On Unstability of Certain Group of ODEs with Quasiderivatives

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Abstract

The article is focused on the unstability of some group of 4-th ODEs with quasi-derivatives. A sufficient condition of the unstability is followed by proper example.

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1. INTRODUCTION

The article is a continuation of [2], where some kind of the stability criterion has been derived for the equations (E) with quasiderivatives (the equations (E) as well as the quasiderivatives see below).

In [2] it was also shown an impossibility of replacement (in general) of the quasi-derivatives by ordinary derivatives in differential equations. Therefore new criteria of stability (as well as unstability) are needed.

The aim of the paper is to investigate the unstability (in Liapunov sense) of arbitrary solutions of equations of the form

$$Q_4 y + R_3(t)Q_3 y + R_2(t)Q_2 y + R_1(t)Q_1 y + R_0(t)Q_0 y = F(t, y), \quad (E)$$

where (a prime means a derivative by t)

$$Q_0 y(t) = y(t), \quad Q_i y(t) = r_i(t)(Q_{i-1} y(t))', \quad i = 1, 2, 3, \quad Q_4 y(t) = (Q_3 y(t))', \\ r_k(t), \quad k = 1, 2, 3, \quad R_j(t), \quad j = 0, 1, 2, 3$$

are continuous functions on $I_b = [b, \infty)$, $b \in R$, $F(t, y)$ is continuous on $I_b \times R$, where $R = (-\infty, \infty)$. The expressions $Q_n y(t)$, $n = 0, 1, 2, 3, 4$ are called *quasiderivatives*.

Let us consider ODE-system (“ODE” means “Ordinary Differential Equation”)

$$\dot{\mathbf{y}} = f_i(t, y_1, y_2, y_3, y_4), \quad i = 1, 2, 3, 4. \quad (\text{S})$$

MAIN ASSUMPTION. Let (S) be rewritten by vectors as $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$. Let there exists c (real or $-\infty$) and an area $A \subset \mathbb{R}^4$, $\mathbf{o} \in A$, $\mathbf{o} = (0, 0, 0, 0)$, such that $\mathbf{f}$ is continuous on an area $B = (c, \infty) \times A$ and for any couple $(s, \mathbf{m}) \in B$ the initial-value problem

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(s) = \mathbf{m}, \quad (1)$$

has the only one solution. Let $\mathbf{f}(t, \mathbf{o}) = \mathbf{o}$ for all $t > c$.

We shall define (by usual way) the concept of Liapunov stability of the trivial solution of (S) (for more details see Definition 1 in [2]) as well as Liapunov instability of the trivial solution of (S) (Definition 2 in [2]).

Let us consider the differential equation

$$\begin{aligned} Q_4 w(x) + R_3(x) Q_3 w(x) + R_2(x) Q_2 w(x) + R_1(x) Q_1 w(x) + \\ + R_0(x) Q_0 w(x) = F(x, u(x) + w(x)) - F(x, u(x)) \end{aligned} \quad (\text{N})$$

as well as the differential system

$$\begin{aligned} \dot{w}_i &= w_{i+1} / r_i(x), \quad i = 1, 2, 3, \\ \dot{w}_4 &= F(x, u(x) + w_1) - F(x, u(x)) - R_0(x)w_1 - R_1(x)w_2 - R_2(x)w_3 - R_3(x)w_4 \end{aligned} \quad (\text{T})$$

One can easily see that $w_1(t)$ solves (N) iff $(z_1(t), L_1 z_1(t), L_2 z_1(t), L_3 z_1(t))$ solves (T)

(“iff” means the usual abbreviation of “if and only if”).

We can also state the stability of the trivial solution of (N) by means of the stability of the trivial solution of (T) (see Definition 5 in [2]). Similarly, we can also define the unstability of the trivial solution of (N) by means of the unstability of the trivial solution of (T).

2. AUXILIARY ASSERTIONS

We determine a matrix norm as a sum of absolute values of the elements of the considered matrix.

AUXILIARY THEOREM. Let us take into account a differential system such that

(2) $\dot{\mathbf{x}} = \mathbf{C}\mathbf{x} + \mathbf{D}(t)\mathbf{x} + \mathbf{h}(t, \mathbf{x}), \mathbf{h}(t, \mathbf{o}) = \mathbf{o}$, where $\mathbf{C}$ is a constant matrix, (3) $\mathbf{D}(t)$ converges to zero matrix as t tends to infinity and $\mathbf{h}$ is continuous on $(b, \infty) \times A$, where $b \in R, \mathbf{o} \in A \subset R^4$ such that (4)

$\|\mathbf{h}(t, \mathbf{x})\|/\|\mathbf{x}\| \rightarrow 0$ uniformly for all $t \geq b$ as $\|\mathbf{x}\| \rightarrow 0$. If there exists an eigenvalue of $\mathbf{C}$ having a positive real part, then the null solution of (2) is unstable in Liapunov sense.

PROOF. See [1], Head 13. $\square$

Let $u(t)$ solves (E) on I_b . Let $y(t)$ be an arbitrary function defined on I_b . Let $z(t) = y(t) - u(t)$, $t \in I_b$. Then an arbitrary $z(t)$ solves (N) on I_b iff $y(t)$ solves (E) on I_b . This assertion can be proved in [2, Lemma 1].

The function $u(t)$ solving (E) is stable in Liapunov sense iff 0 solving (N) is stable in Liapunov sense. It is proved in [2, Lemma 1].

3. RESULTS

AUXILIARY LEMMA. The function $u(t)$ solving (E) is unstable in Liapunov sense iff 0 solving (N) is unstable in Liapunov sense.

PROOF. The assertion simply follows from the last assertion of the foregoing paragraph. $\square$

Now will be proved the instability criterion of (E).

MAIN THEOREM. Consider (E) such that Main assumption is valid. Let $u(t)$ solve (E) on I_b and

$$\lim_{t \rightarrow \infty} r_i(t) = r_i \in (0, \infty), i = 1, 2, 3, \quad \lim_{t \rightarrow \infty} R_j(t) = R_j \in R, j = 0, 1, 2, 3, \quad (a)$$

$$\frac{|F(t, u(t) + z) - F(t, u(t))|}{|z|} \rightarrow 0 \text{ uniformly for all } t \geq b \text{ as } z \rightarrow 0. \quad (c)$$

Let us denote $p(z) = z^4 + R_3 z^3 + \frac{R_2}{r_3} z^2 + \frac{R_1}{r_2 r_3} z + \frac{R_0}{r_1 r_2 r_3}$. If it holds that

$$\text{at least one real part of zeros of } p(z) \text{ is positive,} \quad (b')$$

then $u(t)$ is unstable (in Liapunov sense).

In Main theorem we use the symbols (a), (b') and (c) which are competent to the symbols (a), (b) and (c) in [2].

PROOF OF MAIN THEOREM. From the paragraph "Auxiliary assertions" it is clear that it suffices to check up the unstability of 0-solution of (N), which correspond to (E); 0-solution of (N) is unstable iff (0,0,0,0)-solution of (T) is unstable. (T) can be expressed as (V), where

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} + \mathbf{D}(t)\mathbf{z} + \mathbf{h}(t, \mathbf{z}), \quad \mathbf{h}(t, \mathbf{o}) = \mathbf{o}, \quad (V)$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1/r_1 & 0 & 0 \\ 0 & 0 & 1/r_2 & 0 \\ 0 & 0 & 0 & 1/r_3 \\ -R_0 & -R_1 & -R_2 & -R_3 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix},$$

$$\mathbf{D}(t) = \begin{bmatrix} 0 & 1/r_1(t) - 1/r_1 & 0 & 0 \\ 0 & 0 & 1/r_2(t) - 1/r_2 & 0 \\ 0 & 0 & 0 & 1/r_3(t) - 1/r_3 \\ R_0 - R_0(t) & R_1 - R_1(t) & R_2 - R_2(t) & R_3 - R_3(t) \end{bmatrix},$$

$$\mathbf{h}(t, \mathbf{z}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t, u(t) + z_1(t)) - F(t, u(t)) \end{bmatrix}, \quad \mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If $\|\mathbf{z}\| \neq 0, z_1 \neq 0$, then

$$\begin{aligned}
0 &\leq \frac{\|\mathbf{h}(t, \mathbf{z})\|}{\|\mathbf{z}\|} = \frac{|F(t, u(t) + z_1) - F(t, u(t))|}{|z_1| + |z_2| + |z_3| + |z_4|} = \\
&= \frac{|F(t, u(t) + z_1) - F(t, u(t))|}{|z_1|} \cdot \frac{|z_1|}{|z_1| + |z_2| + |z_3| + |z_4|} \leq \\
&\leq \frac{|F(t, u(t) + z_1) - F(t, u(t))|}{|z_1|}.
\end{aligned}$$

If $\|\mathbf{z}\| \neq 0$, $z_1 = 0$, then $\frac{\|\mathbf{h}(t, \mathbf{z})\|}{\|\mathbf{z}\|} = 0$.

From this and (c) it follows that (4) hold. It can be easily found out that (2), (3) hold. It is not difficult to show that $p(z)$ is the characteristic polynomial of $\mathbf{C}$. From (b') it implies that there exists an eigenvalue $a + bi$ of $\mathbf{C}$ such that $a > 0$. Then from Auxiliary theorem it implies that (0,0,0,0)-solution of (T) is unstable. $\square$

EXAMPLE. Consider (E) such that

$$\begin{aligned}
r_1(t) &= 1 + t^{-1}, \quad r_2(t) = 2 + t^{-1}, \quad r_3(t) = 3 + t^{-1}, \\
R_0(t) &= 6 + t^{-1}, \quad R_1(t) = -24 - t^{-1}, \quad R_2(t) = 18 + t^{-1}, \quad R_3(t) = -4 - t^{-1}, \\
F(t, y) &= 6t^{-1} + 25t^{-2} + 97t^{-3} + 293t^{-4} + 674t^{-5} + 959t^{-6} + 599t^{-7} + 120t^{-8} + (y - t^{-1})^2.
\end{aligned}$$

It can be taken $b = 2$. One can easily seen t^{-1} solves (E) on I_2 as well as $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, $R_0 = 6$, $R_1 = -24$, $R_2 = 18$, $R_3 = -4$. It implies that (a) hold.

If $z \neq 0$, then

$$\frac{|F(t, u(t) + z) - F(t, u(t))|}{|z|} = \frac{z^2}{|z|} = |z|.$$

From this it implies that $|z|$ uniformly converges to zero for $t \in [2, \infty)$ for z approaching to zero, i.e. (c) hold, too. The characteristic polynomial of $\mathbf{C}$ is $(z - 1)^4$. One of its zeros is $1 + 0i$, i.e. (b') hold. Then, owing to Main theorem, t^{-1} is an unstable solution of considered equation.

REFERENCES

- [1] CODDINGTON, E.A. AND LEVINSON, N. 1955. *Theory of ODEs*, McGraw-Hill, NY, USA.
- [2] PALUMBÍNÝ, O. 2016. On stability criterion of 4-th order quasilinear differential equations with quasiderivatives, *Research papers of Faculty of Materials Science and Technology of Slovak University of Technology in Bratislava*, Vol. 24, No. 39, 63–70.

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