.
- [23] Dahmani, Z., Tabharit, L., Taf, S., *New generalizations of Grüss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.

-
- [24] Gorenflo, R., Mainardi, F., *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.
- [25] Miller, S., Ross, B., *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, p.2.
- [26] Podlubni, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [27] Sarikaya, M.Z., Ogunmez, H. , *On new inequalities via Riemann-Liouville fractional integration*, Abstract and Applied Analysis, Volume 2012, Article ID 428983, 10 pages, doi:10.1155/2012/428983.
- [28] Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57 (2013), 2403--2407.
- [29] Set, E., New inequalities of Ostrowski type for mapping whose derivatives are s -convex in the second sense via fractional integrals, Computers and Math. with Appl. 63 (2012), 1147-1154.

Erhan Set,
Department of Mathematics,
Faculty of Science and Arts,
Ordu University, Ordu, TURKEY,
e-mail: erhanset@yahoo.com

Mehmet Zeki Sarikaya,
Department of Mathematics,
Faculty of Science and Arts,
Düzce University, Düzce, TURKEY,
e-mail: sarikayamz@gmail.com

M. Emin Özdemir
Department of Mathematics
K. K. Education Faculty,
Ataturk University, 25640, Kampus, Erzurum, TURKEY,
e-mail: emos@atauni.edu.tr

Hüseyin Yıldırım
Department of Mathematics,
Faculty of Science and Arts,
Kahramanmaraş Sütcü Imam University, Kahramanmaraş-TURKEY,
e-mail: hyildir@ksu.edu.tr