Estimation Methods for a Flexible INAR(1) COM-Poisson Time Series Model

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Abstract

Time series of counts occur in many real-life situations where they exhibit various forms of dispersion. To facilitate the modeling of such time series, this paper introduces a flexible first-order integer-valued non-stationary autoregressive (INAR(1)) process where the innovation terms follow a Conway-Maxwell Poisson distribution (COM-Poisson). To estimate the unknown parameters in this model, different estimation approaches based on likelihood and quasi-likelihood formulations are considered. From simulation experiments and a real-life data application, the Generalized Quasi-Likelihood (GQL) approach yields estimates with lower bias than the other estimation approaches.

Mathematics Subject Classification 2010:

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1. INTRODUCTION

Counts observations are often exposed to over- and under-dispersion relative to their means. In the existing literature, the modeling of such data is handled mainly by some Generalized forms of the Poisson distribution such as the Generalized Poisson Distribution (GPD) [Consul and Shoukri 1984], the family of the Weighted Poisson (WP) distributions [Castillo and Perez-Cassany 2005], the Negative Binomial (NB) model specifically for the over-dispersion case and lately by the COM-Poisson model, among others. For some of the Generalized forms of the Poisson Distribution (PD) such as GPD and WP, the drawback is that such models do not form part of the class of exponential densities and Generalized Linear Models (GLMs) whilst for the renovated COM-Poisson distribution, the properties of GLMs such as the availability of the link function predictor and the forms of its mean and variance functions are compliant with the theory of exponential densities. These features in turn facilitate the use of standard estimation methods such as Maximum Likelihood Estimation (MLE) and Fisher-Scoring (FS) algorithm. As at date, many researchers such as Shmueli et al. [2005], Guikema and Coffelt [2008], Nadarajah [2009], Lord et al. [2010], Sellers et al. [2012] and Dey et al. [2016] have extensively applied the COM-Poisson to analyze many real-life data sets in linguistics, in modelling traffic accidents and number of sales in supermarkets and lately due to an upsurge of

interest in this model, Sellers et al. [2016] have developed the bivariate COM-Poisson. Shmueli et al. [2005] have also compared the consistency of the fitted values obtained from applying COM-Poisson, NB and GPD to real-life data and concluded that COM-Poisson is equally able to provide reliable values under any situation of over-, equi- and under-dispersion. The COM-Poisson structure is made up of a normalizing constant based on an infinite summation, which has been approximated to an asymptotic closed form expression to facilitate the computation of the moments and derivation of the moment generating function. However, an important note here is that the COM-Poisson under this approximated formula is especially suitable for over- and equi-dispersion but may still be used for the under-dispersion case as the means outweigh the variances under this approximation. Based on these elegant advantages, this paper proposes to extend the univariate COM-Poisson to an integer-valued autoregressive (INAR(1)) COM-Poisson time series model.

Time series of counts are usually modeled using parameter-driven (PD) or observation-driven (OD) approaches [Cox 1981]. In the parameter-driven models, the serial-correlation is induced by correlated randomly-distributed latent effects while under observation-driven, the observation at the current time point is related to previous lagged observation through the binomial thinning operation [Steutel and Van Harn 1979] which makes the serial-correlation and an innovation term taken at the current time point. Various studies have shown that it is more appropriate to model the time series using the OD model since parameter estimation becomes less cumbersome (See Ravishanker et al. [2014], Jung et al. [2006], Jung and Tremayne [2003], Jung and Liesenfeld [2001], Durbin and Koopman [2000] and Chan and Ledolter [1995]). One such class of OD model constitutes of the classical INAR(1) process introduced by McKenzie [1986] and Alzaid and Al Osh [1990] in the modeling of discrete time series processes where the innovation terms are Poisson or Negative Binomial marginals. In this paper, we explore the INAR(1) process where the innovation terms are COM-Poisson distributed. Since under such set-up, the counts are also mostly influenced by time-varying covariates as illustrated in many real-life studies, the proposed INAR(1) process in this paper is developed based on the error terms following COM-Poisson under non-stationary moments specified by time-dependent covariates. Thus, it is of worth interest to explore the different estimation methods to estimate the regression effects and the serial-correlations. Firstly, the Conditional Maximum Likelihood Estimation (CMLE) method is

appropriate in this context since conditioned on the previous observation, the likelihood function may be derived using convolution and thereafter, estimation of parameters is carried out by maximizing the conditional likelihood and solve for the unknown parameters using iterative approach such as Newton-Raphson method. The Conditional Least Squares (CLS) is also considered in this paper. In many time series studies conducted by Quoreshi [2008], CLS was used for estimation purpose and yielded consistent parameter estimates. Another popular technique is the Generalized Method of Moments (GMM) which combines a set of moment scores into an objective function and this is minimized and solved iteratively using the Newton-Raphson technique. However, in this paper, we use a modified GMM that is based on only two moment adaptive estimating equations developed by Qu and Lindsay [2003]. Note that the modified GMM has been tested and yields equally consistent and efficient estimates as the traditional GMM with more than two score functions [Qu and Lindsay 2003]. In the last ten years, the Generalized Quasi-Likelihood (GQL) method has also gained a lot of popularity due to its flexibility in requiring only the population means and variances of the model rather than the full likelihood function. The GQL technique is originated from the Generalized Estimating Equations (GEE) [Liang and Zeger 1986] and quasi-likelihood function [Wedderburn 1974]. In fact, in GEE it was difficult to specify the true correlation structure and Liang and Zeger [1986] suggested to approximate this structure by some working correlation matrices. However, Crowder [1995] and Sutradhar and Das [1999] noted that GEE estimates under working structure are comparatively inefficient as compared to the GEE based on independence structure approach and this is a clear contradiction since the observations are serially correlated. Then Sutradhar and Das [1999] developed a robust autocorrelation structure where the correlation coefficients are obtained through the method of moments and this robust correlation structure was incorporated into the GEE to yield the GQL estimating function. The GQL approach has been explored in many longitudinal studies, where consistent and efficient parameter estimates were generated and even in the late findings by Sutradhar et al. [2014], the GQL method was shown to provide asymptotically efficient estimates as the maximum likelihood based method.

This paper underpins all these approaches and compares the efficiency of the estimates under the different methods on the basis of the standard errors. The organization of the paper is as follows: In the next section, the INAR(1)

COM-Poisson time series model is developed and its autocorrelation structure is derived taking into account that the series of observations is influenced by some time-varying covariates. Section 3 of the paper focuses on the estimation methods. In section 4, a simulation study is presented and the different approaches are compared. In section 5, the INAR(1) COM-Poisson model is fitted to a real-life dataset in Mauritius and the paper is concluded in section 6.

2. THE INAR(1) COM-POISSON TIME SERIES PROCESS

The COM-Poisson probability function is expressed as

$$P(Y = y_j) = \frac{(\lambda_j)^{y_j}}{(y_j!)^\nu} \frac{1}{Z(\lambda_j, \nu)}, \quad y_j = 0, 1, 2, 3, \dots, \quad \text{for } \lambda_j \geq 0, \nu \geq 0 \quad (1)$$

where $Z(\lambda_j, \nu) = \sum_{k=0}^{\infty} \frac{(\lambda_j)^k}{(k!)^\nu}$.

The moments of the COM-Poisson are calculated iteratively using

$$E(Y^{r+1}) = \begin{cases} \lambda_j E(Y+1)^{1-\nu} & r = 0, \\ \lambda_j \frac{d}{d\lambda_j} E(Y^r) + E(Y)E(Y^r) & r > 0. \end{cases} \quad (2)$$

From Shmueli et al. [2005], Minka et al. [2003] and Sellers et al. [2012], the first two moments can be approximated by

$$E(Y_j) = (\lambda_j)^{1/\nu} - \left(\frac{\nu-1}{2\nu}\right) \quad \text{and} \quad V(Y_j) = \frac{(\lambda_j)^{1/\nu}}{\nu} \quad (3)$$

where the normalizing constant $Z(\lambda_j, \nu)$ is approximated by

$$Z(\lambda_j, \nu) \simeq \frac{\exp(\nu \lambda_j^{\frac{1}{\nu}})}{\lambda_j^{\frac{\nu-1}{2\nu}} (2\pi)^{\frac{\nu-1}{2}} \sqrt{\nu}} \quad (4)$$

From numerical studies, Shmueli et al. [2005] showed that these approximations work suitably well for all $\nu > 0$ but in particular for $\nu < 1$ or $\lambda_j > 10^\nu$. Under these expectations and variances, it can easily be verified that for $\nu < 1$, $V(Y_j) > E(Y_j)$ (over-dispersion) and for $\nu > 1$, $E(Y_j) > V(Y_j)$ (under-dispersion) since $\lambda_j > 0$. Under a regression set-up, these assumptions about over- and under-dispersion hold since $\lambda_j = \exp(x_j^T \beta)$ for $x_j = [x_{j1}, x_{j2}, \dots, x_{jp}]^T$ and $\beta = [\beta_1, \beta_2, \dots, \beta_p]^T$. Using McKenzie [1986] and Al-Osh and Alzaid [1988] approach, the

observation-driven equation for the first-order AR(1) process is given by

$$y_t = \rho * y_{t-1} + d_t \quad (5)$$

where d_t corresponds to the innovation term, $\rho * y_{t-1} | y_{t-1} \sim \text{Binomial}(y_{t-1}, \rho)$. d_t and y_{t-k} are independent, for $k = 1, 2, \dots, T$.

Assuming that Y_t is a COM-Poisson observation taken at the t^{th} time point, then the mean and variance at the t^{th} time point is given by

$$E(Y_t) = \theta_t = \lambda_t^{1/v} - \frac{v-1}{2v} \quad \text{and} \quad \text{Var}(Y_t) = \sigma_t^2 = \frac{\lambda_t^{1/v}}{v}, \quad \text{for } t = 1, 2, 3, \dots \quad (6)$$

Under a re-parametrization, $\text{Var}(Y_t) = \frac{\theta_t}{v} + \frac{v-1}{2v^2}$. Note that, under non-stationary assumption which is due to the presence of time-dependent covariates;

$$\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_t \neq \dots \neq \lambda_T \Rightarrow \theta_1 \neq \theta_2 \neq \dots \neq \theta_t \neq \dots \neq \theta_T \Rightarrow \sigma_1^2 \neq \sigma_2^2 \neq \dots \neq \sigma_t^2 \neq \dots \neq \sigma_T^2 \quad (7)$$

From equation (5) and equation (6), the mean and variance of the innovation term d_t are determined using the same procedure as defined by Mamode khan and Jowaheer [2013] for the stationary case. Under the moment expression obtained for d_t , d_t is shown to be COM-Poisson with

$$E(d_t) = \theta_t - \rho \theta_{t-1} \quad (8)$$

and the dispersion

$$\tilde{v} = \frac{(2\theta_t - 2\rho\theta_{t-1} + 1) + \sqrt{(2\theta_t - 2\rho\theta_{t-1} + 1)^2 - 8[(1 - \rho^2)(\frac{v-1}{2v^2}) + \frac{\theta_t}{v} - \frac{\rho\theta_{t-1}}{v}(\rho + v(1 - \rho))]}{4[(1 - \rho^2)(\frac{v-1}{2v^2}) + \frac{\theta_t}{v} - \frac{\rho\theta_{t-1}}{v}(\rho + v(1 - \rho))]} \quad (9)$$

and under these distributional assumptions for d_t , it can easily be re-verified that Y_t has same mean and variance as in equation (6). The serial-correlation is shown to be

$$\begin{aligned} \text{Corr}(Y_t, Y_{t-k}) &= \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\frac{\theta_t}{v} + \frac{v-1}{2v^2}} \sqrt{\frac{\theta_{t-k}}{v} + \frac{v-1}{2v^2}}} \\ &= \rho^k \frac{\sqrt{(\frac{\theta_{t-k}}{v} + \frac{v-1}{2v^2})}}{\frac{\theta_t}{v} + \frac{v-1}{2v^2}} \end{aligned} \quad (10)$$

for $k = 1, \dots, T-1$ and $0 < \rho < \min(\frac{\theta_2}{\theta_1}, \dots, \frac{\theta_T}{\theta_{T-1}}, 1)$.

3. ESTIMATION METHODS

3.1. Conditional Maximum Likelihood Estimation (CMLE)

In this section, the different estimation approaches are presented. The CML approach consists of using a conditional likelihood function of the form where

$$L(\rho, \beta, v|y) = \prod_{t=1}^T Pr(y_t|y_{t-1}, \rho, \beta, v) \\ = \sum_{t=1}^T \sum_{i=0}^m \left(\frac{\lambda_t^{y_t-i}}{(y_t-i)!^v} \right) \left(\frac{\exp(v\lambda_t^{\frac{1}{v}})}{\lambda_t^{\frac{v-1}{2v}} (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \right) \binom{y_{t-1}}{i} \rho^i (1-\rho)^{y_{t-1}-i} \quad (11)$$

where $m=\min(y_t, y_{t-1})$ and by maximizing this function with respect to the regression vector and the serial coefficient using the R optim function or the MatLab software, the likelihood scores are obtained (See Appendix). The objective function in the CLS approach is expressed as

$$Q(\psi) = \sum_{t=2}^T [y_t - E(y_t|y_{t-1})]^2 = \sum_{t=2}^T [y_t - \rho y_{t-1} - (\lambda_t^{1/v} - \frac{v-1}{2v}) + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})]^2$$

and again here, the derivatives are obtained by minimizing the above (See Appendix) and use Newton-Raphson iterative procedure to solve.

3.2. The GQL Approach

The GQL estimating function depends only on the specification of the mean and variance function and does not require the expression of the likelihood function. However, one important component in the GQL formulation is the modeling of the auto-covariance structure. In previous estimation approaches such as GEE, the auto-covariance was constructed using the working correlation structures but in the case of GQL, the auto-covariance is built under the assumption of a robust autocorrelation function [Sutradhar 2003; Sutradhar and Das 1999].

The GQL for the INAR(1) COM-Poisson time series model is given by

$$D^T \Sigma^{-1}(\tilde{y} - \tilde{\theta}) = 0 \quad (12)$$

where $\tilde{y} = [y_1, y_2, \dots, y_T]^T$ and $\tilde{\theta} = [\theta_1, \theta_2, \dots, \theta_T]^T$. Assuming a set of p explanatory variables $\tilde{\beta} = [\beta_1, \beta_2, \dots, \beta_p]^T$, the derivative matrix is a block diagonal $T \times (p+1)$

matrix and

$$D^T = \begin{pmatrix} \frac{\partial \theta_1}{\partial \beta_1} & \frac{\partial \theta_2}{\partial \beta_1} & \cdots & \frac{\partial \theta_T}{\partial \beta_1} \\ \frac{\partial \theta_1}{\partial \beta_2} & \frac{\partial \theta_2}{\partial \beta_2} & \cdots & \frac{\partial \theta_T}{\partial \beta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \theta_1}{\partial \beta_p} & \frac{\partial \theta_2}{\partial \beta_p} & \vdots & \frac{\partial \theta_T}{\partial \beta_p} \\ \frac{\partial \theta_1}{\partial v} & \frac{\partial \theta_2}{\partial v} & \cdots & \frac{\partial \theta_T}{\partial v} \end{pmatrix}_{(p+1) \times T} \quad (13)$$

where $\frac{\partial \theta_t}{\partial \beta_k} = \frac{(\lambda_t)^{\frac{1}{v}} x_t^T}{v}$ and $\frac{\partial \theta_t}{\partial v} = -\frac{(\lambda_t)^{\frac{1}{v}} \ln(\lambda_t)}{v^2} - \frac{1}{2v} + \frac{v-1}{2v^2}$.

The covariance structure Σ is computed using

$$\frac{\sum_{t=1}^{T-1} \tilde{Y}_t \tilde{Y}_{t+1} / (T-1)}{\sum_{t=1}^T \tilde{Y}_t^2 / T} = \rho \frac{\sum_{t=2}^T \sqrt{\frac{\lambda_{t-1}^{\frac{1}{v}}}{\frac{v}{\lambda_{t-1}^{\frac{1}{v}}}}}}{(T-1)} \quad (14)$$

Thus

$$\rho = \frac{T \sum_{t=1}^{T-1} \tilde{Y}_t \tilde{Y}_{t+1}}{[\sum_{t=1}^T \tilde{Y}_t^2][\sum_{t=2}^T \sqrt{\frac{\lambda_{t-1}^{\frac{1}{v}}}{\frac{v}{\lambda_{t-1}^{\frac{1}{v}}}}}] } \quad (15)$$

where $\tilde{y}_t = \frac{y_t - \theta_t}{\sqrt{\theta_t}}$ and this value of ρ is substituted in equation (10) to calculate the serial-correlation. The Newton-Raphson iterative technique is used to solve the GQL in equation (12) which yields

$$\begin{pmatrix} \hat{\beta}_{r+1} \\ \hat{v}_{r+1} \end{pmatrix} = \begin{pmatrix} \hat{\beta}_r \\ \hat{v}_r \end{pmatrix} + [D^T \Sigma^{-1} D]_r^{-1} [D^T \Sigma^{-1} (y - \theta)]_r \quad (16)$$

where $\hat{\beta}_r$ is the value of $\hat{\beta}$ at the r^{th} iteration. $[\cdot]_r$ is the value of the expression at the r^{th} iteration. The algorithm works by assuming an initial value of $\hat{\beta}$ and $\hat{v}$, we calculate $\hat{\rho}$ using equation (15) and then use this parameter to update the values of $\hat{\beta}$ and $\hat{v}$. Then the new set of parameters is used to calculate $\hat{\rho}$ and the iteration continues in this way until convergence. The estimators are consistent and under mild regularity conditions, it may be shown that $((\hat{\beta}, \hat{v}) - (\beta, v))^T$ has an asymptotic normal distribution with mean 0 and covariance matrix

$[D^T \tilde{\Sigma}^{-1} D]^{-1} [D^T \tilde{\Sigma}^{-1} (\tilde{y} - \tilde{\theta})(\tilde{y} - \tilde{\theta})^T \tilde{\Sigma}^{-1} D] [D^T \tilde{\Sigma}^{-1} D]^{-1}$ [Sutradhar 2003; Sutradhar et al. 2014; Mamode khan et al. 2016].

3.3. Generalized Method of Moments (GMM)

For the GMM methodology, we follow the adaptive GMM equation developed by Qu and Lindsay [2003] where only 2 moment conditions are taken into account,

$$g = \begin{bmatrix} D^T (y - \theta) \\ \hat{\alpha}^T D^T V (y - \theta) \end{bmatrix} \quad (17)$$

where $V = \frac{1}{T} (y - \theta)(y - \theta)^T$ and to solve the over-determined system of equations, the moment vector (17) is combined to form an objective function

$$Q(\beta) = g^T C^{-1} g \quad (18)$$

where C is the sample variance of g :

$$\begin{pmatrix} D^T V D & (D^T V^2 D) \hat{\alpha} \\ \hat{\alpha}^T (D^T V^2 D) & \hat{\alpha}^T (D^T V^3 D) \hat{\alpha} \end{pmatrix} \quad (19)$$

where $\hat{\alpha}$ is the product of the inverse of the upper triangular matrix, G , obtained from the Cholesky decomposition of the matrix $(D^T V - L D^T) V (D^T V - L D^T)$ where $L = (D^T V^2 D) (D^T V D)^{-1}$ and the eigenvector corresponding to the largest eigenvalue of the matrix $(G^T)^{-1} \tilde{D} \tilde{D}^T G^{-1}$ where $\tilde{D} = D^T V D - L D^T D$.

By minimizing the function (18) and applying the Newton-Raphson algorithm, we obtain the iterative

$$\hat{\beta}_{r+1} = \hat{\beta}_r - [\ddot{Q}(\hat{\beta}_r)]^{-1} \dot{Q}(\hat{\beta}_r) \quad (20)$$

where asymptotically, $\dot{Q}(\beta) = 2\dot{g}^T C^{-1} g$ and the hessian matrix $\ddot{Q}(\beta) = 2\dot{g}^T C^{-1} \dot{g}$ with $\dot{g} = \frac{\partial g}{\partial \beta^T}$ where $\dot{g}$ is of dimension $(p+1) \times p$. The iterative equation (20) works by assuming an initial value of $\hat{\beta}$ which is used to calculate $\dot{Q}(\beta)$ and $\ddot{Q}(\beta)$. After obtaining an updated value of $\hat{\beta}$, these 2 quantities are re-computed until the previous and current elements of $\hat{\beta}$ converges such that $\|\hat{\beta}_{r+1} - \hat{\beta}\| < 10^{-5}$. Qu and Lindsay [2003] showed that the estimator of β obtained by minimizing the objective function (18) is consistent and asymptotically normal with $\hat{\beta} - \beta \sim N\{0, (E[\dot{g}^T] E[C^{-1}] E[\dot{g}])^{-1}\}$.

Using the converged estimates from these methods, the forecasting equations are given by:

Using the model (5), for $t = T$ and $j = 1, 2, 3, \dots$

$$y_{T+j} = \hat{\rho}^j * y_T + \sum_{i=0}^{j-1} \hat{\rho}^i * d_{T+j-i} \quad (21)$$

where

$$d_{T+j} \sim \text{CMP}\left(\left(\frac{(\theta_{T+j} - \rho\theta_{T+j-1})}{\tilde{v}}, \tilde{v}\right)\right) \quad (22)$$

The conditional expectation and variance of the future observation $Y_{T+j}^{[k]}$ given $y_T^{[k]}$ are as follows:

$$E(Y_{T+j}|y_T) = \theta_{T+j} + \hat{\rho}^j(y_T - \theta_T) \quad (23)$$

$$\text{Var}(Y_{T+j}|y_T) = \left(\frac{\theta_{T+j}}{v} + \frac{v-1}{2v^2}\right) + \hat{\rho}^j(1 - \hat{\rho}^j)(y_T - \theta_T) - (\hat{\rho}^2)^j\left(\frac{\theta_T}{v} + \frac{v-1}{2v^2}\right) \quad (24)$$

Using equations (21), (23) and (24), the variance of the j -step ahead forecasting error, $e_{T+j} = y_{T+j} - E(Y_{T+j}|y_T)$, is given by

$$\begin{aligned} \text{Var}(e_{T+j}) &= \text{Var}[E(e_{T+j}|y_T)] + E[\text{Var}(e_{T+j}|y_T)] \\ &= \left(\frac{\theta_{T+j}}{v} + \frac{v-1}{2v^2}\right) - (\hat{\rho}^2)^j\left(\frac{\theta_{T+j-1}}{v} + \frac{v-1}{2v^2}\right) \end{aligned} \quad (25)$$

4. SIMULATIONS AND RESULTS

Time series data of size $T = 60, 100$, and 500 are generated using equation (5) under $v = 0.3, 0.5, 0.9, 1, 1.5, 2$ and 3 . A 2×1 regression vector is assumed where for the first covariate

$$x_{t1} = \begin{cases} \cos(2\pi t/12) & (t = 1, \dots, T/4), \\ \text{rnorm}(1, 0, 1) & (t = (T/4) + 1, \dots, 3T/4), \\ \sin(2\pi t/12) & (T = (3T/4) + 1, \dots, T), \end{cases}$$

and x_{t2} , the second covariate, follows standard normal distribution with true values of $\beta_1, \beta_2 = 1$. Note that for this simulation study, $\theta_0 = \theta_1$ and for each combination of v, ρ and T , 5000 simulations are run.

v	T	Method	$\rho = 0.5$				$\rho = 0.9$			
			$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{v}$	$\hat{\rho}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{v}$	$\hat{\rho}$
3	60	CMLE	1.0002	0.9956	2.9901	0.4921	0.9912	1.0036	3.0096	0.9012
			(0.1030)	(0.0954)	(0.1384)		(0.1162)	(0.1291)	(0.1515)	
		CLS	0.9848	0.9802	3.1531	0.4800	0.9840	0.9885	2.9828	0.9109
			(0.3114)	(0.3295)	(0.3789)		(0.3822)	(0.3360)	(0.3237)	
		GMM	1.1003	0.9850	2.9882	0.5175	0.9843	1.1371	3.1387	0.8896
			(0.4162)	(0.4702)	(0.4380)		(0.4131)	(0.4674)	(0.4720)	
		GQL	0.9864	0.9992	2.9951	0.4958	0.9985	1.0033	2.9932	0.8964
			(0.1029)	(0.0952)	(0.1383)		(0.1160)	(0.1290)	(0.1515)	
	100	CMLE	1.0120	0.9906	3.0119	0.4971	1.0122	0.9912	3.0028	0.8912
			(0.0971)	(0.0940)	(0.1019)		(0.0949)	(0.1087)	(0.1007)	
		CLS	0.9833	0.9843	2.9820	0.4841	0.9871	0.9829	2.9877	0.9115
			(0.2609)	(0.2518)	(0.2011)		(0.2215)	(0.2473)	(0.2670)	
		GMM	1.1644	0.9853	2.9882	0.5131	0.9819	1.1811	2.9821	0.8871
			(0.3111)	(0.3271)	(0.3202)		(0.3297)	(0.3712)	(0.3110)	
		GQL	0.9971	0.9906	2.9960	0.5047	1.0111	1.0209	3.0189	0.8911
			(0.0970)	(0.0938)	(0.1018)		(0.0949)	(0.1086)	(0.1005)	
	500	CMLE	0.9984	0.9902	2.9904	0.5068	0.9961	1.0005	3.0281	0.9088
			(0.0311)	(0.0403)	(0.0790)		(0.0738)	(0.0969)	(0.0740)	
		CLS	0.9864	0.9808	2.9878	0.4833	0.9806	0.9895	2.9818	0.9113
			(0.1682)	(0.1167)	(0.1111)		(0.1304)	(0.1378)	(0.1620)	
		GMM	1.1669	0.9827	2.9853	0.5103	0.9852	1.1678	2.9861	0.8920
			(0.2433)	(0.2540)	(0.2796)		(0.2490)	(0.2533)	(0.2332)	
		GQL	1.0066	0.9987	3.0108	0.5013	1.0195	0.9978	3.0052	0.9069
			(0.0310)	(0.0401)	(0.0788)		(0.0736)	(0.0967)	(0.0738)	
2.5	60	CMLE	1.0250	1.0120	2.5003	0.4928	1.0182	1.0245	2.4945	0.9002
			(0.1141)	(0.1205)	(0.1740)		(0.1050)	(0.1326)	(0.1651)	
		CLS	0.9857	0.9875	2.4801	0.4803	0.9899	0.9872	2.5171	0.8822
			(0.3282)	(0.3774)	(0.3317)		(0.3677)	(0.3824)	(0.3570)	
		GMM	1.1173	0.9821	2.4832	0.5103	0.9823	1.1293	2.5152	0.8869
			(0.4122)	(0.4361)	(0.4573)		(0.4813)	(0.4052)	(0.4153)	
		GQL	1.0980	1.0166	2.4922	0.4904	1.0153	0.9924	2.5015	0.8925
			(0.1141)	(0.1204)	(0.1739)		(0.1048)	(0.1325)	(0.1650)	
	100	CMLE	1.0203	1.0178	2.5025	0.4996	1.0190	1.0130	2.4920	0.9076
			(0.1022)	(0.1116)	(0.1280)		(0.0902)	(0.1167)	(0.1339)	
		CLS	0.9809	0.9858	2.5170	0.4845	0.9832	0.9807	2.4836	0.8819
			(0.2386)	(0.2507)	(0.2394)		(0.2033)	(0.2744)	(0.2642)	
		GMM	1.1212	0.9850	2.4859	0.5179	0.9840	1.1042	2.5121	0.8866
			(0.3194)	(0.3207)	(0.3667)		(0.3543)	(0.3318)	(0.3201)	
		GQL	1.0248	1.0128	2.5085	0.4983	1.0206	1.0187	2.5047	0.8966
			(0.1021)	(0.1115)	(0.1279)		(0.0900)	(0.1166)	(0.1339)	
	500	CMLE	1.0106	1.0112	2.4971	0.4903	1.0195	1.0190	2.5018	0.8930
			(0.0861)	(0.0725)	(0.0796)		(0.0792)	(0.0810)	(0.0930)	
		CLS	0.9889	0.9826	2.4891	0.4848	0.9888	0.9807	2.5159	0.8819
			(0.2082)	(0.2403)	(0.2273)		(0.2181)	(0.2339)	(0.2698)	
		GMM	1.1231	0.9808	2.4883	0.5192	0.9856	1.1058	2.5189	0.8812
			(0.3120)	(0.3243)	(0.3415)		(0.3316)	(0.3169)	(0.3253)	
		GQL	1.0159	1.0171	2.5013	0.5062	1.0185	1.0134	2.5008	0.8937
			(0.0858)	(0.0723)	(0.0795)		(0.0790)	(0.0810)	(0.0929)	

Table I. Estimates of the parameters and standard errors under different estimation methods for non-stationary Com-Poisson model

v	T	Method	$\rho = 0.5$				$\rho = 0.9$			
			$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{v}$	$\hat{\rho}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{v}$	$\hat{\rho}$
2	60	CMLE	1.0042	0.9991	2.0040	0.4910	0.9914	1.0127	2.0108	0.8907
			(0.1110)	(0.1105)	(0.1278)		(0.1382)	(0.1283)	(0.1100)	
		CLS	0.9855	0.9808	1.8991	0.4815	0.9836	0.9839	2.0174	0.9120
			(0.3323)	(0.3069)	(0.3160)		(0.3153)	(0.3749)	(0.3560)	
		GMM	1.1141	0.9824	2.1157	0.5189	0.9890	1.1137	2.1032	0.8834
			(0.4581)	(0.4999)	(0.4860)		(0.4281)	(0.4442)	(0.4668)	
		GQL	0.9913	0.9901	2.0175	0.4941	0.9970	1.0097	1.9568	0.8955
			(0.1110)	(0.1104)	(0.1277)		(0.1380)	(0.1281)	(0.1100)	
	100	CMLE	1.0131	0.9996	2.0141	0.4924	1.0027	0.9981	2.0160	0.8974
			(0.0761)	(0.0860)	(0.0903)		(0.0978)	(0.0717)	(0.0913)	
		CLS	0.9851	0.9828	2.1291	0.4819	0.9858	0.9821	2.1386	0.9114
			(0.2481)	(0.2906)	(0.2419)		(0.2727)	(0.2586)	(0.2440)	
		GMM	1.1454	0.9834	2.1020	0.5105	0.9867	1.1478	2.1411	0.8889
			(0.3317)	(0.3147)	(0.3022)		(0.3895)	(0.3098)	(0.3252)	
		GQL	0.9913	0.9934	2.0139	0.5022	1.0071	1.0007	2.0111	0.8970
			(0.0760)	(0.0859)	(0.0901)		(0.0975)	(0.0715)	(0.0913)	
	500	CMLE	0.9966	0.9999	2.0130	0.5085	0.9901	1.0017	2.0140	0.9080
			(0.0580)	(0.0578)	(0.0539)		(0.0491)	(0.0503)	(0.0527)	
		CLS	0.9899	0.9873	2.1804	0.4856	0.9863	0.9880	2.1721	0.9102
			(0.1357)	(0.1191)	(0.1695)		(0.1837)	(0.1680)	(0.1487)	
		GMM	1.1550	0.9856	1.9117	0.5133	0.9840	1.1064	2.1128	0.8822
			(0.2605)	(0.2256)	(0.2925)		(0.2108)	(0.2027)	(0.2968)	
		GQL	1.0085	0.9961	2.0121	0.5048	1.0091	0.9958	2.0102	0.9090
			(0.0578)	(0.0575)	(0.0539)		(0.0490)	(0.0502)	(0.0527)	
1.5	60	CMLE	1.0044	1.0156	1.4988	0.4922	1.0102	1.0129	1.5054	0.9007
			(0.1092)	(0.1231)	(0.1042)		(0.1204)	(0.1263)	(0.1142)	
		CLS	0.9809	0.9898	1.5156	0.4890	0.9872	0.9892	1.5117	0.8809
			(0.3432)	(0.3209)	(0.3449)		(0.3821)	(0.3585)	(0.3937)	
		GMM	1.1131	0.9808	1.5182	0.5133	0.9899	1.1117	1.5108	0.8891
			(0.4362)	(0.4218)	(0.4195)		(0.4185)	(0.4033)	(0.4093)	
		GQL	1.0088	1.0111	1.5001	0.4942	1.0115	1.0166	1.5054	0.8903
			(0.1090)	(0.1230)	(0.1041)		(0.1204)	(0.1263)	(0.1141)	
	100	CMLE	1.0101	1.0189	1.5014	0.4903	1.0141	1.0193	1.5007	0.9085
			(0.1045)	(0.1023)	(0.1015)		(0.0997)	(0.1091)	(0.1010)	
		CLS	0.9807	0.9839	1.5136	0.4814	0.9827	0.9804	1.5152	0.8893
			(0.2314)	(0.2913)	(0.2513)		(0.2916)	(0.2328)	(0.2755)	
		GMM	1.1504	0.9839	1.5179	0.5139	0.9880	1.1330	1.5186	0.8898
			(0.3218)	(0.3129)	(0.3412)		(0.3335)	(0.3529)	(0.3405)	
		GQL	1.0171	1.0169	1.5050	0.4993	1.0187	1.0103	1.5077	0.8937
			(0.1044)	(0.1022)	(0.1015)		(0.0996)	(0.1090)	(0.1010)	
	500	CMLE	1.0055	1.0031	1.5008	0.4948	1.0014	1.0071	1.5009	0.8931
			(0.0882)	(0.0940)	(0.0801)		(0.0852)	(0.0936)	(0.0961)	
		CLS	0.9811	0.9855	1.5102	0.4871	0.9819	0.9841	1.5150	0.8899
			(0.2547)	(0.2777)	(0.2141)		(0.2256)	(0.2380)	(0.2177)	
		GMM	1.1674	0.9810	1.5140	0.5183	0.9829	1.1466	1.5190	0.8886
			(0.2768)	(0.2879)	(0.2822)		(0.2889)	(0.2835)	(0.2736)	
		GQL	1.0018	1.0021	1.5050	0.5042	1.0044	1.0092	1.5012	0.8960
			(0.0880)	(0.0940)	(0.0800)		(0.0851)	(0.0935)	(0.0960)	
1	60	CMLE	1.0122	0.9979	0.9991	0.4918	0.9972	1.0101	0.9916	0.8951
			(0.1002)	(0.0917)	(0.1702)		(0.1014)	(0.1121)	(0.1678)	
		CLS	0.9729	0.9761	0.9742	0.4728	0.9761	0.9701	0.9783	0.9311
			(0.3517)	(0.3718)	(0.3850)		(0.3900)	(0.3104)	(0.3061)	
		GMM	1.3325	0.9746	0.9738	0.5340	0.9764	1.3109	0.9790	0.8720
			(0.4072)	(0.4680)	(0.4414)		(0.4259)	(0.4767)	(0.4993)	
		GQL	0.9881	0.9991	0.9989	0.4911	0.9995	1.0097	0.9988	0.8977
			(0.1001)	(0.0915)	(0.1701)		(0.1012)	(0.1120)	(0.1677)	
	100	CMLE	1.0056	0.9991	1.0010	0.4943	1.0035	0.9999	1.0001	0.8984
			(0.0850)	(0.0888)	(0.1112)		(0.0975)	(0.1181)	(0.1101)	
		CLS	0.9721	0.9715	0.9760	0.4726	0.9748	0.9722	0.9788	0.9322
			(0.2887)	(0.2633)	(0.2120)		(0.2060)	(0.2528)	(0.2718)	
		GMM	1.3406	0.9747	0.9700	0.5328	0.9703	1.3355	0.9756	0.8702
			(0.3684)	(0.3034)	(0.3136)		(0.3338)	(0.3960)	(0.3022)	
		GQL	0.9998	0.9995	0.9999	0.5008	1.0020	1.0010	1.0015	0.8912
			(0.0842)	(0.0880)	(0.1110)		(0.0974)	(0.1180)	(0.1100)	
	500	CMLE	0.9995	0.9999	0.9997	0.5002	0.9999	1.0002	1.0125	0.9015
			(0.0438)	(0.0501)	(0.0919)		(0.0870)	(0.1000)	(0.0868)	
		CLS	0.9815	0.9829	0.9844	0.4852	0.9860	0.9832	0.9824	0.9269
			(0.1787)	(0.1019)	(0.1230)		(0.1073)	(0.1202)	(0.1906)	
		GMM	1.2130	0.9871	0.9829	0.5230	0.9832	1.2154	0.9832	0.8815
			(0.2621)	(0.2764)	(0.2900)		(0.2770)	(0.2763)	(0.2511)	
		GQL	1.0025	0.9936	1.0010	0.5014	1.0025	0.9999	1.0010	0.9015
			(0.0438)	(0.0500)	(0.0915)		(0.0869)	(0.0990)	(0.0867)	

Table II. Estimates of the parameters and standard errors under different estimation methods for non-stationary Com-Poisson model

v	T	Method	$\rho = 0.5$				$\rho = 0.9$			
			$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{v}$	$\hat{\rho}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{v}$	$\hat{\rho}$
0.9	60	CMLE	1.0110 (0.1201)	1.0107 (0.1175)	0.8989 (0.1881)	0.4909	1.0110 (0.1111)	1.0105 (0.1230)	0.8990 (0.1776)	0.9028
		CLS	0.9778 (0.3463)	0.9794 (0.3871)	0.9112 (0.3417)	0.4782	0.9779 (0.3718)	0.9734 (0.3961)	0.9132 (0.3850)	0.8714
		GMM	1.3583 (0.4259)	0.9744 (0.4254)	0.9166 (0.4311)	0.5330	0.9750 (0.4926)	1.3340 (0.4180)	0.9187 (0.4041)	0.8710
		GQL	1.0112 (0.1200)	1.0113 (0.1173)	0.9001 (0.1880)	0.4905	1.0108 (0.1110)	1.0106 (0.1227)	0.9008 (0.1775)	0.8945
	100	CMLE	1.0105 (0.1001)	1.0118 (0.1097)	0.9112 (0.1331)	0.4930	1.0104 (0.0909)	1.0103 (0.1021)	0.8883 (0.1512)	0.9073
		CLS	0.9736 (0.2228)	0.9715 (0.2721)	0.9199 (0.2530)	0.4740	0.9742 (0.2156)	0.9763 (0.2895)	0.9112 (0.2923)	0.8714
		GMM	1.3390 (0.3528)	0.9713 (0.3680)	0.9134 (0.3824)	0.5329	0.9783 (0.3728)	1.3350 (0.3613)	0.9181 (0.3472)	0.8709
		GQL	1.0102 (0.1000)	1.0106 (0.1095)	0.8991 (0.1330)	0.4947	1.0103 (0.0907)	1.0104 (0.1020)	0.8980 (0.1512)	0.8919
	500	CMLE	1.0106 (0.0985)	1.0101 (0.0997)	0.8999 (0.0882)	0.4918	1.0104 (0.0825)	1.0102 (0.0922)	0.9001 (0.0901)	0.8962
		CLS	0.9895 (0.2135)	0.9805 (0.2529)	0.9160 (0.2119)	0.4826	0.9805 (0.2011)	0.9837 (0.2267)	0.9109 (0.2821)	0.8816
		GMM	1.2595 (0.3027)	0.9810 (0.3546)	0.9107 (0.3306)	0.5212	0.9840 (0.3299)	1.2246 (0.3026)	0.9141 (0.3313)	0.8829
		GQL	1.0104 (0.0980)	1.0157 (0.0995)	0.9033 (0.0882)	0.5010	1.0103 (0.0824)	1.0102 (0.0921)	0.9083 (0.0900)	0.8991
0.5	60	CMLE	1.0039 (0.1095)	0.9930 (0.1057)	0.4914 (0.1105)	0.4966	0.9959 (0.1260)	1.0113 (0.1131)	0.4944 (0.1244)	0.8917
		CLS	0.9710 (0.3993)	0.9750 (0.3128)	0.4753 (0.3057)	0.4762	0.9711 (0.3293)	0.9710 (0.3203)	0.4755 (0.3263)	0.9350
		GMM	1.3376 (0.4691)	0.9712 (0.4482)	0.4765 (0.4214)	0.5334	0.9777 (0.4513)	1.3145 (0.4632)	0.4727 (0.4130)	0.8787
		GQL	0.9950 (0.1092)	0.9967 (0.1055)	0.4910 (0.1100)	0.4909	0.9935 (0.1260)	1.0045 (0.1130)	0.4960 (0.1243)	0.8943
	100	CMLE	1.0050 (0.0882)	0.9936 (0.0881)	0.4901 (0.0861)	0.4921	1.0085 (0.0971)	0.9938 (0.0808)	0.4917 (0.0885)	0.8990
		CLS	0.9723 (0.2396)	0.9742 (0.2092)	0.4758 (0.2151)	0.4703	0.9791 (0.2371)	0.9731 (0.2110)	0.4714 (0.2450)	0.9305
		GMM	1.3960 (0.3540)	0.9752 (0.3533)	0.4725 (0.3008)	0.5380	0.9749 (0.3586)	1.3355 (0.3921)	0.4771 (0.3052)	0.8786
		GQL	0.9954 (0.0880)	0.9932 (0.0880)	0.4906 (0.0861)	0.5011	1.0015 (0.0970)	1.0050 (0.0807)	0.5094 (0.0884)	0.8942
	500	CMLE	0.9936 (0.0431)	0.9983 (0.0476)	0.4930 (0.0412)	0.5064	0.9905 (0.0400)	1.0061 (0.0560)	0.5096 (0.0497)	0.9035
		CLS	0.9872 (0.1453)	0.9840 (0.1115)	0.4892 (0.1302)	0.4817	0.9859 (0.1484)	0.9835 (0.1817)	0.4836 (0.1813)	0.9200
		GMM	1.2871 (0.2027)	0.9891 (0.2948)	0.4858 (0.2960)	0.5213	0.9862 (0.2198)	1.2066 (0.2740)	0.4864 (0.2422)	0.8802
		GQL	1.0079 (0.0430)	0.9954 (0.0475)	0.5068 (0.0412)	0.5090	1.0049 (0.0400)	0.9956 (0.0559)	0.5022 (0.0496)	0.9035
0.3	60	CMLE	1.0087 (0.1188)	1.0248 (0.1162)	0.2905 (0.1182)	0.4980	1.0261 (0.1118)	1.0223 (0.1115)	0.3097 (0.1106)	0.9018
		CLS	0.9787 (0.3884)	0.9709 (0.3816)	0.2741 (0.3620)	0.4739	0.9759 (0.3512)	0.9720 (0.3048)	0.2715 (0.3810)	0.8701
		GMM	1.3642 (0.4453)	0.9781 (0.4126)	0.2702 (0.4313)	0.5321	0.9796 (0.4275)	1.3327 (0.4330)	0.2729 (0.4730)	0.8700
		GQL	1.0126 (0.1186)	1.0160 (0.1160)	0.3008 (0.1182)	0.4903	1.0107 (0.1118)	1.0183 (0.1114)	0.3040 (0.1105)	0.8971
	100	CMLE	1.0112 (0.1037)	1.0197 (0.1006)	0.3056 (0.1022)	0.4933	1.0130 (0.0985)	1.0115 (0.1052)	0.3045 (0.1022)	0.9015
		CLS	0.9735 (0.2809)	0.9752 (0.2484)	0.2730 (0.2054)	0.4729	0.9714 (0.2400)	0.9774 (0.2307)	0.2734 (0.2464)	0.8700
		GMM	1.3301 (0.3804)	0.9761 (0.3205)	0.2712 (0.3608)	0.5378	0.9714 (0.3421)	1.3340 (0.3204)	0.2703 (0.3735)	0.8796
		GQL	1.0164 (0.1036)	1.0120 (0.1005)	0.2904 (0.1022)	0.4962	1.0148 (0.0984)	1.0175 (0.1052)	0.2991 (0.1020)	0.8930
	500	CMLE	1.0020 (0.0955)	1.0083 (0.0919)	0.2961 (0.0820)	0.4998	1.0050 (0.0838)	1.0014 (0.0969)	0.3064 (0.0931)	0.8924
		CLS	0.9869 (0.2303)	0.9808 (0.2287)	0.2818 (0.2000)	0.4851	0.9804 (0.2206)	0.9865 (0.2085)	0.2880 (0.2231)	0.8862
		GMM	1.2903 (0.2540)	0.9857 (0.2475)	0.2897 (0.2820)	0.5229	0.9880 (0.2760)	1.2178 (0.2806)	0.2898 (0.2406)	0.8832
		GQL	1.0031 (0.0954)	1.0088 (0.0919)	0.3091 (0.0820)	0.5027	1.0090 (0.0837)	1.0048 (0.0967)	0.3087 (0.0930)	0.8988

Table III. Estimates of the parameters and standard errors under different estimation methods for non-stationary Com-Poisson model

The tables display the simulated mean estimates of the regression, dispersion and correlation parameters along with the standard errors in brackets. These converged estimates are obtained by assuming small initial values in the iterative processes described in equations (16) and (20). As the number of time points increases, the standard errors across the different methods decreases, but the standard errors in the GQL approach is lower than those based on the GMM and CLS approaches while comparatively same as in the CMLE approach. This pattern is noted irrespective of the correlation parameter. In term of non-convergent simulations, for $T = 100$ and $\rho = 0.9$, CMLE failed in 2500 simulations, CLS in 1200, GMM in 1000 and GQL in 550, for $\nu = 0.5$. Almost the same number of failures is noted when $T = 500$, for the same level of dispersion. On the other hand, for $T = 500$, $\rho = 0.5$ and $\nu = 0.3$, CMLE flopped in 2700 simulations, CLS in 1350, GMM in 1100 and GQL in 600. When $T = 100$, $\nu = 1.5$ and $\rho = 0.5$, CMLE failed in 1650 simulations, CLS failed in 950 simulations, GMM in 830 and GQL in 425. For $T = 500$, CMLE flopped in 2525 simulations, CLS flopped in 1350, GMM in 1275 and GQL in 525. When $\nu = 2$, $T=100$ and $\rho = 0.5$, the non-convergent simulations are 1775 for CMLE, 975 for CLS, 825 for GMM and 320 for GQL. When $T=500$, the non-convergent simulations are 1550 for CMLE, 875 for CLS, 675 for GMM and 325 for GQL. In addition, when $T = 100$, $\nu = 2.5$ and $\rho = 0.9$, CMLE failed in 1950 simulations, CLS in 1020, GMM in 880 and GQL in 250. When $T = 500$, the number of non-convergent simulations are 2025 for CMLE, 1075 for CLS, 930 for GMM and 315 for GQL for similar values of serial and dispersion parameter. However, the non-convergent simulations are slightly less when $\rho = 0.5$. For $\nu = 3$, $T = 100$ and $\rho = 0.5$, CMLE failed in 2075 simulations, CLS in 1220, GMM in 980 and GQL in 350. As T increases to 500, for similar values of ν and ρ , the non-convergent simulations are 2040 for CMLE, 1180 for CLS, 1050 for GMM and 390 for GQL. Moreover, there were also some non-convergent simulations when $\nu = 1$. For $T = 100$ and $\rho = 0.5$, CMLE flopped in 1525 simulations, CLS in 810, GMM in 720 and GQL in 280. When $T=500$, the non-convergent simulations increases and are 1650 for CMLE, 880 for CLS, 775 for GMM and 320 for GQL. It is noted that the non-convergent simulations increase slightly when $\rho = 0.9$. To summarize, we conclude here that the GQL approach yields far better estimates than CLS and GMM under both high and low serial-correlation and dispersion parameter and is almost equally efficient as CMLE as confirmed by Sutradhar et al. [2014], but computationally more stable.

5. APPLICATION OF THE COM-POISSON INAR(1) PROCESS TO THE MCB DATA

The Mauritius Commercial Bank is one of the most long lasting and proficient banking institution in Mauritius and is also an active institution in the stock exchange market in the country. We have collected some of its daily number of transactions (TRANS) that took place on the Stock Exchange Mauritius (SEM) from the period 01.03.2015 to 01.06.2015 and that makes a total of 68 days excluding Saturdays, Sundays and public holidays. Note that SEM starts its operation at 09.00 a.m and closes at 13.30 p.m local time. The summary statistics show that TRANS has a mean of 7.36 with a variance of 10.36 thus clearly indicating over-dispersion and with an empirical lag 1 correlation of 0.69. The factors influencing TRANS were the news or rumour effect, the Monday effect and the time of the day. The news variable is a categorical variable with 0 indicating that no news were in circulation in day t while 1 indicates either positive or negative news were rumoured in the local or international market that influences investor's decision. The Monday effect is another binary variable with 1 indicating Monday and 0 for the other days in the week. The purpose of this variable is that it is perceived in the global market that Monday is the day where the highest number of investments may occur. The time of day variable is categorized into the morning session coded as 0 and the afternoon session coded as 1. Thus, along with the number of transactions, data related to these three factors were also recorded. The INAR(1) CMP model was fitted to the data and the parameters were estimated by CMLE, GMM, GQL and CLS. The results are displayed below:

Method	INTC	Monday	Time of Day	News	$\hat{\nu}$	$\hat{\rho}$
<i>CMLE</i>	0.2582 (0.4321)	0.1480 (0.0057)	-0.0327 (0.0040)	0.3210 (0.0109)	0.8111 (0.0048)	0.4532
<i>CLS</i>	0.2601 (0.5521)	0.2512 (0.0227)	-0.0418 (0.0190)	0.2133 (0.0311)	0.9120 (0.0178)	0.5529
<i>GMM</i>	0.2799 (0.6101)	0.2410 (0.0360)	-0.0428 (0.0198)	0.2158 (0.0382)	0.9185 (0.0195)	0.5147
<i>GQL</i>	0.2601 (0.4318)	0.1485 (0.0056)	-0.0330 (0.0040)	0.3222 (0.0107)	0.8164 (0.0047)	0.4529

Table IV. Estimates of the regression and over-dispersion parameters using CMLE, CLS, GMM and GQL approaches

From the above results, Monday and the news effects have positive coefficients indicating that there is a significant number of movements in the MCB transactions at SEM on Monday and during the day while the negative sign in the time of day variable shows that more of the transactions is taking place in the morning session rather than in the afternoon. The estimates under the different methods are almost

consistent, but GQL yields lower standard errors than GMM and CLS, however, comparatively equally efficient estimates as CMLE. However, CMLE yielded some computational problems in estimating the hessian entries due to the singularity of the covariance matrix. To avoid these complications, the initial values were refined. The over-dispersion estimates are all significant and the correlation parameter is reliable. Using the correlation coefficient, the non-stationary correlations are easily obtained through equations (10).

6. CONCLUSION

This paper has considered a novel non-stationary INAR(1) discrete time series observation-driven process based on the COM-Poisson model. The INAR(1) generating process was developed along with the derivation of the moments and covariance expressions. As for the estimation of the unknown parameters, the CMLE, CLS, GMM and GQL estimating functions were developed and it was remarked that the CMLE approach was rather computationally cumbersome and yielded a significant number of non-convergent simulations while GQL was computationally more appealing and provides consistent and equally efficient estimates than the other approaches.

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The derivatives of the conditional likelihood function for the CMLE are:

Differentiating with respect to ρ , we have

$$\sum_{t=1}^T \frac{\partial [\log L(\rho, \beta, v|y)]}{\partial \rho} = \sum_{t=1}^T \left[\sum_{i=0}^m \left(\frac{\lambda_t^{y_t-i}}{(y_t-i)!^v} \right) \left(\frac{\exp(v\lambda_t^{\frac{1}{v}})}{\lambda_t^{\frac{v-1}{2v}} (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \right) \binom{y_{t-1}}{i} [-(y_{t-1}-i)\rho^i(1-\rho)^{y_{t-1}-i-1} + i\rho^{i-1}(1-\rho)^{y_{t-1}-i}] / [Pr(y_t|y_{t-1})] \right] \quad (26)$$

Differentiating with respect to v , we have

$$\begin{aligned} \sum_{t=1}^T \frac{\partial [\log L(\rho, \beta, v|y)]}{\partial v} &= \sum_{t=1}^T \left[\sum_{i=0}^m \lambda_t^{y_t-i} \rho^i (1-\rho)^{y_{t-1}-i} \binom{y_{t-1}}{i} \right. \\ &\quad \left[\frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v\lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right. \\ &\quad - \frac{\exp(v\lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \ln((y_t-i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\ &\quad + \frac{\exp(v\lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (-\frac{1}{2v} - \frac{1-v}{2v^2}) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\ &\quad - \frac{\exp(v\lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{2\sqrt{v}} \\ &\quad \left. \left. - \frac{\exp(v\lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{2v^{\frac{3}{2}}} \right] \right] / [Pr(y_t|y_{t-1})] \quad (27) \end{aligned}$$

Differentiating with respect to β_j , we have

$$\begin{aligned} \sum_{t=1}^T \frac{\partial [\log L(\rho, \beta, v|y)]}{\partial \beta_j} &= \sum_{t=1}^T \left[\sum_{i=0}^m \frac{1}{(y_t-i)!^v (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \binom{y_{t-1}}{i} \rho^i (1-\rho)^{y_{t-1}-i} \right. \\ &\quad \left[\frac{\lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} x_t^T \exp(v\lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} + \frac{(x_t^T \beta_j)^{y_t-i} (y_t-i) \lambda_t^{y_t-i} \exp(v\lambda_t^{\frac{1}{v}})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} \right. \\ &\quad \left. \left. - \frac{\lambda_t^{y_t-i} x_t^T \exp(v\lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\lambda_t^{\frac{1-v}{2v}})} \right] \right] / [Pr(y_t|y_{t-1})] \quad (28) \end{aligned}$$

The derivation of the double derivatives in the CMLE approach are shown below:

$$\begin{aligned}
 \sum_{i=1}^T \frac{\partial^2 [\log L(\rho, \beta, v|y)]}{\partial v^2} &= \sum_{i=1}^T \left[\sum_{i=0}^m \lambda_t^{y_i-i} \rho^i (1-\rho)^{y_{i-1}-i} \binom{y_{i-1}}{i} \right. \\
 &\quad \left[- \frac{2 \exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \ln((y_i - i)!) \lambda_t^{\frac{1-v}{2v}} \left(-\frac{1}{2v} - \frac{1-v}{2v^2} \right) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right. \\
 &\quad - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \left(-\frac{1}{2v} - \frac{1-v}{2v^2} \right) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{v^{\frac{3}{2}}} \\
 &\quad - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \left(-\frac{1}{2v} - \frac{1-v}{2v^2} \right) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{\sqrt{v}} \\
 &\quad + \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)^2 \exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{v^{\frac{7}{2}}} \\
 &\quad + \frac{2(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \ln(\lambda_t) \exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \left(-\frac{1}{2v} - \frac{1-v}{2v^2} \right) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
 &\quad + \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \ln((y_i - i)!)^2 (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
 &\quad + \frac{1}{4} \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \ln(2\pi)^2 (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
 &\quad + \frac{3}{4} \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{v^{\frac{5}{2}}} \\
 &\quad + \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \left(\frac{1}{v^2} + \frac{1-v}{v^3} \right) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
 &\quad + \frac{1}{2} \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} \ln(2\pi) (2\pi)^{\frac{1-v}{2}}}{v^{\frac{3}{2}}} \\
 &\quad - \frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{v^{\frac{3}{2}}} \\
 &\quad + \frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v})^2 \exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
 &\quad \left. - 2 \frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v \lambda_t^{\frac{1}{v}}) (y_i - i)!^{-v} \ln((y_i - i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right]
 \end{aligned}$$

$$\begin{aligned}
& - \frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \ln(2\pi) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
& + \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \ln((y_t - i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{v^{\frac{3}{2}}} \\
& + \frac{\ln(\lambda_t)^2 \exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (-\frac{1}{2v} - \frac{1-v}{2v^2})^2 (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
& + \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \ln((y_t - i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{\sqrt{v}} \big] / [Pr(y_t | y_{t-1})] \\
& - \frac{[\sum_{i=0}^m \lambda_t^{y_t-i} \rho^i (1-\rho)^{y_{t-1}-i} \binom{y_{t-1}}{i} \frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}}}{\sqrt{v}} \\
& - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \ln((y_t - i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
& + \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (-\frac{1}{2v} - \frac{1-v}{2v^2}) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
& - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{2\sqrt{v}} \\
& - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t - i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{2v^{\frac{3}{2}}} \big]^2 / [Pr(y_t | y_{t-1})]^2 \quad (29)
\end{aligned}$$

$$\begin{aligned}
\sum_{t=1}^T \frac{\partial^2 [\log L(\rho, \beta, v | y)]}{\partial \rho^2} &= \sum_{t=1}^T \left[\left[\sum_{i=0}^m \left(\frac{\lambda_t^{y_t-i}}{(y_t-i)!^v} \right) \left(\frac{\exp(v \lambda_t^{\frac{1}{v}})}{\lambda_t^{\frac{v-1}{2v}} (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \right) \binom{y_{t-1}}{i} \right] \right. \\
&\quad + (y_{t-1} - i) \rho^i (1-\rho)^{y_{t-1}-i-2} + i(i-1) \rho^{i-2} (1-\rho)^{y_{t-1}-i} \\
&\quad \left. - i(y_{t-1} - i) \rho^{i-1} (1-\rho)^{y_{t-1}-i-1} \right] / \sum_{i=0}^m Pr(y_t | y_{t-1}) \\
&\quad - \left[\sum_{i=0}^m \left(\frac{\lambda_t^{y_t-i}}{(y_t-i)!^v} \right) \left(\frac{\exp(v \lambda_t^{\frac{1}{v}})}{\lambda_t^{\frac{v-1}{2v}} (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \right) \binom{y_{t-1}}{i} \right] \\
&\quad \times \left[- (y_{t-1} - i) \rho^i (1-\rho)^{y_{t-1}-i-1} + i \rho^{i-1} (1-\rho)^{y_{t-1}-i} \right]^2 / Pr(y_t | y_{t-1})^2 \quad (30)
\end{aligned}$$

$$\begin{aligned}
\sum_{t=1}^T \frac{\partial^2 [\log L(\rho, \beta, v|y)]}{\partial \beta_j^2} &= \sum_{t=1}^T \left[\left[\sum_{i=0}^m \frac{1}{(y_t - i)! v (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \binom{y_{t-1}}{i} \rho^i (1-\rho)^{y_{t-1}-i} \right. \right. \\
&\quad \left[\frac{(x_t^T \beta_j)^{y_t-i} (y_t - i)^2 \lambda_t^{y_t-i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j^2 \lambda_t^{\frac{1-v}{2v}})} - \frac{(x_t^T \beta_j)^{y_t-i} (y_t - i) \lambda_t^{y_t-i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j^2 \lambda_t^{\frac{1-v}{2v}})} \right. \\
&\quad + \frac{[(x_t^T \beta_j)^{y_t-i}]^2 (y_t - i)^2 \lambda_t^{y_t-i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j^2 \lambda_t^{\frac{1-v}{2v}})} \\
&\quad + \frac{2(x_t^T \beta_j)^{y_t-i} (y_t - i) \lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} x_t^T \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} \\
&\quad - \frac{2(x_t^T \beta_j)^{y_t-i} (y_t - i) \lambda_t^{y_t-i} x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} + \frac{\lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} (\frac{1}{v}) (x_t^T)^2 \exp(v \lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} \\
&\quad + \frac{\lambda_t^{y_t-i} (\lambda_t^{\frac{1}{v}})^2 (x_t^T)^2 \exp(v \lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} - \frac{2 \lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} (x_t^T)^2 \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\lambda_t^{\frac{1-v}{2v}})} \\
&\quad + \left. \frac{\lambda_t^{y_t-i} (x_t^T)^2 \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})^2}{(\lambda_t^{\frac{1-v}{2v}})} \right] / [Pr(y_t | y_{t-1})] \Big] \\
&\quad - \left[\sum_{i=0}^m \frac{1}{(y_t - i)! v (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \binom{y_{t-1}}{i} \rho^i (1-\rho)^{y_{t-1}-i} \right. \\
&\quad \left[\frac{\lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} x_t^T \exp(v \lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} + \frac{(x_t^T \beta_j)^{y_t-i} (y_t - i) \lambda_t^{y_t-i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} \right. \\
&\quad \left. \left. - \frac{\lambda_t^{y_t-i} x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\lambda_t^{\frac{1-v}{2v}})} \right]^2 / [Pr(y_t | y_{t-1})]^2 \right]
\end{aligned} \tag{31}$$

$$\begin{aligned}
\sum_{t=1}^T \frac{\partial^2 [\log L(\rho, \beta, v|y)]}{\partial \rho \partial v} &= \sum_{t=1}^T \left[\sum_{i=0}^m \lambda_t^{y_t-i} \binom{y_{t-1}}{i} [-(y_{t-1}-i)\rho^i(1-\rho)^{y_{t-1}-i-1} \right. \\
&\quad + i\rho^{i-1}(1-\rho)^{y_{t-1}-i}] \times \left[\frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right. \\
&\quad - \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \ln((y_t-i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad + \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (-\frac{1}{2v} - \frac{1-v}{2v^2}) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad - \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{2\sqrt{v}} \\
&\quad - \left. \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{2v^{\frac{3}{2}}} \right] / [Pr(y_t|y_{t-1})] \\
&\quad - \left[\left(\frac{\lambda_t^{y_t-i}}{(y_t-i)!^v} \right) \left(\frac{\exp(v\lambda_t^{\frac{1}{v}})}{\lambda_t^{\frac{v-1}{2v}} (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \right) \binom{y_{t-1}}{i} [-(y_{t-1}-i)\rho^i(1-\rho)^{y_{t-1}-i-1} \right. \\
&\quad + i\rho^{i-1}(1-\rho)^{y_{t-1}-i}] \\
&\quad \times \left[\lambda_t^{y_t-i} \rho^i(1-\rho)^{y_{t-1}-i} \binom{y_{t-1}}{i} \right. \\
&\quad \left[\frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right. \\
&\quad - \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \ln((y_t-i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad + \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (-\frac{1}{2v} - \frac{1-v}{2v^2}) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad - \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{2\sqrt{v}} \\
&\quad - \left. \left. \frac{\exp(v\lambda_t^{\frac{1}{v}})(y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{2v^{\frac{3}{2}}} \right] \right] / [Pr(y_t|y_{t-1})]^2 \Big] \quad (32)
\end{aligned}$$

$$\begin{aligned}
\sum_{t=1}^T \frac{\partial^2 [\log L(\rho, \beta, v|y)]}{\partial \beta_j \partial \rho} &= \sum_{t=1}^T \left[\sum_{i=0}^m \frac{1}{(y_t - i)!^v (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \binom{y_{t-1}}{i} [-(y_{t-1} - i) \rho^i (1 - \rho)^{y_{t-1} - i - 1} \right. \\
&\quad + i \rho^{i-1} (1 - \rho)^{y_{t-1} - i}] \times \left[\frac{\lambda_t^{y_t - i} \lambda_t^{\frac{1}{v}} x_t^T \exp(v \lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} \right. \\
&\quad + \frac{(x_t^T \beta_j)^{y_t - i} (y_t - i) \lambda_t^{y_t - i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} \\
&\quad \left. - \frac{\lambda_t^{y_t - i} x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\lambda_t^{\frac{1-v}{2v}})} \right] / [Pr(y_t | y_{t-1})] \\
&\quad - \left(\frac{\lambda_t^{y_t - i}}{(y_t - i)!^v} \right) \left(\frac{\exp(v \lambda_t^{\frac{1}{v}})}{\lambda_t^{\frac{v-1}{2v}} (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \right) \binom{y_{t-1}}{i} [-(y_{t-1} - i) \rho^i (1 - \rho)^{y_{t-1} - i - 1} \\
&\quad + i \rho^{i-1} (1 - \rho)^{y_{t-1} - i}] \times \left[\frac{1}{(y_t - i)!^v (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \binom{y_{t-1}}{i} \rho^i (1 - \rho)^{y_{t-1} - i} \right. \\
&\quad \left. \left[\frac{\lambda_t^{y_t - i} \lambda_t^{\frac{1}{v}} x_t^T \exp(v \lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} + \frac{(x_t^T \beta_j)^{y_t - i} (y_t - i) \lambda_t^{y_t - i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} \right. \right. \\
&\quad \left. \left. - \frac{\lambda_t^{y_t - i} x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\lambda_t^{\frac{1-v}{2v}})} \right] \right] / [Pr(y_t | y_{t-1})]^2 \quad (33)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^T \frac{\partial^2 [\log L(\rho, \beta, v|y)]}{\partial \beta_j \partial v} &= \sum_{i=1}^T \left[\left(\sum_{i=0}^m \rho^i (1-\rho)^{y_{t-1}-i} \binom{y_{t-1}}{i} \right. \right. \\
&\quad \left[\frac{\lambda_t^{y_t-i} (y_t-i) x_t^T (\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right. \\
&\quad - \frac{\lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} x_t^T \ln(\lambda_t) \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{v^{\frac{5}{2}}} \\
&\quad - \frac{\lambda_t^{y_t-i} (y_t-i) x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \ln((y_t-i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad + \frac{\lambda_t^{y_t-i} (y_t-i) x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} (-\frac{1}{2v} - \frac{1-v}{2v^2}) \ln(\lambda_t) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad + \frac{\lambda_t^{y_t-i} \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} x_t^T (-\frac{1}{2v} - \frac{1-v}{2v^2}) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad - \frac{1}{2} \frac{\lambda_t^{y_t-i} (y_t-i) x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \ln(2\pi) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad \left. \left. - \frac{1}{2} \frac{\lambda_t^{y_t-i} (y_t-i) x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{v^{\frac{3}{2}}} \right] \right] / [Pr(y_t|y_{t-1})] \\
&\quad - \sum_{i=0}^m \lambda_t^{y_t-i} \rho^i (1-\rho)^{y_{t-1}-i} \binom{y_{t-1}}{i} \\
&\quad \left[\frac{(\lambda_t^{\frac{1}{v}} - \frac{\lambda_t^{\frac{1}{v}} \ln(\lambda_t)}{v}) \exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \right. \\
&\quad - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \ln((y_t-i)!) \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad + \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (-\frac{1}{2v} - \frac{1-v}{2v^2}) \ln(\lambda_t) (2\pi)^{\frac{1-v}{2}}}{\sqrt{v}} \\
&\quad - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}} \ln(2\pi)}{2\sqrt{v}} \\
&\quad \left. - \frac{\exp(v \lambda_t^{\frac{1}{v}}) (y_t-i)!^{-v} \lambda_t^{\frac{1-v}{2v}} (2\pi)^{\frac{1-v}{2}}}{2v^{\frac{3}{2}}} \right] \\
&\quad \times \left[\frac{1}{(y_t-i)!^v (2\pi)^{\frac{v-1}{2}} \sqrt{v}} \binom{y_{t-1}}{i} \rho^i (1-\rho)^{y_{t-1}-i} \right. \\
&\quad \left[\frac{\lambda_t^{y_t-i} \lambda_t^{\frac{1}{v}} x_t^T \exp(v \lambda_t^{\frac{1}{v}})}{(\lambda_t^{\frac{1-v}{2v}})} + \frac{(x_t^T \beta_j)^{y_t-i} (y_t-i) \lambda_t^{y_t-i} \exp(v \lambda_t^{\frac{1}{v}})}{(\beta_j \lambda_t^{\frac{1-v}{2v}})} \right. \\
&\quad \left. - \frac{\lambda_t^{y_t-i} x_t^T \exp(v \lambda_t^{\frac{1}{v}}) (\frac{1-v}{2v})}{(\lambda_t^{\frac{1-v}{2v}})} \right] / [Pr(y_t|y_{t-1})]^2 \Big]
\end{aligned} \tag{34}$$

The derivatives of the criterion function for the CLS estimator are:

$$\frac{\partial Q(\psi)}{\partial \rho} = 2 \sum_{t=2}^T [y_t - \rho y_{t-1} - (\lambda_t^{1/v} - \frac{v-1}{2v}) + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})](-y_{t-1} + (\lambda_{t-1}^{1/v} - \frac{v-1}{2v})) \quad (35)$$

$$\frac{\partial^2 Q(\psi)}{\partial \rho^2} = 2 \sum_{t=2}^T -y_{t-1} + (\lambda_{t-1}^{1/v} - \frac{v-1}{2v}) \quad (36)$$

$$\begin{aligned} \frac{\partial Q(\psi)}{\partial v} &= 2 \sum_{t=2}^T [y_t - \rho y_{t-1} - (\lambda_t^{1/v} - \frac{v-1}{2v}) + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})] (\frac{\lambda_t^{1/v} \ln(\lambda_t)}{v^2} - \frac{v-1}{2v^2} + \frac{1}{2v} \\ &\quad - \rho \frac{(\lambda_{t-1}^{1/v} \ln(\lambda_{t-1}))}{v^2} + \rho \frac{(v-1)}{2v^2} - \rho \frac{1}{2v}) \end{aligned} \quad (37)$$

$$\begin{aligned} \frac{\partial^2 Q(\psi)}{\partial v^2} &= 2 \sum_{t=2}^T [\frac{\lambda_t^{1/v} \ln(\lambda_t)}{v^2} + \frac{1}{2v} - \frac{1}{2} \frac{v-1}{v^2} + \rho(\frac{-\lambda_{t-1}^{1/v} \ln(\lambda_{t-1})}{v^2} + \frac{(v-1)}{2v^2} - \frac{1}{2v})]^2 \\ &\quad + [y_t - \rho y_{t-1} - \lambda_t^{1/v} + \frac{v-1}{2v} + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})] \times [\frac{-\lambda_t^{1/v} \ln(\lambda_t)^2}{v^4} - 2 \frac{\lambda_t^{1/v} \ln(\lambda_t)}{v^3} \\ &\quad - \frac{1}{v^2} + \frac{(v-1)}{v^3} + \rho(\frac{\lambda_{t-1}^{1/v} \ln(\lambda_{t-1})^2}{v^4} + 2 \frac{\lambda_{t-1}^{1/v} \ln(\lambda_{t-1})}{v^3} + \frac{1}{v^2} - \frac{v-1}{v^3})] \end{aligned} \quad (38)$$

$$\frac{\partial Q(\psi)}{\partial \beta_j} = 2 \sum_{t=2}^T [(y_t - \rho y_{t-1} - (\lambda_t^{1/v} - \frac{v-1}{2v}) + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v}))(-\frac{\lambda_t^{(1/v)}}{v} x_t^T + \frac{\rho}{v} \lambda_{t-1}^{(1/v)} x_{t-1}^T)] \quad (39)$$

$$\begin{aligned} \frac{\partial^2 Q(\psi)}{\partial \beta_j^2} &= 2 \sum_{t=2}^T [(-\frac{\lambda_t^{(1/v)}}{v} x_t^T + \frac{\rho}{v} \lambda_{t-1}^{(1/v)} x_{t-1}^T)^2 + [y_t - \rho y_{t-1} - \lambda_t^{1/v} + \frac{v-1}{2v} + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})] \\ &\quad \times [-\frac{\lambda_t^{(1/v)}}{v^2} (x_t^T)^2 + \frac{\rho}{v^2} \lambda_{t-1}^{(1/v)} (x_{t-1}^T)^2]] \end{aligned} \quad (40)$$

$$\begin{aligned} \frac{\partial^2 Q(\psi)}{\partial \beta_j \partial \rho} &= 2 \sum_{t=2}^T [(-y_{t-1} + \lambda_{t-1}^{1/v} - \frac{v-1}{2v}) \times [-\frac{\lambda_t^{(1/v)}(x_t^T)}{v} + \frac{\rho}{v} \lambda_{t-1}^{(1/v)}(x_{t-1}^T)] \\ &\quad + \frac{[y_t - \rho y_{t-1} - \lambda_t^{1/v} + \frac{v-1}{2v} + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})] \times (\lambda_{t-1}^{(1/v)} x_{t-1}^T)}{v}] \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{\partial^2 Q(\psi)}{\partial \beta_j \partial v} &= 2 \sum_{t=2}^T [(\frac{\lambda_t^{1/v} \ln(\lambda_t)}{v^2} + \frac{1}{2v} - \frac{v-1}{2v^2} + \rho(-\frac{\lambda_{t-1}^{1/v} \ln(\lambda_{t-1})}{v^2} - \frac{1}{2v} + \frac{v-1}{2v^2})) \\ &\quad \times (-\frac{\lambda_t^{1/v} x_t^T}{v} + \frac{\rho \lambda_{t-1}^{1/v} x_{t-1}^T}{v}) + [y_t - \rho y_{t-1} - \lambda_t^{1/v} + \frac{v-1}{2v} + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})] \\ &\quad \times [\frac{\lambda_t^{(1/v)} \ln(\lambda_t)(x_t^T)}{v^3} + \frac{\lambda_t^{(1/v)}(x_t^T)}{v^2} - \frac{\rho}{v^3} \lambda_{t-1}^{(1/v)} \ln(\lambda_{t-1})(x_{t-1}^T) - \frac{\rho}{v^2} \lambda_{t-1}^{(1/v)}(x_{t-1}^T)]] \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{\partial^2 Q(\psi)}{\partial v \partial \rho} &= 2 \sum_{t=2}^T [(\lambda_{t-1}^{1/v} - \frac{v-1}{2v}) \times [\frac{\lambda_t^{1/v} \ln(\lambda_t)}{v^2} - \frac{v-1}{2v^2} + \frac{1}{2v} - \rho \frac{(\lambda_{t-1}^{1/v} \ln(\lambda_{t-1}))}{v^2} + \rho \frac{(v-1)}{2v^2} - \rho \frac{1}{2v}] \\ &\quad + [y_t - \rho y_{t-1} - \lambda_t^{1/v} + \frac{v-1}{2v} + \rho(\lambda_{t-1}^{1/v} - \frac{v-1}{2v})] \times [-\frac{\lambda_{t-1}^{1/v} \ln(\lambda_{t-1})}{v^2} + \frac{(v-1)}{2v^2} - \frac{1}{2v}]] \end{aligned} \quad (43)$$

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